1. Introduction

The real world consists of three space dimensions. We need a calculus which is up to the job. By the end of this course we will possess the skills to develop our understanding of:

- General theory of waves – light, Quantum Mechanics, sound, vibrations, strings & membranes.
- Heat flow and diffusion
- Fluid dynamics
- Gravitation

Books:
"Further Mathematics for the Physical Sciences" (Tinker & Lambourne)
"Mathematical Methods in the Physical Sciences" (Mary L Boas)
"Div, Grad, Curl and all that" (H. M. Schey)
Feynman Lectures on Physics vol. 2
NB: Jordan & Smith does not cover the whole course.

2. Scalar & Vector Fields

A field is some quantity that depends on where you are in space.

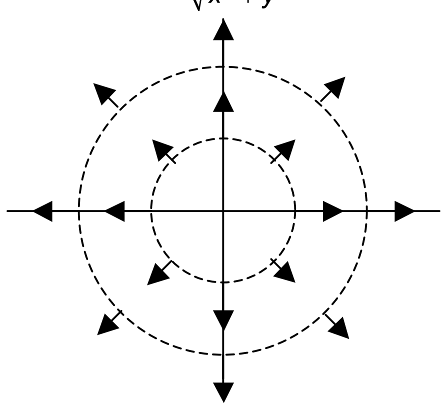
2.1 Scalar field

Characterised with identifying a single number with each point in space (may vary with time) e.g. temperature in this room, T(x,y,z)
Temperature is a scalar field.
How we set up the coordinate system x,y,z is up to us. If we have a different coordinate system, in which the same point in space has coordinate x',y',z', the temperature will be a different function of these coordinates, T'(x',y',z'), but a given point in space has a unique temperature. i.e. $T(x,y,z)=T'(x',y',z')$.
One of our goals is to formulate the laws of physics in such a way that the explicit dependence on the coordinate system drops out.

2.2 Vector Field

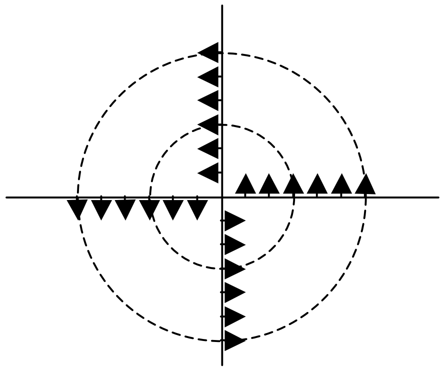
This is characterised by identifying a vector with each point in space, e.g. the velocity of the flow of a liquid (speed and direction). $\underline{v}(x,y,z)$
Another example: the electric field due to some charge distribution $\underline{E}(x,y,z)$.
Examples:

a) $\underline{v}(x,y)=\hat{\underline{r}}=\dfrac{x\hat{\underline{i}}+y\hat{\underline{j}}}{\sqrt{x^2+y^2}}$



All arrows are the same length. This vector field is a diverging field.

b) $\dfrac{-y\underline{\hat{i}}+x\underline{\hat{j}}}{\sqrt{x^2+y^2}}=\underline{v}$

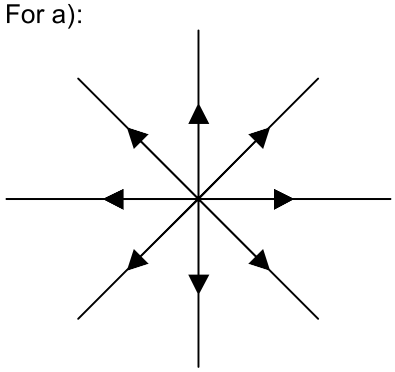


This is a curling vector field.

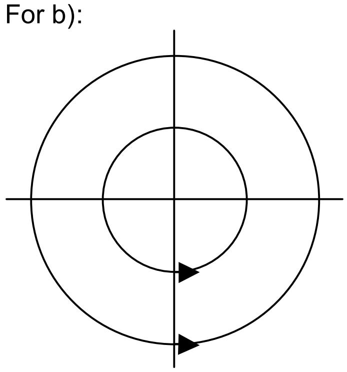
2.3 Field lines ("streamlines")

These show the direction of the vector field at each point in space, but we loose information about the magnitude.

For a):

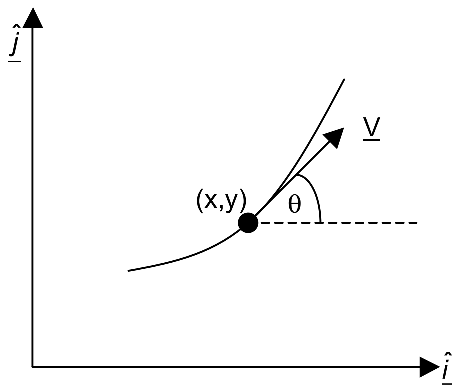


For b):



How do we calculate the field lines for a 2D vector field?

$\underline{v}(x,y)=V_x(x,y)\underline{\hat{i}}+V_y(x,y)\underline{\hat{j}}$



Let the field line be $y = f(x)$. For points on the field line:

Slope of the tangent $= \tan\theta$.

$$\frac{dy}{dx} = \frac{V_x(x,y)}{V_y(x,y)}$$

This shows the shape of the field but not the direction of the arrows.
Solve this to get the field lines.
Example: for our picture of a curling field (b):

$$\underline{v} = \frac{-y\hat{\underline{i}} + x\hat{\underline{j}}}{\left(x^2+y^2\right)^{1/2}}$$

$$\frac{dy}{dx} = \frac{x/\sqrt{x^2+y^2}}{-y/\sqrt{x^2+y^2}} = -\frac{x}{y}$$

$$\Rightarrow \int ydy = \int xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$C = \frac{R^2}{2}$$

$$x^2 + y^2 = R^2$$

Equation of a circle.

3. The Gradient ("Grad")

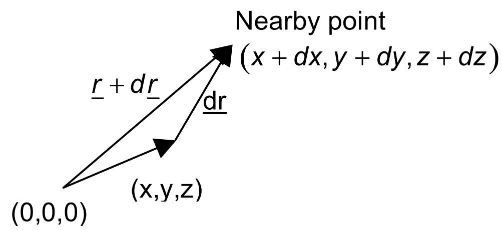
3.1 Definition

What is the derivative of a scalar field?

Consider $T(x,y,z)$.

$\frac{\partial T}{\partial x}$ tells us how T varies with x (fixed y and z)

This is not a scalar, as it depends on the choice of axis.



$\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

Temperature difference between $\underline{r}$ and $\underline{r}+d\underline{r}$.

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$dT = \left(\hat{\underline{i}}\frac{dT}{dx} + \hat{\underline{j}}\frac{dT}{dy} + \hat{\underline{k}}\frac{dT}{dz}\right)\cdot\left(\hat{\underline{i}}dx + \hat{\underline{j}}dy + \hat{\underline{k}}dz\right) = \underline{\nabla}T\cdot\underline{dr}$$

dT is a scalar.

