1) Introduction

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1.1 Introduction

The course concentrates on astrophysics, not on descriptive astronomy. We use first year physics as much as possible and make many quantitative calculations. However, some of the physics of the Solar System is either too complicated (e.g. planetary structure) or not yet understood (e.g. the solar neutrino problem) for us to do more than describe current ideas.

1.2 Recommended books:

Planetary Science, Cole & Woolfson (Institute of Physics)
The New Solar System, Bealty, J.k. & Chaikin (CUP)

1.3 The Solar System

The Solar System is "a tiny island in the galaxy". The overall size corresponds to 1/30,000th of the distance to the nearest star. It is two thirds of the way out from the centre of our spiral galaxy.
The solar system is dominated by the Sun; an unexceptional "yellow dwarf" star containing 99.87% of the total mass of the solar system, mostly as ionized Hydrogen and Helium. The planets, in comparison, are tiny. Jupiter is the dominant planet.
If the sun is modeled as a beach ball, Jupiter would be a golf ball 150m away, moving at 8mm/hour. Earth would be a small pea about 20m away from the beach ball.

The Solar System has many regularities:

1. The orbits are roughly circular, apart from Mercury and Pluto
2. It is mostly coplanar, i.e. a 2D system
3. Most of the moves and spins are in the same direction
4. It obeys Keppler's Laws
5. "Bode's Law": the planetary distances from the sun can be expressed as simple ratios. The significance of this remains contentious.

1.4 The Major Constituents of the Solar System

There are four major constituents:

1. "Terrestrial Planets".
   These consist of rocky silicates. Mercury and the Moon are airless, with no geological activity and heavy cratering. The Earth and Venus have atmospheres, giving them the "greenhouse" effect. They also have geological activity. Mars has intermediate properties.
2. "Gas Giants"
   These consist mostly of Hydrogen, Helium, Methane, Ammonia, CO_2 etc. Jupiter, Saturn, Uranus and Neptune all have complex atmospheres, probably rocky cores, and complex satellite and ring systems which can be viewed as mini-solar systems.
3. Asteroids
   There are 10^4 to 10^6 asteroids, with numbers increasing all the time. They are rocky objects with diameters ranging from 0.3 to 350km (Ceres), and <1/1000th of the mass of Earth. Their origins are not certain.
4. Comets
   "Dirty snowballs". They contain primitive material from the formation stage of the solar system. Comet Halley is a few km in diameter, and most comets evaporate near to the sun. They spend most of their time well outside the orbit of Pluto.

Our knowledge of the various bodies in the solar system has been extremely limited for most of human history, however in the last 30 years or so a series of spacecraft, e.g. Pioneer, Mariner, Voyager, Venra, Phobos, Giotto, Galileo, Cassini and Mars probes have given us our first glimpses of the outer planets, have studied Mars, the Moon and Venus in some detail with surface landers as well as remote sensing instruments.

1.5 The Study of the Solar System in ancient Times

Astronomy was the first science, i.e. an activity involving a body of systematic knowledge about the natural world. For most of history it consisted of a study of the solar system because the motions of the Sun, Moon and the planets are easily discernible from systematic observations with the unaided eye.

Mankind learned the motions for calendrical purposes: predicting the seasons gave the planting times, but also for religious, mystical (e.g. predictions of eclipses) and astrological purposes (NB: all current astronomical signs are "wrong" by one with respect to the calendar because of the precession of the equinoxes – astrology is nonsense!)

Scientific study began with the Greeks. Many of their ideas were ahead of their time. They determined, from geometric measurements, that:

1. The Earth is a sphere (Pythagoras) whose absolute diameter could be calculated (Eratosthenes) with reasonable accuracy (500-600BC)
2. The relative sizes / distances of the Earth, Moon and Sun could be calculated (Aristarchus, 300BC)
3. The eclipses could be explained.

However, they believed in an Earth-centered universe (apart from Aristarchus) which is the most obvious hypothesis given the lack of any sensation of circular movement, and thus to explain the complicated motion of the planets e.g. Mars they had to resort to a system of cycles and epicycles (given enough parameters one can "account" for anything!). Retrograde motion was a major problem.

1.5.1 Copernicus

Poland, early 16th Century.

He led an intellectual revolt against the ~80 epicycles needed to "explain" the solar, lunar and planetary motions against the background of stars. He also noted that the calendar had slipped out of synchronism with the stars by ~ 1 week. He proposed that the current model was wrong, and that a far simpler system can be invented with the Sun at the centre (heliocentric). The Church didn't like this!

1.5.2 Tycho Brahe

Denmark, late 16th Century.

Tycho constructed large visual instruments for measuring elevations of stars and planets, obtaining the time of transit from the positions on the sky. He set up two observatories on an island donated by the King of Denmark, using up 5% of the country's gross national product. This was the first "big science project". He took 20 years of systematic observations and observed 777 stars and the naked-eye planets, and compared the two independent sets of data from the 2 observatories.

He understood errors, including systematic ones, e.g. instrumentational flexure, atmospheric refraction as a function of elevation. He obtained positions to 1-2 minutes of arc (1/60th of a degree) i.e. about the resolution of the human eye.

Tycho gave an observational and technological breakthrough.

1.5.3 Kepler

Prague, ~1600.

Worked with Tycho's data on the orbit of Mars. He tried to fit the data to a circular orbit by moving the Sun from the centre slightly. This gave a better fit, but still the error between the observed and calculated orbits differed by 6-8 arc-minutes, i.e. 4 sigma error. He trusted Tycho's data, and discarded the model. Eventually, after much trial and error and laborious hand calculation, he arrived at his "laws". But even Keppler was a mystic and had not completely rejected mediaeval thinking.

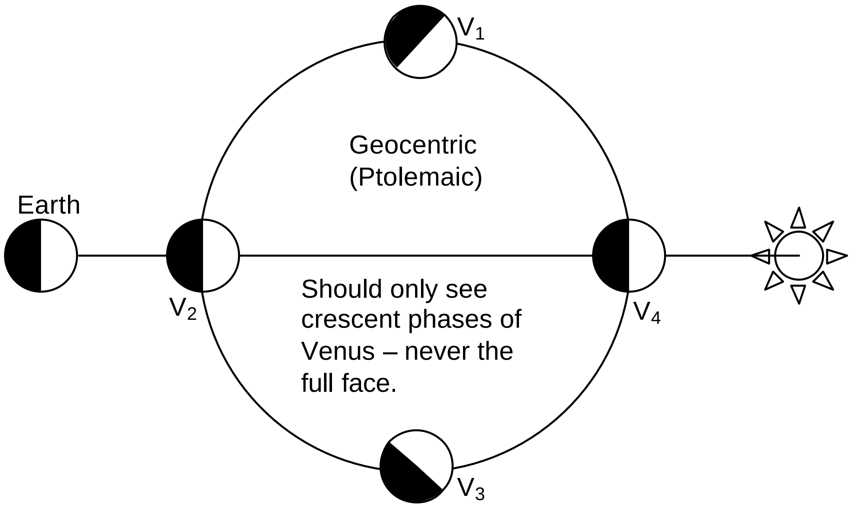
He codified results in empirical laws framed mathematically (phenomenology).

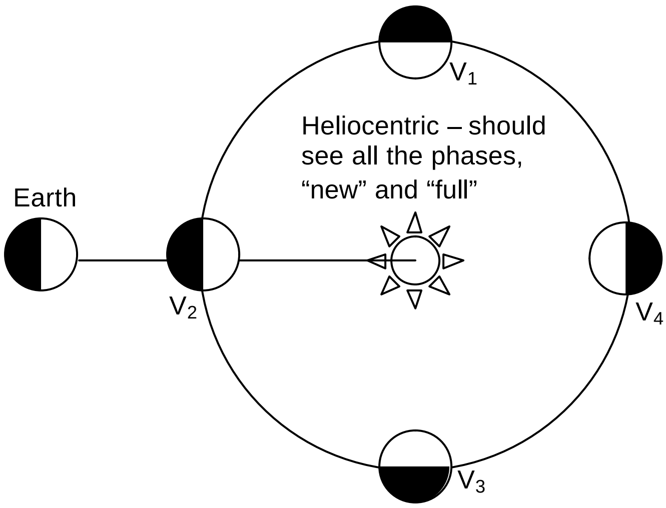
1. Planets revolve around the sun in ellipses with the sun at one focus.
2. Planets (Sun→planet) trace out equal areas in equal times
3. Sidereal period squared is proportional to the distance from the sun cubed $T^2 \propto r^3$.

1.5.4 Galileo

Italy, early 17th Century.

He was the first person to study the motion of bodies systematically, by simplifying problems (e.g. balls and inclined planes). He founded classical mechanics, and was the first person to use the telescope for astronomical as opposed to military use. He found that the Moon is not a perfect crystalline sphere, Saturn has rings, and Jupiter has moons. He also saw the phases of Venus.





He also codified the motion of bodies (mechanics) and confirmed the heliocentric solar system.

1.5.5 Newton

England, 17th Century.

He developed calculus and systematised previous knowledge on mechanics (Galileo, Descartes) → Newton's Laws. He applied the new mechanics to the motions of the moon and planets → deduced the law of universal gravitation and explained the tides. Newton accounted for empirical evidence with a theory based on a minimum of assumptions

Steps to the inverse square law (~1665/6):

From his study of circular motion, force $\propto \frac{v^2}{r}$. From Keppler III, $T \propto r^{3/2}$. Therefore since $v \propto \frac{r}{T}$:

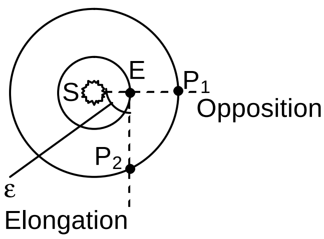
$$v \propto \frac{r}{r^{3/2}} = \frac{1}{r^{1/2}} \quad \therefore F \propto \left(\frac{1}{r}\right)\left(\frac{1}{r^{1/2}}\right)^2 = \frac{1}{r^2}$$

So a single, central force from the sun accounts for planetary motions.
(Ellipses are more complicated → 20 more years)