$dT = \underline{\nabla}T\cdot\underline{dr}$ (Scalar / dot product)

So:

$\underline{\nabla}T = \hat{\underline{i}}\frac{\partial T}{\partial x} + \hat{\underline{j}}\frac{\partial T}{\partial y} + \hat{\underline{k}}\frac{\partial T}{\partial z}$ is a vector.

Consider a change in coordinate systems from (x,y,z) to (x',y',z').

Then $T(x,y,z) = T'(x',y',z')$

$$dT = \underline{\nabla}T\cdot\underline{dr} = \underline{\nabla}'T'\cdot\underline{dr}'$$

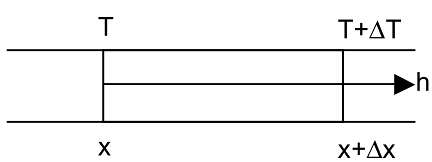
$$\underline{dr}' = \hat{\underline{i}}'dx' + \hat{\underline{j}}'dy' + \hat{\underline{k}}'dz'$$

$$\underline{\nabla}'T' = \hat{\underline{i}}'\frac{\partial T'}{\partial x'} + \hat{\underline{j}}'\frac{\partial T'}{\partial y'} + \hat{\underline{k}}'\frac{\partial T'}{\partial z'}$$

We expect to see gradients of scalar fields appearing in physical laws.
e.g. heat flow equation:

$$h = -k\frac{\Delta T}{\Delta x}$$

h = heat per unit area per unt time flowing across Δx.



ΔT: temperature difference
k: thermal conductivity

In the limit $\Delta x \to 0$, $\Delta T \to 0$:

$$h = -k\frac{dT}{dx}$$

In three dimensions:

$$\underline{h} = -k\underline{\nabla}T$$

Heat flows in the direction of $-\underline{\nabla}T$ i.e. normal to the isotherms.

Properties of $\underline{\nabla}T$:

– $\underline{\nabla}T$ is a vector

– $dT = \underline{\nabla} T \cdot \underline{dr}$ (Change in T between $\underline{r}$, $r + dr$).

Let $\underline{\hat{u}}$ be a unit vector (vector of unit length) in the direction parallel to $\underline{dr}$.
$\underline{dr} = \underline{\hat{u}} \cdot ds$ where ds is the magnitude of $\underline{dr}$ is the distance between the points
→ $dT = \underline{\nabla} T \cdot \underline{\hat{u}} ds$

$$\left.\frac{dT}{ds}\right|_{\underline{\hat{u}}} = \underline{\nabla} T \cdot \underline{\hat{u}}$$

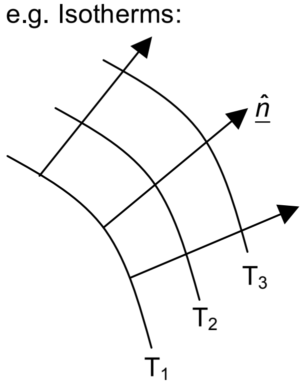
"Directional derivative"
Rate of change of T in the direction $\underline{\hat{u}}$.
- $\underline{\nabla} T \cdot \underline{\hat{u}}$ is a maximum when $\underline{\hat{u}}$ is parallel to $\underline{\nabla} T$

Maximum rate f change $\left.\frac{dT}{ds}\right|_{\max} = \left|\underline{\nabla} T\right|$

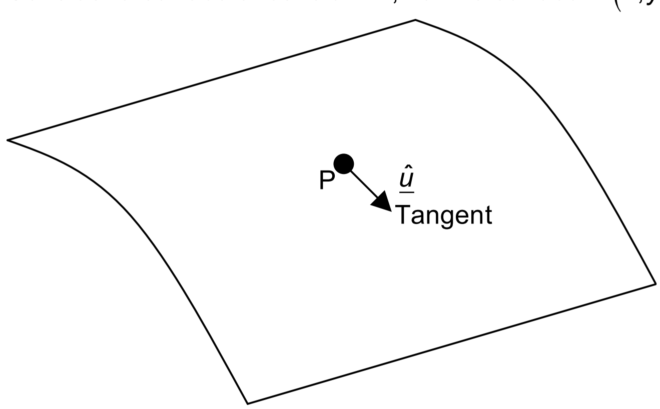
Direction of the maximum change is when $\underline{\hat{u}}$ is parallel to $\underline{\nabla} T$

$\underline{\hat{n}} = \frac{\underline{\nabla} T}{\left|\underline{\nabla} T\right|}$ (a unit vector parallel to $\underline{\nabla} T$).

e.g. Isotherms:



Consider a surface of constant T, i.e. the surface $T(x, y, z) = const.$



At P_1, $\left.\frac{dT}{ds}\right|_{\underline{\hat{u}}} = 0$ if $\underline{\hat{u}}$ is tangential to the surface at P

→ $\underline{\nabla} T \cdot \underline{\hat{u}} = 0$ for any vector $\underline{\hat{u}}$ tangent to the surface.

$\underline{\nabla} T$ is normal to the surface $T(x, y, z) = const.$ at any point.

Example:

$ax + by + cz = d$ (equation of a plane)

Define $g(x, y, z) = ax + by + cz - d = 0$

$$\underline{\nabla} g = \frac{\partial g}{\partial x}\hat{\underline{i}} + \frac{\partial g}{\partial y}\hat{\underline{j}} + \frac{\partial g}{\partial z}\hat{\underline{k}}$$

$$= a\hat{\underline{i}} + b\hat{\underline{j}} + c\hat{\underline{k}}$$

normal to the plane.

Unit vector normal to the plane is:

$$\hat{\underline{n}} = \frac{\underline{\nabla} g}{|\underline{\nabla} g|} = \frac{a\hat{\underline{i}} + b\hat{\underline{j}} + c\hat{\underline{k}}}{\sqrt{a^2 + b^2 + c^2}}$$

3.2 Cylindrical and Spherical Polar Coordinate Systems

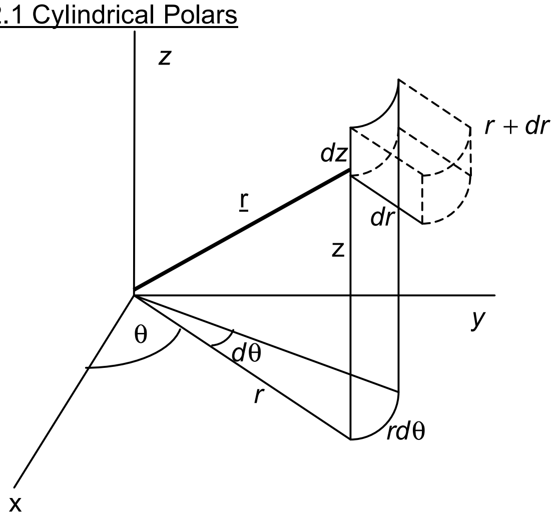
$\hat{\underline{r}}$ = unit vector in the direction of increasing r

$\hat{\underline{\theta}}$ =unit vector in the direction of increasing θ

$\hat{\underline{k}}$ =unit vector in the direction of increasing z

These vectors are mutually orthogonal unit vectors.