2. Dynamics of the Solar System

2.1 Definitions

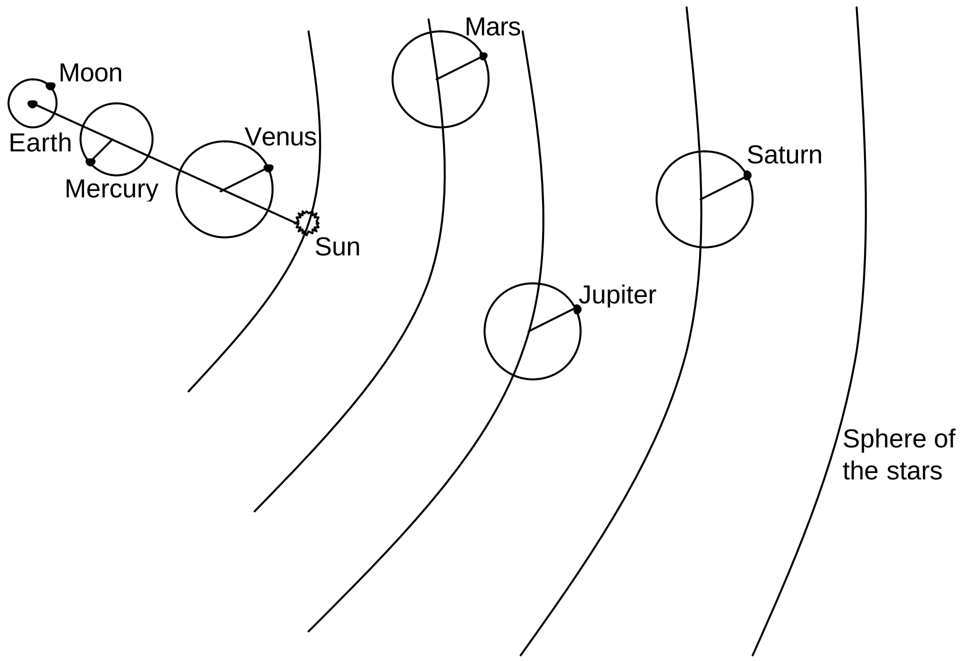
	Ecliptic	Apparent annual path of the sun on the celestial sphere
	Celestial Equator	Intersection of the plane of the Earth's equator with the celestial sphere
*	Inferior Planet	Planet with orbit lying inside Earth's orbit (Mercury, Venus)
*	Superior Planet	Planet with orbit lying outside the Earth's orbit (Mars, Jupiter, Saturn, …)
*	Opposition	Alignment when sun, Earth and superior planet lie in line, with the earth and planet on the same side of the sun
	In Conjunction	A planet whose direction is the same as the sun is said to be in conjunction. *Inferior conjunction* is on the same side of the sun. *Superior conjunction* is on the opposite side of the sun.
*	Elongation	The angle between the vectors from the Earth to the Sun and the Earth to the planet.
	In Quadrature	When the elongation is 90 degrees, the planet is said to be in quadrature (east or west).



Opposition: the planet crosses due south (southern meridian) at midnight i.e. exactly opposite to the sun. However at more general times the sun-earth-planet is not lined up (elongation).

2.2 The Ptolemaic System

The main theory of the Solar System left by the Greeks to post-Roman Europe was a geocentric one. It was given in Ptolemy's Almagest and so bears his name. Its' acceptance by astronomers lasted about 15 centuries.
The Ptolemaic theory sought to describe the apparent movement of all the heavenly bodies and indeed predict their future positions. It did so successfully: all the apparent motions of the Sun, Moon, planets and stars were adequately accounted for. The main features of the Ptolemaic system are below:



- The Earth was the fixed centre of the Universe.
- The stars were fixed to the surface of a transparent sphere which rotated westwards in a period of one sidereal day.
- The Sun and the Moon revolved about the Earth.
- The large circles centred on the Earth were called deferents; the small circles centred on the large circles or on the line joining the Earth and Sun were called epicycles. The planets moved in the epicyclic orbits whose centres themselves moved in the directions shown. The added circular motion explains the "wobbles" in observed motions, which was really caused by the orbits being elliptical.
- Because Mercury and Venus were never seen far from the sun (they were always evening or morning objects), the centres of their epicycles were fixed on the line joining Sun to Earth.

2.3 Heliocentric Model

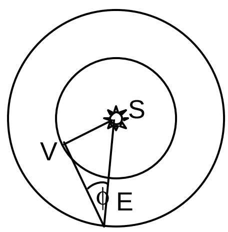
Why was the heliocentric model finally accepted?

1. It provides a natural explanation of retrograde motion of planets as Earth overtakes.
2. It is simpler and more elegant than many cycles and epicycles. (not a proof!)
3. Correlation between sidereal periods with orbital distances.
4. Phases of Venus – all are seen.
5. Jupiter has its own "mini-solar system" – set of moons.

2.4 Synodic Period and Planetary Distances

Having established the "correct" reference frame (heliocentric not geocentric), one can establish the relative distances from naked-eye observations and basic geometry. This was done by Copernicus and Keppler (This is a simplified version).
Assume circular orbits.
T = Sidereal Period: the period as seen from an observer on the sun, or from "above" the solar system.

2.4.1 Inferior Planets

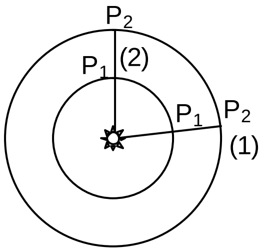


We can measure ϕ_{max} when Venus' angle from the sun is greatest in the morning or evening sky.

$$\sin\phi_{max} = \frac{SV}{SE}$$

$$SV = SE \sin\phi_{max}$$

This gives the relative distance for Venus with respect to the Earth-Sun distance.



S = time between successive similar configurations of two planets (Geocentric frame); easiest is to look for successive "oppositions".

2 planets P_1 and P_2. Sidereal periods T_1 and T_2 as seen from "outside" or heliocentric.

S = time (days) between situations (1) and (2).

P_1 gains on P_2.

$$\omega_1 = \frac{2\pi}{T_1},\ \omega_2 = \frac{2\pi}{T_2}$$

Therefore gains at $\omega_1 - \omega_2$ radians per day.

Gains 2π in S the Synodic period i.e. back to planet passing due south at midnight if P_1 is Earth.

P_1 is lapping P_2.

$$(\omega_1 - \omega_2)S = 2\pi$$

$$S\left(\frac{2\pi}{T_1} - \frac{2\pi}{T_2}\right) = 2\pi$$

$$\underline{\frac{1}{S} = \frac{1}{T_1} - \frac{1}{T_2}}$$

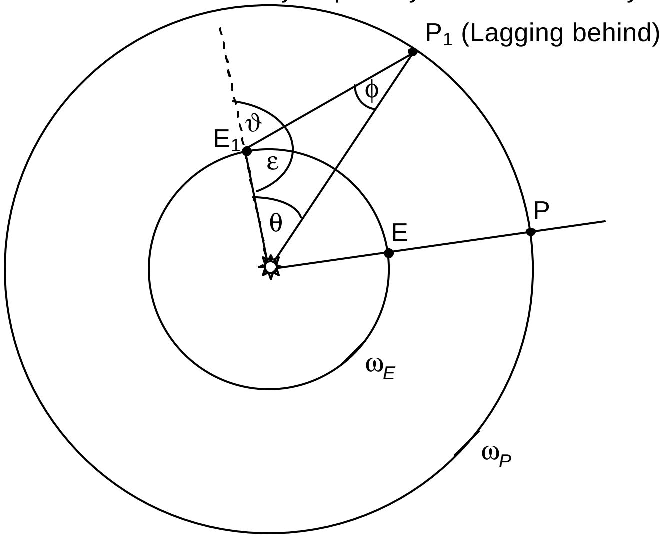
T_1: inner planet (365.25 days for Earth)

T_2: outer planet.

Difference in angular rates is just like difference in frequencies as in stroboscope.

2.4.2 Superior Planet

This was first realized by Kepler if you measure the synodic period.



Looking t days after an opposition for simplicity, you see the planet at P_1.

Look at $\Delta S E_1 P_1$.
We know
$$\theta = \left(\omega_E - \omega_P\right)t$$
$$= \frac{2\pi t}{S}$$
(as above, S = Synodic Period.)
Therefore we know θ since we know S and t.
ϑ is the angle between due south and the planet at midnight. Therefore:
$$\varepsilon = \pi - \vartheta$$
Thus we get $\phi = \pi - \varepsilon - \theta$ and from the sine rule.
$$\frac{\sin\varepsilon}{SP_1} = \frac{\sin\phi}{SE_1}$$
$$SP_1 = SE_1 \frac{\sin\varepsilon}{\sin\phi}$$

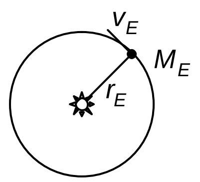
This means we can construct an accurate model of the solar system in terms of relative distances with respect to the sun-earth distance.
Examples 1 leads to additional order in the solar system.

Planet	**S (Days)**	**T (Days)**
Mercury	116	88
Venus	584	225
Earth	-	365.25
Mars	780	686
Jupiter	399	4330
Saturn	378	10800

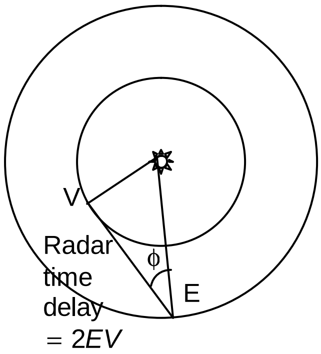
No apparent order in S, while order appears in T.
T correlates with the planetary distances.
Kepler III → $T^2 \propto d^3$

2.5 Masses

2.5.1 The Sun



$$\frac{M_E v^2}{r_E} = \frac{GM_\odot M_E}{{r_E}^2}$$
$$\Rightarrow M_\odot = \frac{{v_E}^2 r_E}{G}$$
$$M_\odot = \frac{4\pi^2 {r_E}^3}{G{T_E}^2}$$
T_E → 1 Earth year
NB: we have ignored the reduced mass, i.e. effect of planets, but since $M_J < 0.1\% M_\odot$ it's a tiny effect. What we have done is a simplification.
We still need to know the distance r_E. This was hard to measure accurately until the advent of planetary radar.



Measure V at max elongation. From the triangle ΔSEV → SE knowing EV from radar.

$SE \equiv$ "Astronomical Unit" (AU) = $1.496\text{x}10^{11}$m (150 million km)

(Accurate to >8 significant figures ~ 1km)

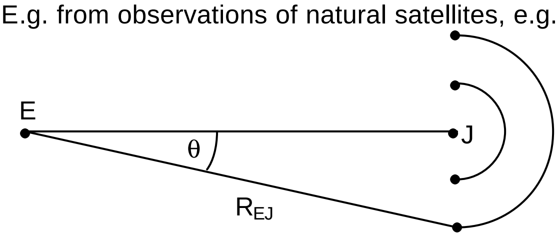
$v_E \sim 30kms^{-1}(67,000mph)$

$M_\odot \sim 2x10^{30}\,kg$

Essential point: you get a mass in astronomy by observing its' effect on another body in its gravitational field e.g. Earth in orbit around the sun)

__2.5.2 Planets__

E.g. from observations of natural satellites, e.g. Jupiter's moons.

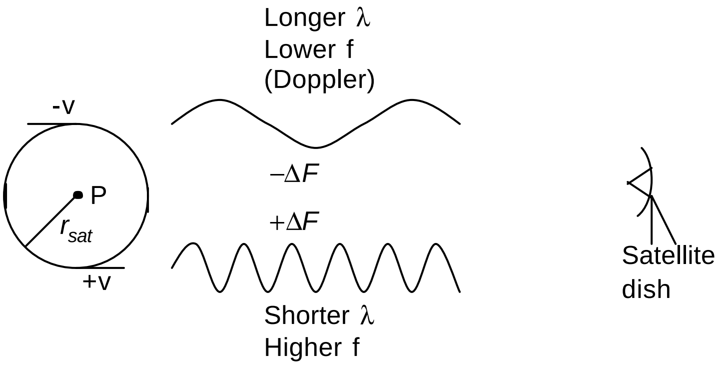


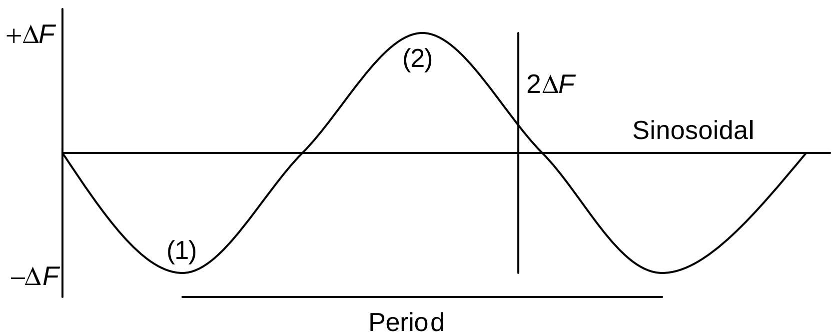
Can observe the period of the moons directly in a telescope.

$\theta r_{EJ} = r_{moonaround.jupiter}$

Got to know r_{EJ} !

E.g. From observations of man-made satellites





The "observable" is the frequency (or wavelength) of the satellite transmission.

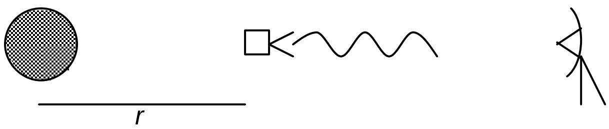
Δf = Doppler shift $= \dfrac{2\Delta f}{2}$

Non-relativistic Doppler: $\dfrac{\Delta f}{f} = \dfrac{\Delta v}{v} = \dfrac{v}{c}$

Therefore as in sun-mass calculation using Newtonian dynamics $M_P = \dfrac{T_{Sat} V_{Sat}^{\ 3}}{2\pi G}$

Recast equations in terms of things we can measure.

E.g. from a moving spacecraft.

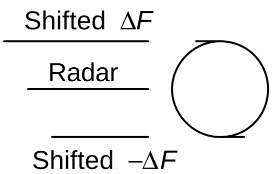


See examples 1 for a simplified calculation.
What are the observables?
We can't see the spacecraft, can't see distance. But can see the frequency of the transmissions and how they change with time.
Two "observables" are needed.
Force comes from the potential energy in the system.

2.6 Other basic information about the solar system

You can get:

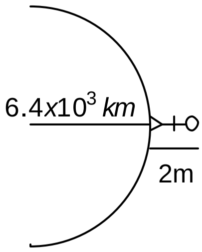
– The sizes of objects from their angular diameters and their distances.
– Rotational rates: just by observing surface features, and now from radar, e.g. for cloud-covered Venus in the 1960's.



2.7 The Sun-Earth-Moon System

2.7.1 Tides

These are due to differential gravitational effects (gravitational field gradient)
Planets and moons have finite size and the gravitational field changes from one side to the other.

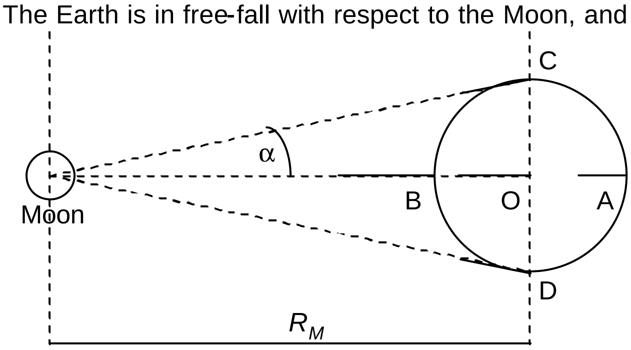


Feet are more strongly affected than your head. $F = \dfrac{GM_E M_{man}}{r^2}$.

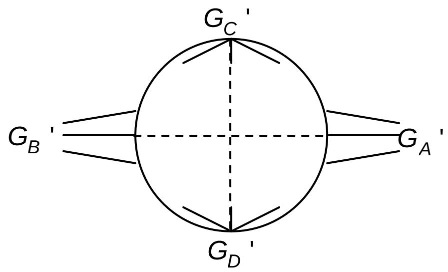
Differential effect: in terms of a ratio $\left(\dfrac{6400002}{6400000}\right)^2 =\sim 1.0000008$. It is equivalent to 4 drops of water.

Tidal effect of the moon on the earth:
The Earth is in free-fall in orbit about a common centre-of-mass on the Earth-Moon system, which is inside the body of Earth but is not fixed.

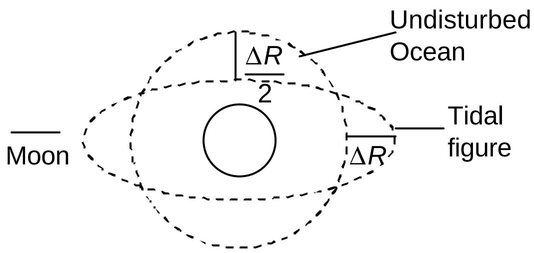
Strength of Tidal Effects:
The Earth is in free-fall with respect to the Moon, and vice-versa.



The above is a schematic of the gravitational field due to the Moon at different points on the Earth's surface.



The differential field with respect to the field at the centre.



The resultant shape – highly exaggerated – of the oceans with respect to the solid Earth (in the absence of the continents). The rotation of the Earth "underneath" these tidal bulges results in a twice-daily tide.

– Shape of the fluid ocean distorts into an ellipsoid
– Only forces act between the line of centres
– Two tides per day as the Earth rotates under the "figure".

The gravitational field at the centre of the Earth $= G_o = \frac{GM_m}{R_M{}^2}$ (i.e. force on a <u>unit</u> mass)

The gravitational field at B $= \frac{GM_m}{(R_M - r_E)^2}$ (it's closer!)

Therefore the apparent field at B with respect to the centre of the Earth, O,

$$G_B' = G_B - G_o = \frac{GM_m}{(R_M - r_E)^2} - \frac{GM_m}{R_M{}^2}$$

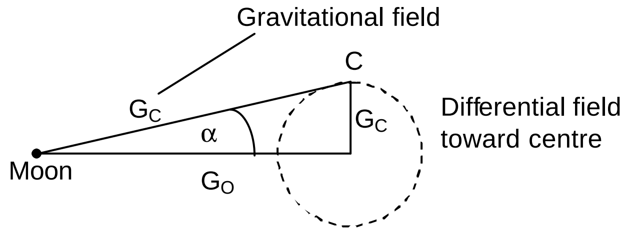
$$G_B' = \frac{GM_m}{R_M{}^2}\left(\frac{1}{\left[1 - \frac{r_E}{R_M}\right]^2} - 1\right)$$

And since $r_E << R_M$ $(6{,}000km << 400{,}000km)$ use the Binomial expansion.

$$G_B' \approx \frac{GM_m}{R_M{}^2}\left[1 + \frac{2r_E}{R_M} \ldots - 1\right] \approx \frac{2GM_m r_E}{R_M{}^3}$$

The tide raising force is proportional the 1/distance cubed; the mass of the tide raising body; the size of the body affected.

Similarly $G_A' = -\frac{2GM_m r_E}{R_M{}^3}$; minus as it is on the other side, so differential force changes sign. It's a weaker pull here. (with respect to free-falling centre of Earth)



$|G_C| \approx |G_o|$ same distance (very slight difference)

Vector difference $= G_C'$

$\alpha = \frac{r_E}{R_M}$ (see Figure 9)

This implies $G_C' \approx G_o \alpha \approx \frac{GM_m}{R_m{}^2}\alpha = \frac{GM_m r_E}{R_M{}^3}$, and similarly for G_o' on Fig 9)

$G_C' \approx \frac{1}{2} G_A'$ (These are the differential fields of figure 9b)

It is a simple dimensional argument to get actual magnitudes in meters. Compare differential effect / fields to normal surface gravity.

Surface gravity $g = \frac{GM_E}{r_E{}^2}$ (downwards)

Differential field at sub-lunar point (Fig 9b) $G_B' = \frac{2GM_m r_E}{R_M{}^3}$ (upwards).

Effectively there is slightly less downward gravity on the surface at this point due to the pull of the moon.

$\frac{G_B'}{g} = 1.13x10^{-7}$.

What rise in height above the Earth's surface ΔR would give the same decrease in downward pull i.e. in g (to decrease by $1.13x10^{-7}$)

i.e. $g_B' = \frac{GM_E}{(r_E + \Delta R)^2}$ → (Binomial expansion) $\frac{GM_E}{r_E{}^2}\left(1 - \frac{2\Delta R}{r_E}\right)$

$$g_B' = g\left(1 - \frac{2\Delta R}{r_E}\right)$$

$\frac{2\Delta R}{r_E} = 1.13x10^{-7}$ from above.

$\Delta r = 0.36m$

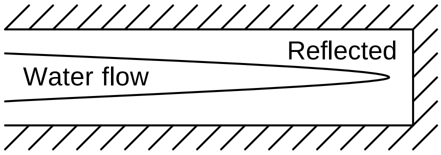
This is the height of the bulges at A and B due to the moon. But remember that at C and D the downward field is ½ that at A and B. Therefore the depth of the troughs at C and D is 0.18m (see figure 9). This means that the peak-trough amplitude for lunar tides $= 0.36 + 0.18 = 0.54m$ in the open ocean.

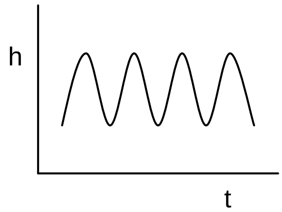
The tidal bulges point approximately towards and away from the moon. This means there is a high tide as the moon passes over or is at the other side of the Earth, and a low tide when the moon rises or sets.

NB: The Earth does one revolution under the tidal figure every 24 hours and 50 minutes since it has to catch up with the moon moving in its' orbit. This means there are tides every 12 hours 25 minutes.