3.2.1 Cylindrical Polars



$$\underline{dr} = \hat{\underline{r}}dr + \hat{\underline{\theta}}rd\theta + \hat{\underline{k}}$$

Write:

$$\underline{\nabla T} = b_r\hat{\underline{r}} + b_\theta\hat{\underline{\theta}} + b_z\underline{k}$$

then

$$dT = \underline{\nabla T} \cdot \underline{dr} = b_r dr + b_\theta r d\theta + b_z dz \text{ (1)}$$

But also,

$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial z}dz \text{ (2)}$$

Compare (1) and (2)

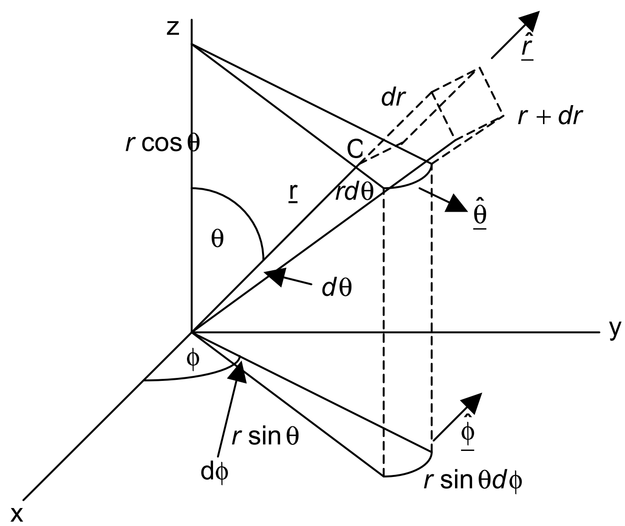
$$b_r = \frac{\partial T}{\partial r}$$

$$b_\theta = \frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$b_z = \frac{\partial T}{\partial z}$$

$$\rightarrow \ \underline{\nabla T} = \frac{\partial T}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\underline{\hat{\theta}} + \frac{\partial T}{\partial z}\underline{\hat{k}}$$

3.2.2 Spherical Polars



$$\underline{dr} = \underline{\hat{r}}dr + \underline{\hat{\theta}}rd\theta + \underline{\hat{\phi}}r\sin\theta d\phi$$

Write $\underline{\nabla T} = a_r\underline{\hat{r}} + a_\theta\underline{\hat{\theta}} + a_\phi\underline{\hat{\phi}}$

Then $dT = \underline{\nabla T}\cdot\underline{dr} = a_r dr + a_\theta r d\theta + a_\phi r\sin\theta d\phi$ (3)

Also $dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \phi}d\phi$ (4)

From (3) and (4)

$$a_r = \frac{\partial T}{\partial r}$$

$$a_\theta = \frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$a_\phi = \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

$$\rightarrow \ \underline{\nabla T} = \frac{\partial T}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\underline{\hat{\phi}}$$

Example 3-1

Evaluate $\underline{\nabla y}$ in spherical polars

$$y = r\sin\theta\sin\phi$$

$$\underline{\nabla y} = \frac{\partial y}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial y}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r\sin\theta}\frac{\partial y}{\partial \phi}\hat{\underline{\phi}}$$

$$= \sin\theta\sin\phi\hat{\underline{r}} + \cos\theta\sin\phi\hat{\underline{\theta}} + \cos\phi\hat{\underline{\phi}}$$

In Cartesians $\underline{\nabla y} = \hat{\underline{j}}$.

Therefore $\hat{\underline{j}} = \sin\theta\sin\phi\hat{\underline{r}} + \cos\theta\sin\phi\hat{\underline{\theta}} + \cos\phi\hat{\underline{\phi}}$

4. The ∇ Operator

For any scalar field T:

$$\underline{\nabla}T = \hat{\underline{i}}\frac{\partial T}{\partial x} + \hat{\underline{j}}\frac{\partial T}{\partial y} + \hat{\underline{k}}\frac{\partial T}{\partial z}$$

→ $\underline{\nabla} = \hat{\underline{i}}\frac{\partial}{\partial x} + \hat{\underline{j}}\frac{\partial}{\partial y} + \hat{\underline{k}}\frac{\partial}{\partial z}$ "del" ("Nabla")

It is a vector differential operator.

The "operator" $\underline{\nabla}$ is "hungry" for something to "operate" on, i.e. differentiate. Feed it a scalar field and it returns a vector field.

We can take scalar products of vectors to form scalars. Can we make a scalar product with $\underline{\nabla}$?

$\underline{\nabla}\cdot\underline{V}$ is a scalar field (the divergence of V, or "div V".

Similarly,

$\underline{\nabla}x\underline{V}$ is a vector field (the curl of V ("Curl V"))

(a) Divergence of V

$$\underline{\nabla}\cdot\underline{V} = \left(\hat{\underline{i}}\frac{\partial}{\partial x} + \hat{\underline{j}}\frac{\partial}{\partial y} + \hat{\underline{k}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\underline{i}}V_x(x,y,z) + \hat{\underline{j}}V_y(x,y,z) + \hat{\underline{k}}V_z(x,y,z)\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial x} + \frac{\partial V_z}{\partial x}$$

Scalar.

(b) Curl of V

$$\underline{\nabla}x\underline{V} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{\underline{j}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{\underline{k}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Vector.

Summary:

Object	Name	Operator "Food"	Output
$\underline{\nabla}T$	Gradient ("grad")	Scalar	Vector

$\underline{\nabla} \cdot \underline{V}$	Divergence ("div")	Vector	Scalar
$\underline{\nabla} x \underline{V}$	Curl	Vector	Vector

<u>Examples:</u>

$\underline{F} = xy\underline{i} + yz\underline{j} + xz\underline{k}$

(a) Divergence of <u>F</u>

$$\underline{\nabla} \cdot \underline{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial x}(xz)$$
$$= y + z + x$$

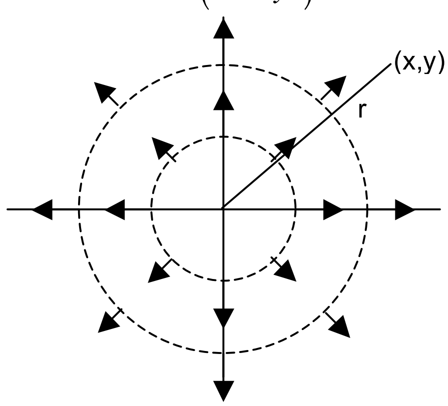
(b) Curl of <u>F</u>

$$\underline{\nabla} x \underline{F} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

$$\underline{\nabla} x \underline{F} = \underline{i}(-y) - \underline{j}(z) + \underline{k}(-x) = -y\underline{i} - z\underline{j} - x\underline{k}$$