2.7.2 Complications

1) The effect of the sun. "Spring" tides occur when the Sun, Moon and Earth are all lined up, meaning that the effect of the sun on the tides is added to that of the moon, leading to the largest tides. "Neap" tides are when the Earth-Moon line is perpendicular to the Earth-sun line, i.e. when the Sun and the Moon are in quadrature, so the effects partly cancel.
   The tide raising effect of the sun is only about half that of the moon, due to the inverse square law (see Examples 1)
   The relative tide heights $\frac{spring}{neap} = \frac{1+0.5}{1-0.5} \approx 3:1$
2) Local geography
   The land masses interrupt the free flow of the "tidal bulge wave" → can greatly affect the times and the heights due to interference effects and resonances with the 12 hour driving period. (Forced oscillator)





This can produce local heights greater or equal to 10 meters depending on the time it takes the waves to be reflected. E.g. the Bay of Fundy in Nova Scotia, Canada (the extreme case), and the Bristol Channel.
Results: in the open ocean you get tides to around 0.8m. At coasts, it can be 2-3m, in extreme cases >10m.

2.7.3 The effects of the tides on the Earth's spin rate

There is friction between land and sea as the Earth rotates under the tidal bulges (especially in shallow seas).
Energy dissipation rate:

$$E = \frac{1}{2} I\left(\omega_1^2 - \omega_2^2\right)$$

$$I = \frac{2}{5} M_E r_E^2$$

ω_1 and ω_2 represent different spin rates at different geological times.
From counting growth rings in ancient coral you can see that 300 million years ago the day was 22 hours long. The year hasn't changed.

Ancient $\omega_1 = \frac{2\pi}{22x60x60} rads^{-1}$

Now $\omega_2 = \frac{2\pi}{24x60x60} rads^{-1}$

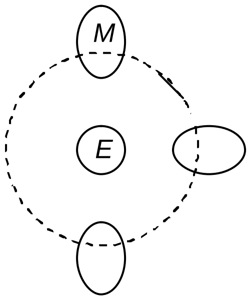
→ change in rotational E_K : $\frac{1}{2} I\omega^2 \approx 10^{29} J$ in $300x10^6 years$ $\approx 10^4$ $1GW$ power stations.

Angular momentum loss:
$I\left(\omega_1 - \omega_2\right) \approx 7x10^{32} J.\text{sec}$ (Earth is slowing)

Where does this go? Angular momentum must be conserved. It is transferred to the Moon's orbital angular momentum (see Examples 1) The moon is moving away from Earth at a rate of 4cm/year.

2.7.4 Spin orbit locking of the Moon:

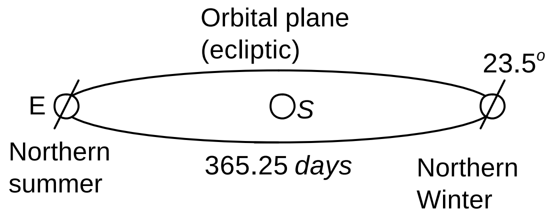
The current spin rate of the moon ≡ the orbital rate. Therefore the same face always points toward us. It is due to tidal effects on the solid body of the moon. (see examples 1 for the effect of the Earth on the moon >> vice versa)



If the spin rate was not equal to the orbital rate the tidal bulges in the solid moon would be raised in different parts as a function of time. This would mean differential motions in the rocks → internal heating → energy loss to space via radiation
Result: system attains lowest energy state consistent with conserving total angular momentum → circular orbit of moon around Earth, and the spin rate of the moon is locked to the orbital rate. This is rather common in the solar system, e.g. Jupiter and Io, the Sun and Mercury (more complicated – 2 spins for three orbits).

2.7.5 Precession of the Earth's Spin Axis

To a good approximation there are no external torques on the Earth. This means that the angular momentum vector remains fixed in space.



Sun appears higher in the sky in summer compared with winter due to the tilt of the spin axis with respect to the orbital plane. But the Earth is not quite spherical. It has a slight equatorial bulge due to its' rotation (>>>greater than tidal bulges) (See Figure 12 caption).
And the Sun and the Moon exert torques on this asymmetry. Result is that the Earth wobbles (precesses) like a Gyroscope. This is called Luni-Solar Precession.
Fig. 13a shows the precession of the equinoxes (spring and autumn when day and night are the same length)
The position of the sun against the stars at a certain time of the year (e.g. equinoxes) shifts as a function of t.
→ accurate astronomical measurements need to take precession into account.
→ Hipparchus first detected this in ~ 150BC from comparing star positions written down over hundreds of years.
Astrology: Sun has "moved" through the Zodiac about 1.5 signs (44 degrees) since it was invented by the Babylonians ~ 3,000 years ago (36 degrees in 2,600).

2.8 Orbits in the Solar System

Newtonian dynamics – motion under the inverse square law.
As the object moves in the larger object's gravitational field, energy is conserved.
$E_K + E_P = const. = E_{total}$, but the balance between them changes.
For Perigee, E_K increases, E_P decreases (moving fastest)
For Apogee, E_K decreases, E_P increases (moving slowest)
The overall type of orbit depends on E_{total}.

1) "Bound" orbits – shape is an ellipse. $E_K < E_P \Rightarrow E_{total} < 0$ i.e. negative. i.e. you have to supply positive energy to the system to separate the bound objects to infinity.

$$\varepsilon = \frac{CS}{CA}$$
$$0 < \varepsilon < 1$$

An ε of 1 is a parabola which does not close.

2) "Critical" orbits e.g. long-period comets. The shape is a parabola, not closed.
$E_K = E_P \Rightarrow E_{total} \to 0$. Zero energy – hardly bound at all.
$\varepsilon \to 1$
3) "Unbound" orbits e.g. a spacecraft with $v > v_{escape}$. Shape is a hyperbola.
Sling-shot. Bends round planet, almost in straight lines.
$E_K > E_P \Rightarrow E_{total} > 0$ (fast-moving).
We mainly consider bound i.e. elliptical orbits. See figure 15.

<u>*2.8.1 Velocity at any point in a bound orbit*</u>

$$E_K + E_P = E_{total} = const.$$

$\frac{1}{2}mv^2 - \frac{GMm}{r} = E_{total}$ (at any radius vector in fig14)

Critical point: E_{total} is the same for all orbits with the same mean distance (see Figure 15). It does not depend on the shape i.e. the eccentricity, which is set by L (angular momentum). This ties in closely with atomic physics in years 2 and 3.

→ without a loss of generality, you can consider a circular orbit of radius a.

(N.B. we can also always get the <u>orbital period</u> by considering a circular orbit of radius a regardless of the shape.)

We want E_{total}.

$\frac{mv^2}{a} = \frac{GMm}{a^2}$ (acceleration towards the centre of a circle) → $v^2 = \frac{GM}{a}$

$$E_K = \frac{1}{2}mv^2 = \frac{GMm}{2a}$$

$$\Rightarrow E_{total} = \frac{GMm}{2a} - \frac{GMm}{a} = -\frac{GMm}{2a}$$

$$E_{total} = -\frac{GMm}{2a}$$

This applies to any orbit of mean distance = a.

Now for an elliptical orbit:

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = E_{total} = -\frac{GMm}{2a}$$

$$\Rightarrow \boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

This is very useful.

This means that the planet or satellite moves faster at closest approach (perihelion for the sun, perigee for the Earth) than at aphelion or apogee.

From Figure 14, where r_P is the radius to the perihelion and r_a the radius to the aphelion;

$$r_p \equiv q \equiv a(1-\varepsilon)$$

$$r_a \equiv q' \equiv a(1+\varepsilon)$$

$$\Rightarrow \frac{v_p}{v_a} = \frac{1+\varepsilon}{1-\varepsilon}$$

An alternative argument at aphelion and perihelion, the velocity is purely transverse therefore conserving angular momentum.

$$mv_p r_p = mv_a r_a$$

$$\Rightarrow \frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1+\varepsilon}{1-\varepsilon}$$

Planets	Î
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.055
Uranus	0.051
Neptune	0.007
Pluto	0.252

These are mostly circular. Mars was chosen by Kepler to study in detail. It has a relatively high eccentricity.
Pluto: very high. Is it a “real” planet?
Mercury is also odd.

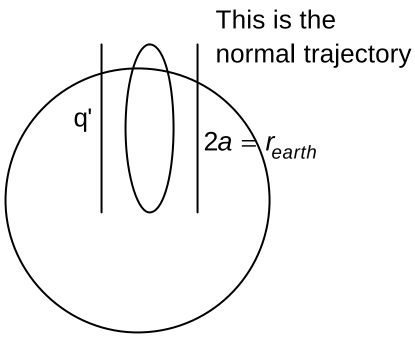
2.9 Orbits of artificial satellites

Throw a ball into the air with $v_{vertical} = 10ms^{-1}, v_{horizontal} = 10ms^{-1}$. What is its’ path (neglecting air resistance and coriolis forces)
Normally assume a parabolic path.

$s \propto \frac{1}{2}gt^2$ vertically

$s \propto vt$ horizontally

But assumes g is constant with height which is not correct → should do a proper orbit calculation.



$$v^2 = GM_E\left(\frac{2}{r} - \frac{1}{a}\right)$$

So at apogee $v_{ap} = 10ms^{-1}$

$$100 = GM_E\left(\frac{2}{\underbrace{a(1+\in)}_{q'}} - \frac{1}{a}\right)$$

$a = 3168km$

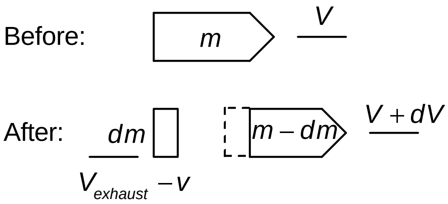
$\in = 0.9999991$

i.e. very close to a parabola $\in = 1$.

How to change orbits:
Use a rocket burn – short duration impulse to change velocity instantaneously by Δv at the same position. i.e. E_K and L changes but E_P doesn’t at that point.

2.9.1 The Rocket Equation

We want to change an orbit in the same plane via a “burn”, i.e. a short duration impulse. This will lead to a change in the satellite’s velocity, but not in its’ position, i.e. there will be a change in E_k but not E_p.



NB: this is in the observer’s frame. $v_{exhaust}$ is the exhaust speed with respect to the rocket.

Assume $v_{exhaust}$ is constant, and there is a constant mass ejected per second, i.e. there is a constant thrust.
Conserving linear momentum:

$$\begin{array}{ll} Before \rightarrow & After \\ mv & = (m - dm)(v + dv) - (v_{exhaust} - v)dm \end{array}$$

Simplifying (and ignoring small terms:

$$mdv = -v_{exhaust} dm$$

Hence:

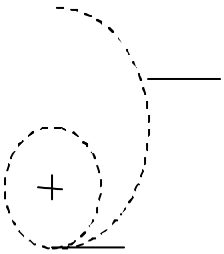
$$\Delta V = (V - V_o) = V_{exhaust} \ln\left(\frac{m_o}{m}\right)$$

where V_o is the initial velocity, m_o is the initial mass, and m is the final mass after the burn.