Return to the 2D vector fields we looked at before:

(a) $\underline{V}(x, y) = \hat{\underline{r}} = \dfrac{x\underline{i} + y\underline{j}}{\left(x^2 + y^2\right)^{1/2}}$



$$\underline{\nabla} \cdot \underline{V} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right)$$

where $r = \left(x^2 + y^2\right)^{1/2}$, so $\dfrac{\partial r}{\partial x} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2} 2x = \dfrac{x}{r}$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

So:

$$\underline{\nabla} \cdot \underline{V} = \frac{1}{r} + x\left(-\frac{1}{r^2}\right)\frac{x}{r} + \frac{1}{r} + y\left(-\frac{1}{r^2}\right)\frac{y}{r} = \frac{2}{r} - \frac{x^2 + y^2}{r^3} = \frac{2}{r} - \frac{1}{r} = \frac{1}{r}$$

Left as an exercise to show that curl <u>V</u> is zero.

"$\underline{V}$ is irrotational".

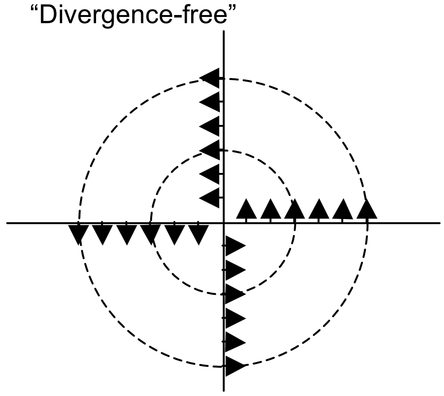
(b) $\underline{V} = \frac{-y}{r}\underline{i} + \frac{x}{r}\underline{j}$

$$\underline{\nabla}\cdot\underline{V} = \frac{\partial}{\partial x}\left(-\frac{y}{r}\right) + \frac{\partial}{\partial y}\left(\frac{x}{r}\right)$$

$$= -y\left(-\frac{1}{r^2}\frac{x}{r}\right) + x\left(-\frac{1}{r^2}\frac{y}{r}\right)$$

$$= 0$$

"Divergence-free"

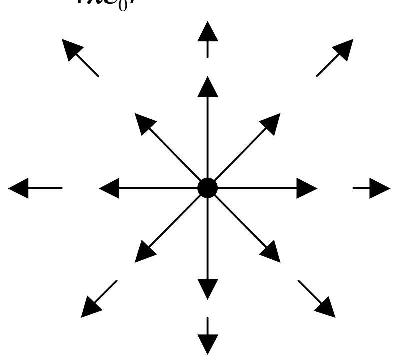


$$\underline{\nabla}x\underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{r} & \frac{x}{r} & 0 \end{vmatrix} = \underline{i}(0) - \underline{j}(0) + \underline{k}\left(\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right)\right) = \underline{k}\left(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3}\right) = \frac{1}{r}\underline{k}$$

(Out of the board)

<u>But</u>: we have to be careful…

$\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\underline{\hat{r}}$ (3 dimensions)



$$\underline{E} = \frac{Q}{4\pi\varepsilon_0 r^3}\underline{r}$$

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

What is $\underline{\nabla} \cdot \underline{E}$?
In general,

$$\underline{\nabla}(\phi \underline{V}) = \frac{\partial}{\partial x}(\phi V_x) + \frac{\partial}{\partial y}(\phi V_y) + \frac{\partial}{\partial z}(\phi V_z) = \phi\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) + \frac{\partial \phi}{\partial x}V_x + \frac{\partial \phi}{\partial y}V_y + \frac{\partial \phi}{\partial z}V_z$$

ϕ is a scalar. This is the product rule.
$\underline{\nabla} \cdot (\phi \underline{V}) = \phi \underline{\nabla} \cdot \underline{V} + \underline{V} \cdot \underline{\nabla}\phi$ - product rule.

$$\phi = \frac{Q}{4\pi\varepsilon_o r^3}$$

$\underline{V} = \underline{r}$

$$\underline{\nabla} \cdot \underline{E} = \frac{Q}{4\pi\varepsilon_o}\underline{\nabla} \cdot \left(\frac{1}{r^3}\underline{r}\right) = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\underline{\nabla} \cdot \underline{r} + \underline{r} \cdot \underline{\nabla}\left(\frac{1}{r^3}\right)\right) \text{(1)}$$

But $\underline{\nabla} \cdot \underline{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$ (2)

What about $\underline{\nabla}\left(\frac{1}{r^3}\right)$?

For any function f(r) (independent of θ and ϕ) then $\underline{\nabla} f(r) = \frac{\partial f}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\underline{\phi}}$

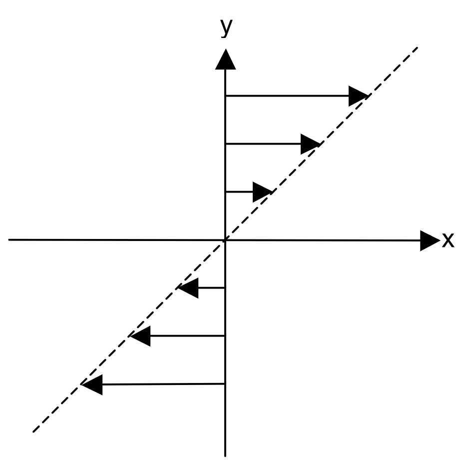
$$\underline{\nabla}\left(\frac{1}{r^3}\right) = \frac{-3}{r^4}\hat{\underline{r}} = \frac{-3}{r^5}\underline{r} \text{ (3)}$$

Putting (2) and (3) into (1):

$$\underline{\nabla} \cdot \underline{E} = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r^3}3 - \frac{3}{r^5}\underline{r} \cdot \underline{r}\right) = 0 \text{ as } \underline{r} \cdot \underline{r} = r^2 \text{., providing } r \neq 0.$$

Divergence is a measure of the divergent nature of a vector field <u>at a point</u>. In this case, there is no divergence except at r=0.

Looking at curl:
Rectilinear fluid flow with a velocity field.
$\underline{V} = \omega y \underline{i}$



$$\underline{\nabla} x \underline{V} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega y & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial y}\omega y \hat{\underline{k}} = -\omega \hat{\underline{k}}$$

Curl is the measure of the rotational nature of a vector field at a point.