NB: $\ln\left(\frac{m_o}{m}\right)$ is the usage of the fuel.

2.9.2 Changing Orbits

Going “outwards” is done by increasing a spacecraft’s apogee, i.e. when the spacecraft’s energy is all E_p. Therefore the total energy of the system needs to be increased. This can be done using a burn at the perigee, thus increasing the E_k.

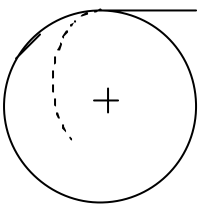


Orbit shape changes since $E_k \propto v^2$, and the angular momentum is $\propto v$

Δv in the same direction as v

Subtle point: with assumptions in the rocket equation, the max E_k transfer by a burn is when v is the largest, i.e. at perigee.

Going inwards:

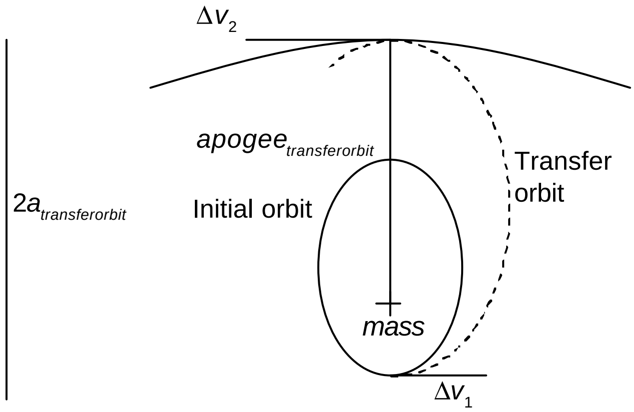


Δv is in the opposite direction

This looses E_k energy, therefore lowers the perigee or perihelion.

2.9.3 Hohmann (Least Energy) Transfer Orbits

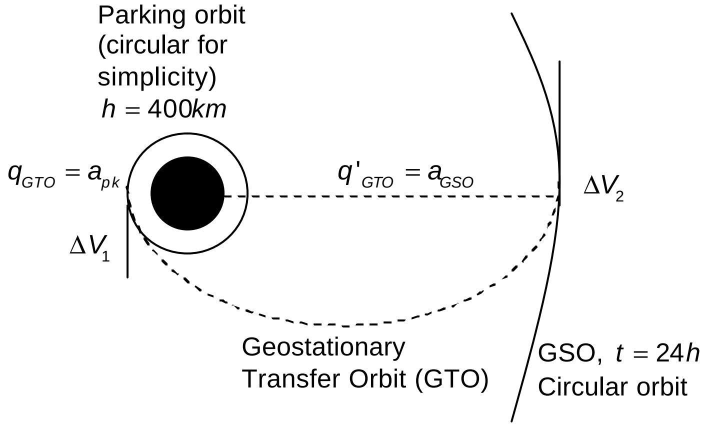
The goal is to transfer as efficiently as possible from the initial to the final orbit. Here we are going out.



Both burns are tangential, i.e. in the same direction as the instantaneous velocity vector at the perigee. The transfer and initial orbits share their perigee, while the transfer and final orbits share an apogee.
This means that there are two burns from the initial to the final orbit.
ΔV_1 gives initial E_K → E_{total} goes up → E_P at apogee goes up.
ΔV_2 at the apogee of the transfer orbit increases (in the case shown) the $\text{angular momentum}/E_{tot}$ ratio to that in the required final orbit (which is more circular as drawn).
Doing a Least Energy Transfer Orbits (LETO) means that minimum fuel is used, so the maximum payload is retained. However, there are short burns but long “coasting” phases; therefore there are long transfer times.

For example, take the transfer to a geostationary orbit (GSO). These are used for satellite TV, Apollo missions to the moon, missions to nearby planets, …

Worked example: transfer to GSO following an IOM (Orbital Insertion Maneuver) at an altitude of 400km.



NB: this is not to scale.
Dimensions of the orbits:

$a_{pk} = r_E + 400km = 6440km$

$q_{GTO} = a_{pk} + a_{GSO}$

Get a_{GSO} from Kepler 3 (Simple Newtonian Calculation)

$$T = \frac{2\pi a_{GSO}^{3/2}}{(GM_E)^{1/2}}$$

But $T = 24$ hours → $a_{GSO} = 42250km$

Therefore $a_{GTO} = \frac{6440 + 42250}{2} = 24345km$

Velocity changes ΔV_1, ΔV_2 via "burns"

Tangential burn in the parking orbit ΔV_1 becomes the location of the perigee into GTO since $v > v_{circl}$ at this point.

$$V_{escape} = \left(\frac{GM_E}{a_{pk}}\right)^{1/2 = 7.88kms^{-1}}$$

This is a circular orbit calculation. $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$, $r = a$

$$v^2{}_{perigeeGTO} = GM_E\left(\frac{2}{q_{GTO}} - \frac{1}{a_{GTO}}\right)$$

$$v_{GTO} = 10.385kms^{-1}$$

→ $\Delta V = V_{GTOperigee} - V_{pk} = 2.51kms^{-1}$

The spacecraft then swings out into an elliptical orbit. What is the eccentricity?
See figure 14.

$$q_{GTO} = a_{GTO}(1 - \varepsilon)$$

$$\varepsilon = 1 - \frac{q_{GTO}}{a_{GTO}} = 1 - \frac{6440}{24354} = 0.735$$

Transfer into the GSO via ΔV_2 :

At apogee in GTO, $v < v_{circ}$ at this distance, therefore must increase v to get into a circular geostationary orbit.

$$v_{GTOapogee}{}^2 = GM_E\left(\frac{2}{q'} - \frac{1}{a_{GTO}}\right)$$

$$v_{GTOapogee} = 1.55kms^{-1}$$

$v_{GSO} = 3.07kms^{-1}$ (simple circular orbit or application of $v^2 = GM\left(\frac{2}{r} - \frac{1}{r}\right)$)

Therefore $\Delta V_2 = V_{GSO} - V_{GTOapogee} = 1.49kms^{-1}$

What about the transfer time?

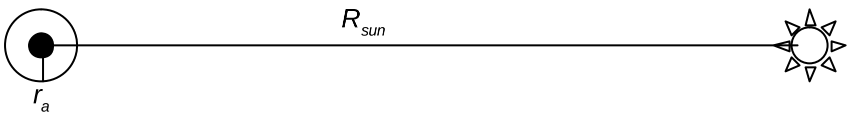
Recall that the period is the same for all orbits with the same semi-major axis (a) regardless of their shape. Therefore $1/2$ complete period for a circular orbit of radius a_{GTO} .

$$\Rightarrow t_{trasnfer} = \frac{1}{2}\left[\frac{2\pi a_{GTO}^{3/2}}{(GM_E)^{1/2}}\right] = 5.29hours$$

See examples 2 for other calculations of this type.

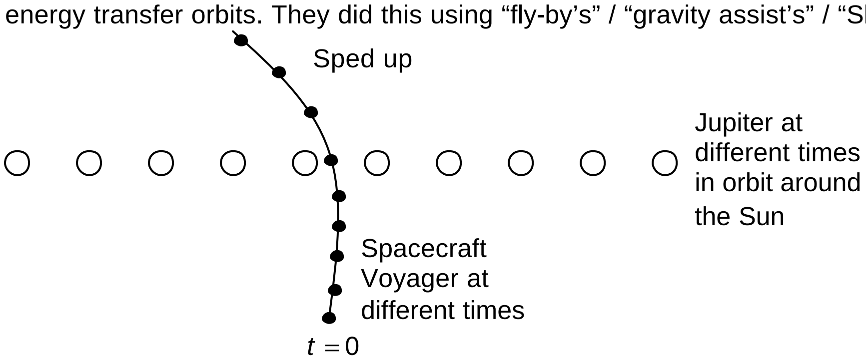
2.9.4 Gravity Assists using Planets

The time to transfer to the outer planets (Uranus, Neptune and Pluto) are too long for normal people's careers. We need to use gravity assists to shorten the times. Therefore we need to get within the "sphere of influence" of planets to allow useful gravity assists / slingshots.

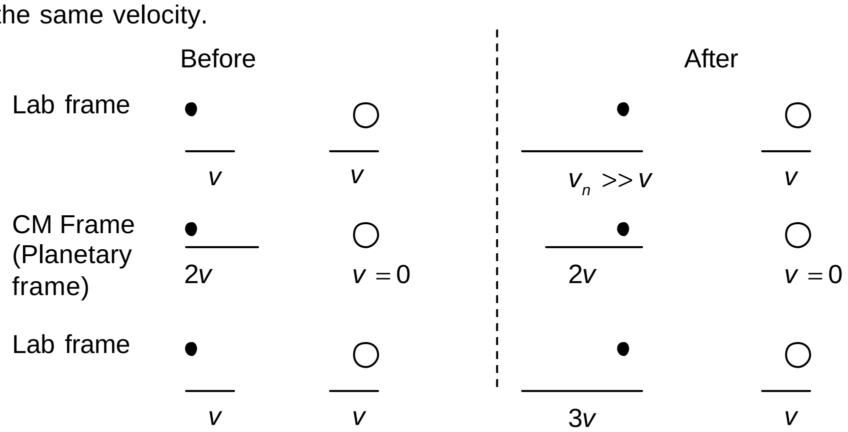


$$\frac{GM_p}{r_a^2} >> \frac{GM_{sun}}{r_{sun}^2}$$

Planet's influence >> sun's influence
$\sim 1\%$ of the mean distance of planet from sun
Transfer times using least energy transfer orbits are too long for the outer planets, but Voyager spacecraft got out to Neptune in 12 years compared to 30 years via least-energy transfer orbits. They did this using "fly-by's" / "gravity assist's" / "Slingshots".

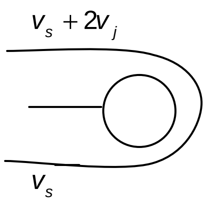


The spacecraft is on a hyperbolic orbit, i.e. $v >> v_{escape}$ for Jupiter. $E_k >> E_p$ at greatest approach.
Spacecraft falls into the planetary potential and climbs out again. With respect to the planet, $v_{in} = v_{out}$, but with respect to the sun $v_{out} > v_{in}$. This is best illustrated as an elastic collision problem, e.g. a head-on collision of two balls of very unequal mass with the same velocity.



i.e. colliding with a brick wall. This is not 100% accurate ($v - dv$ but $dv \sim 0$ if $v_n << v$)

In the case of a spacecraft-planet system, the bodies do not actually touch but it is similar to an elastic collision.