Second Derivatives of Vector Fields

a) $\underline{\nabla} \cdot \left(\underline{\nabla} T\right)$ Scalar

b) $\underline{\nabla} x \left(\underline{\nabla} T\right)$ Vector

c) $\underline{\nabla}\left(\underline{\nabla} \cdot \underline{V}\right)$ Vector

d) $\underline{\nabla} \cdot \left(\underline{\nabla} x \underline{V}\right)$ Scalar

e) $\underline{\nabla} x \left(\underline{\nabla} x \underline{V}\right)$ Vector

Look at each of these in turn.

a) $\underline{\nabla} \cdot \left(\underline{\nabla} T\right)$

$$= \frac{\partial}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This is called “Laplacian”. It is said as “del squared”.

[Aside: Maxwell (1) $\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_o}$ =0 in free charges. $\underline{E} = -\underline{\nabla}\phi \rightarrow \nabla^2\phi = 0$.]

NB: $\nabla^2 \underline{v} \equiv \hat{\underline{i}}\nabla^2 V_x + \hat{\underline{j}}\nabla^2 V_y + \hat{\underline{k}}\nabla^2 V_z$

b) $\underline{\nabla} x \left(\underline{\nabla} T\right)$

$\underline{\nabla} x \left(\underline{\nabla} T\right) = 0$ (like $\underline{a} x \underline{a} = 0$)

Check:

$$\begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} = \hat{\underline{i}}\left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial z \partial y}\right) + \hat{\underline{j}}(0) + \hat{\underline{k}}(0) = 0$$

“Curl grad = 0”

c) $\underline{\nabla}\left(\underline{\nabla} \cdot \underline{V}\right)$

This is just a vector field. It cannot be simplified any futher.

d) $\underline{\nabla} \cdot \left(\underline{\nabla} x \underline{V}\right)$

$= 0$

$$\underline{\nabla} \cdot \left(\underline{\nabla} x \underline{V}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \text{which all cancel.}$$

"Div curl = 0"

e) $\underline{\nabla}x(\underline{\nabla}x\underline{V})$

=?

Recall $\underline{a}x(\underline{b}x\underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - (\underline{a}\cdot\underline{b})\underline{c}$

Use $\underline{a}\to\underline{\nabla}, \underline{b}\to\underline{\nabla}, \underline{c}\to\underline{V}$ and keep $\underline{c}$ to the right of the del operators.

$= \underline{\nabla}(\underline{\nabla}\cdot\underline{V}) - (\underline{\nabla}\cdot\underline{\nabla})\underline{V}$

$= \underline{\nabla}(\underline{\nabla}\cdot\underline{V}) - \nabla^2\underline{V}$

"Curl curl = grad div – del squared"
Important for electromagnetism.

<u>Summery Table</u>

$\underline{\nabla}\cdot(\underline{\nabla}T) = \nabla^2 T$

$\underline{\nabla}x(\underline{\nabla}T) = 0$

$\underline{\nabla}\cdot(\underline{\nabla}x\underline{V}) = 0$

$\underline{\nabla}x(\underline{\nabla}x\underline{V}) = \underline{\nabla}(\underline{\nabla}\cdot\underline{V}) - \nabla^2\underline{V}$

<u>Other useful identities:</u>

$\underline{\nabla}\cdot(\phi\underline{V}) = \phi\underline{\nabla}\cdot\underline{V} + \underline{V}\cdot\underline{\nabla}\phi$

$\underline{\nabla}x(\phi\underline{V}) = \phi\underline{\nabla}x\underline{V} - \underline{V}x\underline{\nabla}\phi$

$\underline{\nabla}\cdot(\underline{V}x\underline{W}) = (\underline{\nabla}x\underline{V})\cdot\underline{W} - (\underline{\nabla}x\underline{W})\cdot\underline{V}$

$\underline{\nabla}x(\underline{V}x\underline{W}) = (\underline{\nabla}\cdot\underline{W})\underline{V} + (\underline{W}\cdot\underline{\nabla})\underline{V} - (\underline{\nabla}\cdot\underline{V})\underline{W} - (\underline{V}\cdot\underline{\nabla})\underline{W}$

a) Already proved (product rule)

b)

$$\underline{\nabla}x(\phi\underline{V}) = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi V_x & \phi V_y & \phi V_z \end{vmatrix}$$

$$= \hat{\underline{i}}\left(\frac{\partial}{\partial y}\phi V_z - \frac{\partial}{\partial z}\phi V_y\right) + \hat{\underline{j}}(\ldots) + \hat{\underline{k}}(\ldots)$$

$$= \hat{\underline{i}}\left(\frac{\partial\phi}{\partial y}V_z + \phi\frac{\partial V_z}{\partial y} - \frac{\partial\phi}{\partial z}V_y - \phi\frac{\partial V_y}{\partial z}\right) + \hat{\underline{j}}(\ldots) + \hat{\underline{k}}(\ldots)$$

$$= \phi\left(\hat{\underline{i}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{\underline{j}}(\ldots) + \hat{\underline{k}}(\ldots)\right) + \hat{\underline{i}}\left(\frac{\partial\phi}{\partial y}V_z - \frac{\partial\phi}{\partial z}V_y\right) + \hat{\underline{j}}(\ldots) + \hat{\underline{k}}(\ldots)$$

$$= \phi(\underline{\nabla}x\underline{V}) + (\underline{\nabla}\phi)x\underline{V}$$

c)

$$\underline{\nabla}\cdot\left(\underline{V}x\underline{W}\right)=\underline{\nabla}\cdot\begin{vmatrix}\hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}}\\ V_x & V_y & V_z\\ W_x & W_y & W_z\end{vmatrix}$$

$$=\frac{\partial}{\partial x}\left(V_yW_z-V_zW_y\right)-\frac{\partial}{\partial y}\left(V_zW_x-V_xW_z\right)+\left(V_xW_y-V_yW_x\right)$$

$$=\underbrace{\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)W_x+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)W_y+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)W_z}_{V\left(\underline{\nabla}x\underline{V}\right)\cdot\underline{W}}$$

$$+\underbrace{\left(\frac{\partial W_z}{\partial x}-\frac{\partial W_x}{\partial z}\right)V_y+\left(\frac{\partial W_x}{\partial y}-\frac{\partial W_y}{\partial x}\right)V_z+\left(\frac{\partial W_y}{\partial z}-\frac{\partial W_z}{\partial y}\right)V_x}_{-\left(\underline{\nabla}x\underline{W}\right)\cdot\underline{V}}$$

d)

$$\underline{\nabla}x\left(\underline{V}x\underline{W}\right)=\left(\underline{\nabla}\cdot\underline{W}\right)\underline{V}+\left(\underline{W}\cdot\underline{\nabla}\right)\underline{V}-\left(\underline{\nabla}\cdot\underline{V}\right)\underline{W}-\left(\underline{V}\cdot\underline{\nabla}\right)\underline{W}$$

$$\underline{\nabla}x\left(\underline{V}x\underline{W}\right)=\left[\underline{\nabla}\cdot\underline{W}\right]\underline{V}-\left[\underline{\nabla}\cdot\underline{V}\right]\underline{W}$$