You have to go very close to the planet to transfer the most velocity possible $(2v_j)$.
This is an extreme case of gravity assist; it is usually only around 45^o at most.
From this, you can get a large Δv "kick" like a powerful extra rocket burn, e.g. $\Delta v > 10 kms^{-1}$.

Voyager's kick from Jupiter $\Delta v = 16kms^{-1}$ ($\Delta v_{jupiter}$ was the equivalent of a difference in orbital position of 0.3m in 10^{12} years).
All interplanetary missions beyond Mars use this effect to speed up transfer and / or allow heavy spacecraft; therefore they can carry a lot of payload.

3. Physics of the Sun

The sun is our nearest star, and is the source of all our energy. It is almost a spherically symmetric ball of plasma held together by self-gravity generating energy from thermonuclear fusion reactions.
Basic properties:

$M_\odot = 2x10^{30}kg$

$R_\odot = 7x10^{6}m$

$\bar{\rho} \approx 1.5x10^{3}kgm^{-3}$

i.e. 50% greater than water.
NB: all this depends on knowing the AU precisely.

3.1 Luminosity and Temperature

One easy thing is to measure the "solar constant" i.e. the total energy falling on $1m^2$ perpendicular to the solar direction just outside the Earth's atmosphere.

$$\Omega = \frac{L_\odot}{4\pi A^2} = 1.36kWm^{-2}$$

Ω is the solar constant, and A is the astronomical unit AU.

Consider the sun as a black body (see also Thermal Physics)
A black body is a perfect absorber and perfect radiator. Regardless of the material which it is made of it always has the same spectrum at a given temperature.
Wein's Law $\lambda_{\max}T = 2.9x10^{-3}mk$
Stefan's Law $E_{total} = \sigma T^4 Wm^2$ (per unit area). $\sigma = 5.7x10^{-8}Wm^{-2}k^{-4}$
The sun is approximately a black body, therefore we can define its' effective temperature. What black body temperature would produce the same energy output?

$L_\odot = 4\pi R_\odot{}^2 \sigma T_{eff}{}^4 = E_{tot}$

We can get $L_\odot$ from Ω i.e. $L_\odot = 4\pi A^2\Omega = 3.9x10^{26}W = 60MWm^{-2}$ i.e. a small power station.
$\Rightarrow T_{effective} = 5780k$.

3.2 Solar Interior

This can be studied via:

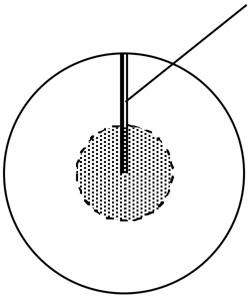
- Solar Neutrinos
- Solar Oscillations (solar seismology)
- Theoretical inference

We know $M_\odot, L_\odot, \bar{\rho}_\odot, T_{eff}$. From this we can get the energy generated per unit mass

$$\frac{L_\odot}{M_\odot} = 2x10^{-4}Wkg^{-1}.$$

We also know that the sun is in a steady state (approximately) from geological records, and we can infer that the sun is gaseous all the way to the centre due to known gas laws.
We can also infer that the sun is in hydrostatic equilibrium; it is a compressible gas, and the pressure balances the self-weight → central regions are hotter and denser.
See Fig. 20 & 21.
The sun is in thermal equilibrium: the amount of energy generated is equal to the amount of energy radiated away.
The sun generates the energy in the core via nuclear reactions (later on we will worry about the resistance to energy flow outwards and energy transport mechanisms).

We need to construct a mathematical mode, i.e. equations governing the behavior layer by layer (see fig 20).
This is "too hard" for a first year course, but we can get the basic ideas easily: all the layers are governed by gas laws.
We need to know P, T and the number density.
$P = nkT$ where n is the number density i.e. number per unit volume. This is the ideal gas law, which will work as it turns out.
What is the typical pressure in the middle of the sun?
We can get a rough idea as follows:



Column of gas of unit area $(1m^2)$ (at the top – tapers down towards middle)

What is the weight of this column of gas?

$$Mass = \frac{M_\odot}{4\pi R_\odot^{\,2}}$$

The $4\pi R_\odot^{\,2}$ is the number of square metres of surface, therefore the number of columns like this.
Now we need to know the weight: typical gravitational attraction on the column $W = mg$
We need to make a guess that say half way down the column $R_\odot/2$ the mass within this radius is $M_\odot/2$ i.e.

$$M\left(<R_\odot/2\right) = M_\odot/2$$

This is a guess since we know it is not a uniform density.
Note that this is already implying that the central density is higher since if the density were uniform then $M\left(<R_\odot/2\right) = M_\odot/8$
Therefore since only the material inside $R_\odot/2$ is effective (the rest cancels out) to the gravitational attraction.

$$\frac{G\left(M_\odot/2\right)}{\left(R_\odot/2\right)^2} = \frac{2GM_\odot}{R_\odot^{\,2}}$$

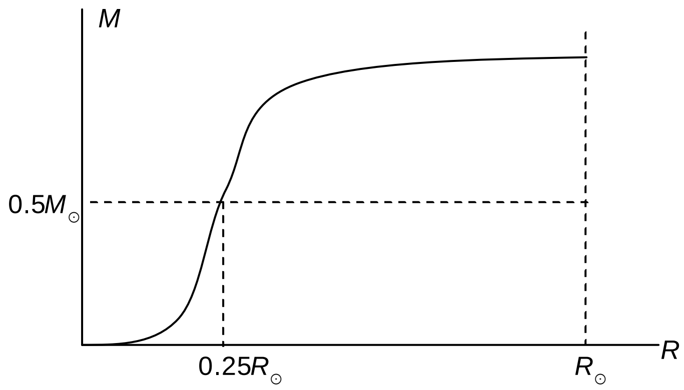
Crudely, the weight of the column is the mass times the gravity half way down.

$$\left[\frac{M_\odot}{4\pi R_\odot^{\,2}}\right]\left[\frac{2GM_\odot}{R_\odot^{\,2}}\right] = \frac{1}{2\pi}\frac{GM_\odot^{\,2}}{R_\odot^{\,4}}$$

The $\frac{1}{2\pi}$ is just a numerical factor depending on the accuracy of the assumptions.

$$\rightarrow P_{typical} \sim 2x10^{14}\,Nm^{-2}$$

For an accurate calculation, this comes out to be $\sim 10^{15}\,Nm^{-2}$.
NB: one Earth atmosphere $\sim 10^{5}\,Nm^{-2}$.
In reality all the parameters P, T and ρ are a smooth function of the radius – need large computer programs to calculate the "standard solar model". The result is:



Solar Core

The enormous central pressures break down the atomic electron shells → plasma.
Results more quantitavely:
Mass within $0.25R_{\odot} \approx 0.5M_{\odot}$ from graph.

$\rightarrow \rho_{core} \geq 40x10^4 kgm^{-3}$ (40x water) (compare with $\bar{\rho}_{\odot} = 1.5x10^3 kgm^{-3}$

$\rightarrow n_{core} \geq \frac{40x10^3}{1.67x10^{-27}}$ (using the mass of hydrogen for simplicity) ie. $m \geq 2.5x10^{31} m^{-3}$.

$P_{core} \geq 3x10^{15} Nm^{-2}$ (higher than our simple estimate)

Therefore assuming it behaves as an ideal gas:

$P = nkT$

$T = \frac{P}{nk}$

$\rightarrow T_{core} \geq \frac{3x10^{15}}{2.5x10^{31} x 1.38x10^{-23}}$

$\rightarrow T_{core} \geq 8x10^6 k$

Actually, right at the centre:

$T = 1.5x10^7 k$

$P = 2x10^{16} Nm^{-2}$

$\rho = 160x10^3 kgm^{-3}$

These conditions are self-consistant with the gas in a plasma state i.e. protons, neutrons and electrons are all separated → perfect gas laws.

Equation of state of gas in the core?
Check for self-consistancy with perfect gas assumption.

$E_k = \frac{3}{2}kT_{core} = \frac{3}{2}\frac{1.38x10^{-23}x6x10^8}{1.6x10^{-19}} \sim 1KeV$

>> ionization potential of electrons in H $(13.6eV)$ and He $(24.6, 54.4eV)$

→ gas is fully ionized, i.e. a plasma.

E_p of the electrostatic interaction between two charged particules separated by

$\approx \left[\frac{1}{3x10^{31}}\right]^{1/3} \sim 3.2x10^{-11} m$ (Closer than the 1st Bohr radius in Hydrogen)

$E_P = \frac{q^2}{4\pi\varepsilon_o \bar{r}} \sim \frac{(1.6x10^{-19})^2}{4\pi x 8.85x10^{-12} x 3.2x10^{-11}} \sim 45eV$

$E_{P,interaction} << E_{k,translation}$

Therefore the charged particles have little effect on each other. Also, the size of the particles is much smaller than their separation. This is consistent with the perfect gas assumptions.

3.2 The Source of Solar Energy

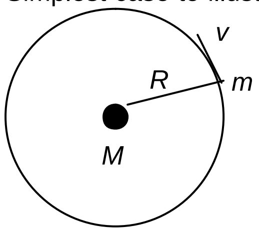
We have to account for $L_{\odot} \sim 4x10^{26} W$ over the lifetime of the sun $\sim 4.5x10^{9}$ years

What about gravitational contraction?

Use the Viral theorem for a system of bound particles.

$$E_{k,total} = -\frac{1}{2}E_{p,total}$$

Simplest case to illustrate:



$$E_k = \frac{GMm}{2r}$$

$$E_p = -\frac{GMm}{r}$$

i.e. $E_k = -\frac{1}{2}E_p$.

But true also in general cases e.g. self-bound systems: atoms, planets, stars, clusters of galaxies and the universe.

Apply this to the sun:

$E_k = E_{thermal}$ E_k of ions in the solar plasma added up.

$E_p = E_{gravitational}$ Potential attraction of the ions. NB: coulomb forces cancel.

Viral: $E_k = -\frac{1}{2}E_p$

$$E_{thermal} = -\frac{1}{2}E_{grav}$$

Hence the total energy $E_{total} = E_{thermal} + E_{grav} = \frac{1}{2}E_{grav}$, which is negative.

→ the total amount of energy needed to completely split up the sun's material to infinity (no interaction) i.e. its' binding energy is $\frac{1}{2}E_{grav}$ since half is already there in the form of E_k of the motion of the particles.

As a star contracts it must obey the Viral theorem to remain stable. E_{grav} and $E_{thermal}$ can change however. Thus only half of the gravitational E_p released during the contraction can remain in the star as $E_{thermal}$. The rest must be radiated away.

Contraction → loss of E_{grav} , gain in $E_{thermal}$ but only ${}^{1}\!/_{2}E_{grav,lost}$ is retained.