[…] means that the del operator also acts on anything to the right. (…) shows that it is entirely separate.

$$\left[\underline{\nabla}\cdot\underline{W}\right]\underline{V}=\left[\frac{\partial}{\partial x}W_x+\frac{\partial}{\partial y}W_y+\frac{\partial}{\partial z}W_z\right]\underline{V}$$

Use $\frac{\partial}{\partial x}\left(W_x\underline{V}\right)=\frac{\partial W_x}{\partial x}\underline{V}+W_x\frac{\partial \underline{V}}{\partial x}$ (Product rule) etc…

$$\left[\underline{\nabla}\cdot\underline{W}\right]\underline{V}=\left(\underline{\nabla}\cdot\underline{W}\right)\underline{V}+\underbrace{\left(W_x\frac{\partial}{\partial x}+W_y\frac{\partial}{\partial y}+W_z\frac{\partial}{\partial z}\right)}_{\underline{W}\cdot\underline{\nabla}}\underline{V}=\left(\underline{\nabla}\cdot\underline{W}\right)\underline{V}+\left(\underline{W}\cdot\underline{\nabla}\right)\underline{V}$$

Similarly $\left[\underline{\nabla}\cdot\underline{V}\right]\underline{W}=\left(\underline{\nabla}\cdot\underline{V}\right)\underline{W}+\left(\underline{V}\cdot\underline{\nabla}\right)\underline{W}$.

Polar Coordinate System

Cylindrical Polars

$$\underline{\nabla}f=\hat{\underline{r}}\frac{\partial f}{\partial x}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\underline{k}}\frac{\partial}{\partial z}$$

$$\underline{\nabla}\cdot\underline{V}=\frac{1}{r}\frac{\partial}{\partial r}\left(rV_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}V_\theta+\frac{\partial}{\partial z}V_z$$

$$\underline{\nabla}x\underline{V}=\frac{1}{r}\begin{vmatrix}\hat{\underline{r}} & \hat{\theta} & \hat{\underline{k}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ V_r & rV_\theta & V_z\end{vmatrix}$$

$$\nabla^2 f=\underline{\nabla}\cdot\left(\underline{\nabla}f\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}$$

Spherical Polars

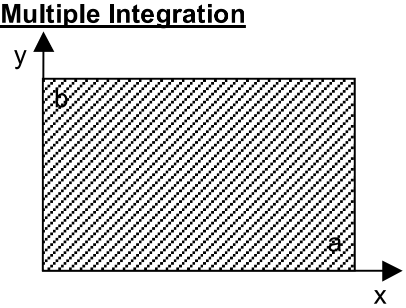
$$\underline{\nabla} f=\hat{\underline{r}}\frac{\partial f}{\partial r}+\hat{\underline{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta}+\hat{\underline{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\underline{\nabla}\cdot\underline{V}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2V_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta V_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

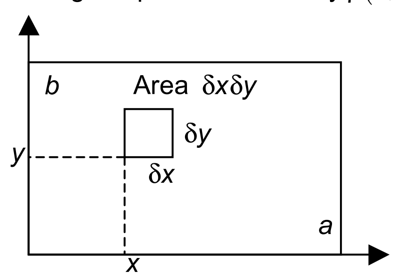
$$\underline{\nabla}x\underline{V}=\frac{1}{r\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ V_r & rV_\theta & r\sin\theta V_\phi\end{vmatrix}$$

$$\nabla^2 f=\underline{\nabla}\cdot\left(\underline{\nabla} f\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

5. Multiple Integration



Rectangular plate with density $\rho(x,y)=Cxy$ $(C=const.)$. What is its' mass?



Consider a small area on the rectangle.

$\delta m=\rho(x,y)\delta x\delta y$

Total mass:

$$M=\lim_{\substack{\delta x\to 0\\ \delta y\to 0}}\sum\rho(x,y)\delta x\delta y$$

Imagine keeping x fixed and summing the rectangles in a column. The mass of the vertical column is given by a 1-D integral.

$$\delta x\int_0^b dy\rho\left(x_i,y\right)=\delta x\int_0^b dyCx_iy=Cx_i\delta x\int_0^b dyy=Cx_i\delta x\left(\frac{b^2}{2}\right)$$

This is the mass of the i^{th} column. Now need to sum over all the columns.

$$M=\lim_{\delta x\to 0}\sum_i Cx_i\delta x\left(\frac{b^2}{2}\right)=C\left(\frac{b^2}{2}\right)\int_0^a dxx=C\left(\frac{b^2}{2}\right)\left(\frac{a^2}{2}\right)=\frac{1}{4}Ca^2b^2$$

We could instead have kept y fixed, and summed over the rows, and then finally summed over y.

The mass of the i^{th} row $= \delta y \int_0^a dx C x y_i = C y_i \delta y \int_0^a dx x = C y_i \delta y \left(\frac{a^2}{2}\right)$

Summing over the rows

$$M = \lim_{\delta y \to 0} \sum_i C y_i \delta y \left(\frac{a^2}{2}\right)$$
$$= \frac{1}{2} a^2 C \int_0^b dy y = \frac{1}{4} C a^2 b^2$$

The mass of the plate is given by the double-integral

$$M = \int_0^a dx \left(\int_0^b dy \rho(x,y) \right) = \int_0^b dy \left(\int_0^a dx \rho(x,y) \right)$$

Usually one writes:

$$M = \int_0^a dx \int_0^b dy \rho(x,y) = \int_0^b dy \int_0^a dx \rho(x,y)$$

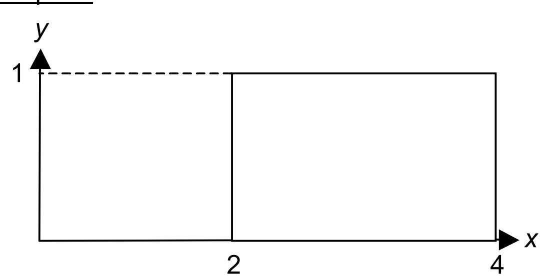
Alternative notations:

$$M = \int_0^a \int_0^b \rho(x,y) dy dx = \int_0^b \int_0^a \rho(x,y) dx dy$$
$$M = \iint_A dx dy \rho(x,y)$$
$$M = \int_A dx dy \rho(x,y)$$

A is a label telling us which region (domain) to integrate over.

5.1 Examples

Example 1



Mass of a rectangular plate in the region:

$2 < x < 4$

$0 < y < 1$

Density $\rho(x,y) = xy + y^2 - 1$

Divide the plate into horizontal strips.

$$M = \iint_A dxdy\rho(x,y)$$
$$= \int_0^1 dy \int_2^4 dx\left(xy + y^2 - 1\right)$$
$$= \int_0^1 dy \left[\frac{x^2y}{2} + xy^2 - x\right]_2^4 = \int_0^1 dy\left(\left(8y + 4y^2 - 4\right) - \left(2y + 2y^2 - 2\right)\right) = \int_0^1 dy\left(6y + 2y^2 - 2\right)$$
$$= \left[3y^2 + \frac{2}{3}y^3 - 2y\right]_0^1 = 3 + \frac{2}{3} - 2 = \frac{5}{3}$$

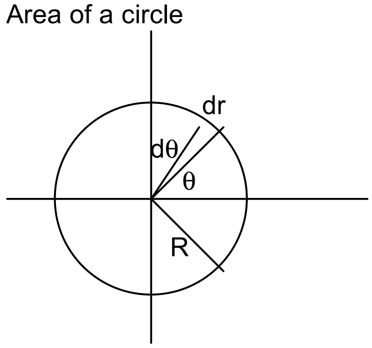
Check this by integrating over the columns first.

$$M = \int_2^4 dx \int_0^1 dy\left(xy + y^2 - 1\right)$$
$$= \int_2^4 dx\left[\frac{xy^2}{2} + \frac{y^3}{3} - y\right]_0^1 = \int_2^4 dx\left(\frac{x}{2} + \frac{1}{3} - 1\right)$$
$$= \left[\frac{x^2}{4} - \frac{2x}{3}\right]_2^4 = \left(4 - \frac{8}{3}\right) - \left(1 - \frac{4}{3}\right) = 3 - \frac{4}{3} = \frac{5}{3}$$

This is the same as before.

<u>Example 2</u>

Area of a circle



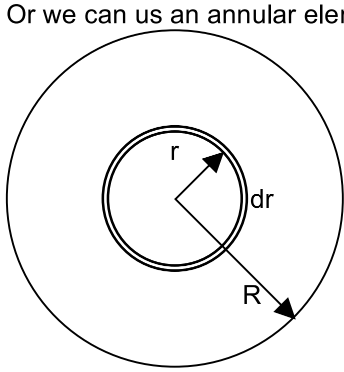
Small area on the end of the lines, area dA.

$dA = rd\theta dr = rdrd\theta$

Total area

$$\int_0^{2\pi} d\theta \int_0^R drr = 2\pi\left[\frac{1}{2}r^2\right]_0^R = \pi R^2$$

Or we can us an annular element (exploiting the circular symmetry)

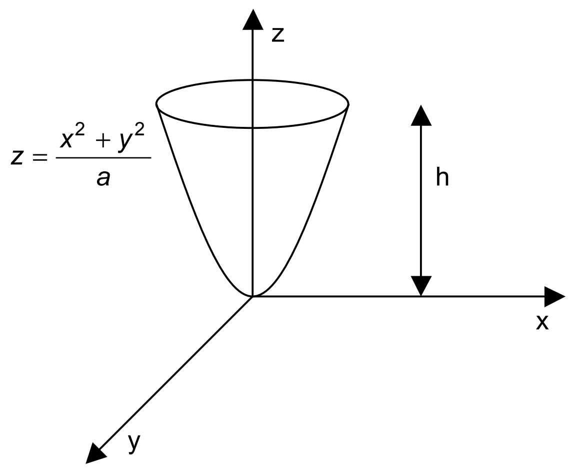


$$dA = 2\pi r dr$$

$$A = \int_0^R 2\pi r dr = \pi r^2$$

Example 3

Calculate the volume of a bowl defined by $z = \dfrac{x^2 + y^2}{a}$ which has depth h.



Take a column in the bowl. For small dx and dy, the curvature of the bowl is very small and can be neglected. Area $= dxdy$

The volume of this elementary column

$$dV = dxdy\left(h - \frac{x^2 + y^2}{a}\right)$$

$$V = \int_A dxdy\left(h - \frac{x^2 + y^2}{a}\right)$$

Use polar coordinates to do the integration.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$dxdy \to rdrd\theta$$

The maximum radius is when $z = h$ into the equation of the bowl.

$$h = \frac{r^2}{a}$$

$$r = \sqrt{ah}$$

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{ah}} rdr\left(h - \frac{r^2}{a}\right)$$

$$= 2\pi\left[\frac{hr^2}{2} - \frac{r^4}{4a}\right]_0^{\sqrt{ha}}$$

$$= 2\pi\left(\frac{h^2 a}{2} - \frac{h^2 a^2}{4a}\right) = \frac{\pi}{2}h^2 a$$

Alternatively the elementary volume could have been a hollow cylinder dropped down from the top surface.

$$dV = \underbrace{2\pi r}_{circumference}\ \underbrace{dr}_{thickness}\ \underbrace{\left(h - \frac{r^2}{a}\right)}_{height}$$

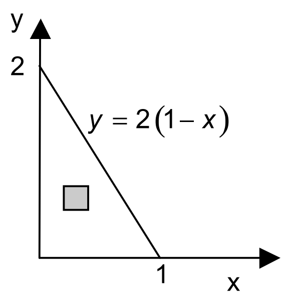
$$V = \int_0^{\sqrt{ha}} dr 2\pi r \left(h - \frac{r^2}{a}\right)$$

as before.

5.2 Integrals over more complicated regions

Example

Calculate the moment of inertia if thus triangular plate about the y-axis:



Block has area $dxdy$.

Suppose the plate has density $\rho(x,y) = 1 + y$

Mass of element $dm = \rho(x,y)dxdy$

Moment of inertia about y-axis:

$$dI = \rho(x,y)x^2 dxdy$$

Total moment of inertia:

$$I = \iint_A dxdy\rho(x,y)x^2$$

Integrating over y first:

$$I = \int_0^1 dx \int_0^{2-2x} dy(1+y)x^2$$

$$= \int_0^1 dx x^2 \left[y + \frac{y^2}{2}\right]_0^{2-2x}$$

$$= \int_0^1 dx x^2 \left(2 - 2x + \frac{1}{2}\left(4 - 8x + 4x^2\right)\right)$$

$$= \int_0^1 dx x^2 \left(4 - 6x + 2x^2\right)$$

$$= \left[\frac{4}{3}x^3 - \frac{3}{2}x^4 + \frac{2}{5}x^2\right]_0^1$$

$$= \frac{4}{3} - \frac{3}{2} + \frac{2}{3} = \frac{7}{30}$$

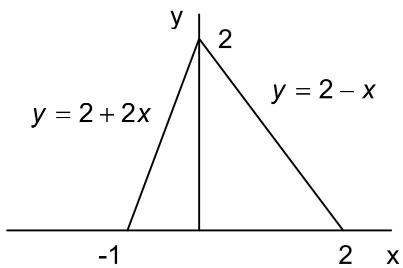
Swapping the order of the integrals:

$$\int_0^2 dy \int_0^{1-\frac{y}{2}} dx\left(1+y\right)x^2 = \int_0^2 dy\left(1+y\right)\left[\frac{1}{3}x^3\right]_0^{1-\frac{y}{2}}$$
$$= \frac{1}{3}\int_0^2 dy\left(1+y\right)\left(1-\frac{y^2}{2}\right)^3$$

After a few steps → 7/30 as before.