Example:

	Start	Finish
E_k	$-E_{grav}$	$-1.5E_{grav}$ (Star is smaller, therefore ${}^{1}\!/_{r^2}$ is bigger, therefore more bound)
E_p	$+\frac{E_{grav}}{2}$	$+0.75E_{grav}$ (Hotter)
E_{total} in the star $\left(E_p + E_k\right)$	$-0.5E_{grav}$	$-0.75E_{grav}$ (Star is more tightly bound even though it is hotter)

Change in binding every is only $-0.25E_{grav}$ but compared to the change in gravitational E_p it is $-0.5E_{grav}$ → $\frac{1}{2}$ of the change in gravitational E_p as E_p contracts remains in the star as increased binding every, and half must be lost (radiated away) if the contraction is to be stable or the contraction is to continue.
Can contraction continuously power the sun?
Need to estimate E_{grav}.

Take 2 point masses each of $\frac{M_\odot}{2}$ separated by R. Therefore the work required to separate them to infinity $= -\frac{G\frac{M_\odot}{2}\frac{M_\odot}{2}}{R} = \frac{-GM_\odot^2}{4R}$. More realistically, consider a spherical distribution made up of concentric shells.

$$dE_{grav} = -\frac{GM(r)}{r} x \underbrace{4\pi\rho(r)r^2dr}_{massofshell}$$

The first part of this equation is the potential due to the mass within the shell. Outside mass cancels out (little near the point, and much on the other side).

$$E_{grav} = \int_0^R dE_{grav}$$

Onion-skin model: split off the shells one by one and count up the result.
See Examples 3 for a uniform density sphere $\left(\rho(r) = const.\right)$

$$E_{grav} = -\frac{9}{15}\frac{GM_\odot^2}{R}$$

For a centrally condensed sun $\left(\rho(r)\ changing\right)$

$$E_{grav} = -2\frac{GM_\odot^2}{R_\odot}$$

→ $E_{grav} \sim 4x10^{41}J$ for the current sun.
Therefore if it collapses half of this is available to radiate away and provide the luminosity.
Time scales?

Current E_{grav} could supply the current $L_\odot$ for $\frac{1}{2}\frac{4x10^{41}}{4x10^{26}}seconds = 10^7 years$.

This is far too short a timescale, therefore gravitational energy can't be a major source of power of the current luminosity.

So can chemical energy power the sun?
Each kg of the sun radiates $> 2x10^{13}J$ over the estimated lifetime. $\left(L_\odot.lifetime/mass\right)$.
Chemical reactions release a few eV per atom (compare with typical ionization energies $\sim 10eV$)
Estimate:
Say the sun consists of Hydrogen and Oxygen in equal proportion, so complete combustion to H_2O could occur. Water molecule's mass is $18m_h = 18x1.67x10^{-27}kg$
Therefore how many water molecules would make up the solar mass?

$$\frac{M_\odot}{RM_H} = \frac{2x10^{30}}{3.5x10^{-26}} \sim 9x10^{55} \text{ molecules.}$$

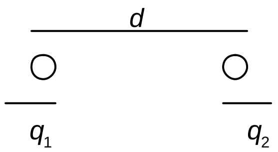
Therefore the total energy available from combustion is:
$6x10^{55}x1.6x10^{-19}x10$
(Number of molecules x eV in Joules x 10eV per reaction)
→ $10^{38}J$ available from chemical reactions. Therefore per kg:

$$\frac{10^{38}}{2x10^{30}} = 5x10^7 Jkg^{-1}$$

i.e. fails by $10^5 - 10^6$ (needed $> 2x10^{13}\, Jkg^{-1}$) to account for the actual output over the lifetime of the sun → chemical reactions cannot be the major source of solar energy.

What about thermonuclear reactions in the sun?
When a heavier nucleus forms from lighter ones (fusion) binding energy is released.
Nuclear reactions $\sim 10^6 eV$
Chemical reactions $\sim 10^0 eV$
This is a factor of a million.
For the binding energy of nuclei, see fig. 23.

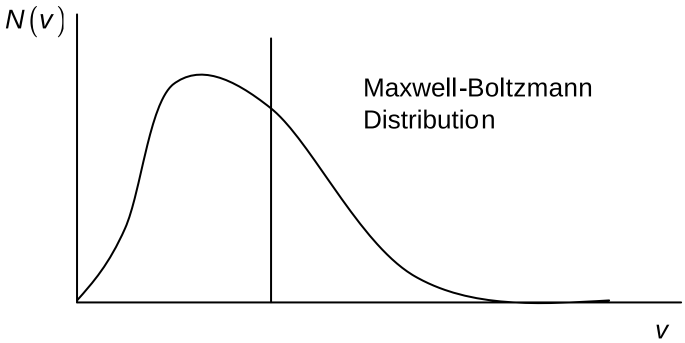
How do reactions occur?
Two positively charged nuclei have to collide and overcome the Coulomb barrier (see fig. 22)



For fusion $d < 2x10^{-15}\, m$ i.e. within range of the strong nuclear force.
Can they do it?
If so then we expect $E_k \sim E_p$ at the closest approach. What is E_p?

$$E_{coulomb} = \frac{q^2}{4\pi\varepsilon_o d} = 2x10^{-15}\, J = 1.5 MeV$$

But our earlier calculation about E_k in the centre of the sun said that the E_k in the core region $\sim 1 - 2KeV$. Therefore, the particles can't get close enough to fuse.
How does the sun avoid this problem? Not all of the particles have "typical" (average) energies.



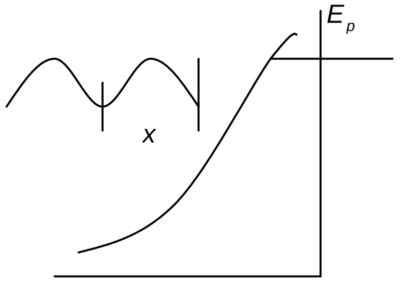
$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

The distribution curve shows that some molecules have much higher velocities. A tiny fraction of the molecules have $v \sim 10 v_{rms}$ → $E_K \sim 100 E_{k,rms}$
This still means that $E_k \approx 100 kEv$.
Essentially none have $v > 30 v_{rms}$ ie.e $E_k > 1000 E_{k,rms}$ due to the exponential fall-off.

We need Quantum Mechanical tunneling.
De Broglie said that particles have wave-like properties.

$$\lambda = \frac{h}{p}$$

and as seen in year 2, they can tunnel through potential barriers which are classically "too high"



There is a finite probability of the wave-particle passing through the barrier "like a ghost". (Me thinking: can the particle have a certain amount of mass / wave-iness related to probability?)

3.4 Stability of the Sun

The sun is in thermal equilibrium. The amount of energy generated is exactly equal to the amount of energy radiated. But what if the energy generation rate increased?
E ↑, T_{core} ↑, → P_{core} ↑. (Pressure = energy / unit volume)

But $P_{core} \propto \frac{GM_{\odot}^{2}}{R_{\odot}^{4}}$, therefore if $R_{\odot}$ goes up, P_{core} decreases.

$p = nkT$.

i.e. works against the change.
Therefore there is a natural feedback mechanism which works against rapid changes in radius. It is possible to argue the exact opposite if the core produces less heat → $R_{\odot}$ decreases, therefore the sun heats up; again opposite reaction to change.

→ the sun is a controlled thermonuclear reactor with natural stability.

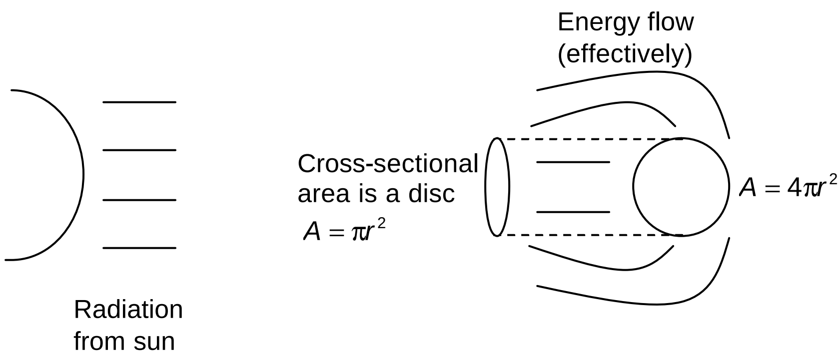
4. Planets

4.1 Energy balance in the planets

The main effect is the solar photon flux which falls mainly (40%) in the visible band and infra red (Fig. 19).
$350 \rightarrow 700 nm$
See fig. 25. The earth's atmosphere transmits this band → atmosphere as a whole is not strongly heated by solar radiation directly.
Solar radiation strikes the surface and heats it. The surface then re-radiates back into space.
At what λ? Use Wien's Law $T\lambda_{\max} = const. = 2.9x10^{-3} mk$.

$T_{earth} \sim 300k$
$T_{sun} \sim 6000k$
- factor 20 in k → factor 20 in λ.

Sun peak at 500nm
Earth peak at $20x500nm = 10000nm = 10\mu m$
Fortunately the earth's atmosphere is nearly transparent in this band also ($10 - 12\mu m$ window)
Equilibrium temperature with no atmosphere:

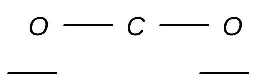


Because we have oceans the energy is spread around the sphere of the Earth, and also the Earth is rotating.
Solar constant (the amount of energy falling on a unit area outside the atmosphere) $\Omega = 1.4kWm^{-2}$. Therefore the input, averaged over the whole surface,
$= \frac{\Omega \pi R_E^{\ 2}}{4\pi R_E^{\ 2}} = \frac{\Omega}{4} = 350Wm^{-2}$.
The Earth is partly reflective i.e. from the sea and clouds. $\sim 30\%$ of the input light is reflected straight back. This is called the "Albedo" $= 0.3$ (Relative reflectivity).
The average energy absorbed $= (1-a)\frac{\Omega}{4} =$ flux in $(F_{in}) = 0.7x350 \approx 250Wm^{-2}$.
Earth retransmits solar energy approximately as a black body.
Flux out $= \sigma T_E^{\ 4} \ \ Wm^{-2}$.
For equilibrium $\sigma T_E^{\ 4} = (1-a)\frac{\Omega}{4}$

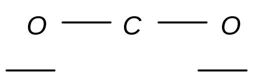
$$\rightarrow T_{Earth} = \left[\frac{0.7x1.4x10^3}{4x5.67x10^{-8}}\right]^{1/4} \sim 255k = -18^oC$$

Therefore we would expect Earth to be frozen but in fact $T_{earth} \sim 290k = +17^oC$.
This is due to the Greenhouse effect (fig. 26)
Not all of the radiation from the surface is transmitted by the atmosphere (in the $8-12\mu$ window in the Infra-Red). Some is absorbed and retransmitted up (out into space) and down (back to the Earth's surface → warmer).
NB: a different photon is re-emitted from a different molecule i.e. statistical effect creating an overall energy balance.
Mechanism?
Quantised vibrational normal modes (not electron orbital changes). General motion is the sum of the normal modes – phase or antiphase.
e.g. for CO_2:

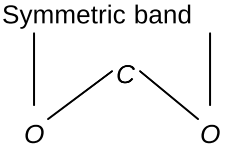
a) Symmetrical stretch (Antiphase)



b) Asymmetric effect in phase.



c) Symmetric band



(a) does not absorb photons since the electrical center of gravity does not change through the cycle. (b) and (c) do absorb since the electrical centre of gravity does change through the cycle.

All the modes correspond to different quantised frequencies but typically correspond to the energy of an infra-red photon.
What are the greenhouse gasses?

- H_2O is the dominant one, as all the molecules absorb.
- CO_2
- CH_4 (Methane) which has more modes than CO_2
- $CFC's$ (Refrigeration) which have lots of vibrational modes.

H_2O gives rise to the basic $+\sim 30^oC$ warming, while the rest give the additional few Kelvin.
Note that O_2 and N_2 do not absorb in infra-red since the vibrational mode keeps the molecule electrically symmetrical.

4.2 Atmospheres and Temperatures of Terrestrial Planets

$$T_{eff,black-body} = \left[\frac{(1-a)\Omega_{planet}}{4\sigma}\right]^{1/4}$$

For equilibrium, $F_{in} = F_{out}$
See handout for measured temperatures of planets.
The central issue is atmospheric retention. Can the atmosphere be retained by the planet's gravity?
The critical balance is between the E_k of the molecule and the gravitational escape velocity from the gravitational field.

$$V_{thermal,RMS} = \left[\frac{3kT}{m_{molecule}}\right]^{1/2}$$

$$\left(\frac{1}{2}kTx3 = \frac{1}{2}mv_{rms}{}^2\right)$$

$$v_{esc} = \left[\frac{2GM_{planet}}{R_{planet}}\right]^{1/2} = \int \frac{GM}{R^2}dr$$

The ratio of these is:

$$x = \frac{V_{thermal}}{V_{escape}}$$

(See handout)
This determines whether or not the molecule is captured or escapes.
e.g. the Moon.

$$v_{esc} = 2.38x10^3 ms^{-1}$$

$$V_{thermal} = 0.48x10^3 ms^{-1}$$

$$\rightarrow x \sim \frac{1}{5}$$

Therefore it looses its' atmosphere in $\sim 10^8$ years (see table in handout)
Therefore lost atmosphere over $4x10^9$ years (age of moon)
Therefore the factor of 4 in the T_{eff} calculation, from the transfer of heat all around the planet, goes.

$$\sigma T_{eff}{}^4 = (1-a)\Omega$$

For the moon, $a = 0.1$ and Ω is the same as for the Earth.
$T_{eff} = 385k$ for a sub-solar point on the moon's surface.
Similar arguments apply to Mercury – both the moon and Mercury are too small to retain a significant atmosphere. Therefore there are large temperature differentials on the surface.

Earth:
$T_{eff} < T_{actual}$ (see earlier)

Cannot retain H, He atoms (calculate $V_{thermal}$ correctly with the right atomic mass)
Light atoms move too fast and leak out, N_2, O_2 etc. can be retained.

Venus:
$T_{eff} << T_{actual surface}$
$232k << 750k$
T_{eff} looks low but the Albedo is 0.75 compared with 0.3 for the earth, due to the highly reflective clouds.
$T_{actual surface}$ was measured by a soviet spacecraft which landed on the surface. It lasted the whole of half an hour.
See handout.

Mars:
Very small atmosphere $\sim 1\%$ of the Earth's surface pressure, therefore very large day-night variations.

Temperatures of Jovian planets:
See handout.

Planetary rings:
See handout.
Self-gravity between the fragments:

$$F_{grav} = \frac{Gm^2}{(2r)^2}$$

Differential tidal force due to planet:

$$F_{tidal} = \frac{GMm}{(D_r - r)^2} - \frac{GMm}{(D_r + r)^2}$$

Therefore at the Roche distance D_r, $F_{grav} = F_{tidal}$

$$\frac{GM^2}{4r^2} = GM_p m \left[\frac{1}{(D_r - r)^2} - \frac{1}{(D_r + r)^2} \right]$$

Same binomial expansion as in "tides" calculation earlier in lectures and take $\frac{r}{D_r} << 1$ to get:

$$\frac{Gm^2}{4r^2} = \frac{GM_p m 4r}{D_r^3}$$

$$D_r^3 = 16r^3 \frac{M_p}{m}$$

but since $m = 0.5Moon$ and $r = (0.5)^{1/3} R_{moon}$

$$\rightarrow D_r^3 = 16R_{moon}^3 \frac{M_{planet}}{M_{moon}}$$

$$D_r \sim 2.5 R_{moon} \left(\frac{M_{planet}}{M_{moon}} \right)^{1/3}$$

If the planet and the moon share the same density, which is plausible (i.e. their composition is roughly the same):
$mass = \frac{4}{3}\pi r^3 \rho$ for both.
Therefore the equation simplifies to:
$D_r \sim 2.5R_{planet}$ (Full calculation gives $2.44R_{planet}$)
e.g. Earth $D_r \sim 16{,}000km$

Moon distance $= 400{,}000km$ i.e. $>> D_r$.
All planetary ring systems are within the Roche limit for their planets.

Comet breakup: Shoemaker Levy 9
1992-1994
See fig. 29 for schematic explanation.
The comet got pulled apart into 1-2km sized fragments due to the tiday forces within the Roche limit.
The breakup indicates the low cohesive strength of the comet → poorly compacted snowball.
Impact: effect depends on the relative velocity with respect to Jupiter.

$V_{jup} \sim 13kms^{-1}$

$\pm V_{comet}$ (Typically of Jupiter distance from the sun $v \sim 20kms^{-1}$

$\pm V_{escJupiter}$ ($\sim 60kms^{-1}$ starting from zero velocity)

Add quadratically i.e. for a random set of vector directions.

v for a typical impact $\sim \sqrt{v_{jup}^2 + v_{comet}^2 + v_{esc}^2} \sim 70kms^{-1}$

Note for the Earth:

$v_{earth} \sim 30kms^{-1}$

$v_{comet} \sim 30-40kms^{-1}$

$v_{escape} \sim 11kms^{-1}$

$v_{impactsEarth} \sim 40-50kms^{-1}$

Look at the impact energy / unit mass.

$E_k = \frac{1}{2}mv^2 = \frac{1}{2}\left(49x10^8\right) = 2.5x10^9 Jkg^{-1}$

c.f. with chemical energy of TNT (few eV/atom) $= 4x10^6 Jkg^{-1}$

→ $E_k \big/ unit\ mass \sim 600xTNT$

Largest fragment of Shoemaker-Levy ~ 2km

$\rho \sim 0.2x10^3 kgm^3$ (1/5 water since it is a loose snowball)

→ mass $= 7x10^{12} kg$

$E_k \sim 4x10^6$ million tones of TNT

→ one of a series of enormous impacts in Jupiter's atmosphere.

5. Origin and Fate of the Solar System

5.1 When did it form?

Radioactive dating of rocks:

$^{238}U \rightarrow {}^{206}P \quad T_{1/2} \sim 4.5x10^9 years$

$^{40}k \rightarrow {}^{40}Ar \quad T_{1/2} = 1.3x10^9 years$

Complicated clocks – gives consistent results.

Oldest rocks on Earth $\sim 3.8x10^9 years$

Oldest rocks from the Moon $\sim 4.2x10^9 years$

Oldest meteorite $\sim 4.5x10^9 years$

Earth / Moon are younger because the reheating due to impacts can 'reset' the clocks.

NB: this is much younger than the universe $\sim 13x10^9 years$

→ Many stars pass through their life cycles before the solar system was born – some stars end up in supernovae.

→ Sun was formed from "processed" materials which has been enriched by heavier elements - $> {}^7Li$ were produced in stellar nuclear reactions.

5.2 How did it form?

This is less certain.
Handout gives the list of facts that any theory must explain.
Start out with a giant molecular cloud.
Cold, thin, tenuous. 2% heavy elements (by weight)
Dust, water etc.
$T \sim 15k$, $R \sim 20$ light years
$\text{Mass} = 10^3 M_\odot$.
Gas collapses and fragments – small condensations form.
- Like pre-solar nebula.

NB: must radiate energy to keep steadily collapsing (obeys Viral theorem $\frac{1}{2}E_{grav}$ that is liberated in the collapse has to be lost.
Mostly energies in the far infra-red since the cloud is cold (Wien's Law)

Collapse – density increases, temperature increases – protostar ~ over 10^7 years.
Opacity in outer layers rises, traps heat, T ↑ → nuclear reactions star.
Complication: rate of angular momentum.
Distribution of angular momentum (Fact 4)
Sun:
At the centre of mass (to high accuracy)
→ all angular momentum will be in its' spin.

$L_\odot = I_\odot \omega_\odot$

$I_\odot \sim \frac{2}{5} M_\odot R_\odot{}^2$

Approx. uniform sphere

$\omega_\odot = 2.6x10^{-6} rad$

→ $L_\odot = 10^{42} kgm^2 s^{-1} \left(Js^{-1}\right)$

Jupiter: dominates the planets but still has only $10^{-3} M_\odot$
→ all angular momentum is orbital.

$L_{jup} = m_j v_j r_j \ kgm^2 s^{-1}$

Values in table 1.

$L_{jupiter} \approx 2x10^{42} Js^{-1}$

Total angular momentum of all the planets $\sim 3x10^{43} Js^{-1}$
i.e. >> spin of angular momentum of the sun.
- Expect the sun to be spinning very fast (~1 second) if contracted from a gas cloud (see handout) like a skater pulling in their arms and conserving angular momentum.
This is a significant puzzle for any theory.

5.3 Fate of the Solar System

- Sun will run out of hydrogen fuel – deep in interior - $5x10^9 years$
- Energy output will fall – contracts – heats it up again (Viral theorem)
- Central temperature $>10^8 k$ - can fuse He

Rapid energy output – expands – cools

$R_\odot \sim 10 - 100 R_\odot$

$L_\odot \sim 1000 L_\odot$

$T_{surface} \sim 3000k$

This is a red giant, like Betelgeuse in Orion.

Earth may be engulfed in the red giant.
After a few $x10^8 years$ the helium is exhausted.
→ energy output falls → contracts, heats up again → (depending on mass):

$R_{\odot} = 0.01R_{\odot}$
$T \sim 20{,}000k$
This is a white dwarf.

There is still a lot to learn about the formation and evolution of the solar system. The current excitement is solar systems around other stars. We may find life elsewhere in the universe.

Exam:
- Short compulsory questions
- 2 out of 3 of the longer questions.
- Easier than illustrative examples.