Be careful to get the limits right?

e.g. evaluate $\iint_A ydxdy$ where A is the triangle with vertices (-1,0), (0,2), (2,0).



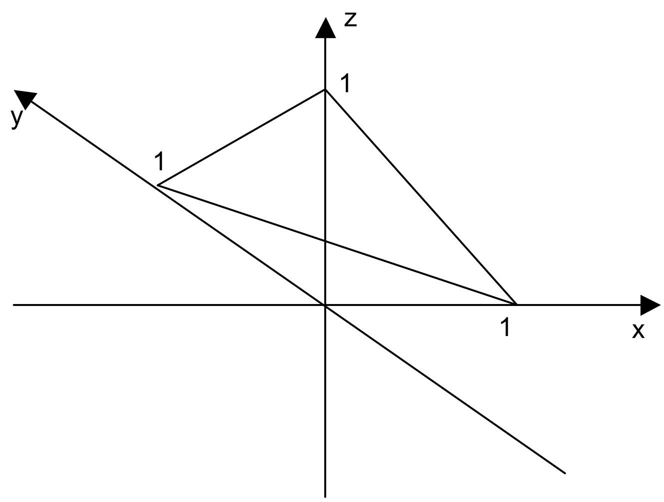
Integrate over x first, then y.

$$\iint_A ydxdy = \int_0^2 dy \int_{\frac{y}{2}-1}^{2-y} dxy$$
$$= \int_0^2 dyy\left[x\right]_{\frac{y}{2}-1}^{2-y} = \int_0^2 dyy\left(\left(2-y\right)-\left(\frac{y}{2}-1\right)\right) = \int_0^2 dy\left(3y-\frac{3y^2}{2}\right)$$
$$= \left[\frac{3y^2}{2}\right]_0^2 - \left[\frac{y^3}{2}\right]_0^2$$
$$= 6 - 4 = \underline{2}$$

Reversing the order, we would have to split the integral into two parts.

A final example:

Find the volume enclosed by the plane $x + y + x = 1$ and the xy, yz and xz planes.



With the surfaces, this forms a tetrahedron with points (0,0,0), (1,0,0), (0,1,0), (0,0,1).

$x > 0, y > 0, z > 0$

$x + y + z < 1$

The height of the column located at (x,y) in the xy plane is $z = 1 - x - y$.

A is the area in the xy plane. The equation of the line here is $y = 1 - x$.

Volume:

$$= \iint_A (1 - x - y)\, dxdy = \int_0^1 dx \int_0^{1-x} dy\,(1 - x - y) = \int_0^1 dx (1 - x)\left(\left[y \right]_0^{1-x} - \left[\frac{y^2}{2} \right]_0^{1-x} \right)$$

$$= \int_0^1 dx \left((1 - x)^2 - \frac{(1 - x)^2}{2} \right) = \frac{1}{2}\int_0^1 dx (1 - x)^2 = \frac{1}{2}\left[-\frac{1}{3}(1 - x)^3 \right]_0^1 = \frac{1}{6}$$

Alternatively, we can do this as a <u>triple integral</u>.

Take an elementary box.

$$\text{Volume} = \iiint_V dxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz = \int_0^1 dx \int_0^{1-x} dy\,(1 - x - y)$$

as before.

Example:

Calculate the moment of inertia of a solid sphere of radius R, with density ρ , about an axis through its centre.

$z = r\cos\theta \qquad 0 \le \theta \le \pi$

$x = r\sin\theta\cos\phi \quad 0 \le \phi \le 2\pi$

$y = r\sin\theta\sin\phi$

Volume element $dV = r^2 \sin\theta\, dr d\theta d\phi$

Moment of inertia about the z-axis:

$$
\begin{aligned}
I &= \rho\int_V r^2 \sin^2\theta dV \\
&= \rho\int_0^R dr\int_0^\pi d\theta\int_0^{2\pi} d\phi r^2 \sin\theta\left(r^2\sin^2\theta\right) \\
&= \rho\underbrace{\int_0^R dr r^4}_{\frac{R^5}{5}}\int_0^\pi d\theta\sin^3\theta\underbrace{\int_0^{2\pi} d\phi}_{2\pi} \\
&= \frac{2\pi}{5}\rho R^5\int_0^\pi d\theta\sin^3\theta \\
&= \frac{2\pi}{5}\rho R^5\int_0^\pi d\theta\sin\theta\left(1-\cos^2\theta\right) \\
&= \frac{2\pi}{5}\rho R^5\left(\left[-\cos\theta\right]_0^\pi + \left[\frac{1}{3}\cos^3\theta\right]_0^\pi\right) \\
&= \frac{2\pi}{5}\rho R^5\left(\left(1+1\right)+\frac{1}{3}\left(-1-1\right)\right) \\
&= \frac{8\pi}{15}\rho R^5 \\
M &= \frac{4}{3}\pi\rho R^3 \\
I &= \frac{2}{5}MR^2
\end{aligned}
$$

6. Flux Integrals and the meaning of Divergence

Flux integrals are;

– Useful in their own right
– Allow us to uncover the physical meaning of divergence.

5.1 Flux Integrals

Example: Heat flow out of a region of space

A closed surface is one which has an inside and an outside, and no link between the two.

$\hat{\underline{n}} =$ Unit outward normal

$\underline{h} =$ Heat current = rate of heat flow per unit time $\left(Jm^{-2}s^{-1}\right)$

How much heat flows across the surface per unit time?
Rate of heat flow (H) across the elementary area is:

$$dH = \left(\underline{h}\cdot\hat{\underline{n}}\right)dA$$

$$H_{total} = \int_{surface}\left(\underline{h}\cdot\hat{\underline{n}}\right)dA$$

This is called the <u>flux</u> of the vector field h out of the surface. It is usually written as;

$\int \underline{h}\cdot\underline{dA}$ (or sometimes $\int \underline{h}\cdot\underline{ds}$)

$\underline{dA}$ is a vector whose magnitude is equal to dA , and whose direction is $\hat{\underline{n}}$.

i.e. $\underline{dA} = \hat{\underline{n}}dA$

We can generalise this to any vector field e.g. $\underline{E}$ (electric field).
Gauss' Law states:

"The total flux of $\underline{E}$ out of a closed surface is equal to the total charge enclosed by that surface times $\frac{1}{\varepsilon_o}$."

i.e. $\int_{surface} \underline{E} \cdot \underline{dA} = \frac{Q}{\varepsilon_o}$.

e.g. a point charge Q at the origin. Imagine it surrounded by a sphere centred on Q.
Elemental area $\underline{dA} = \hat{\underline{r}} dA$. $\underline{E}$ points radially outwards at all points. By symmetry:

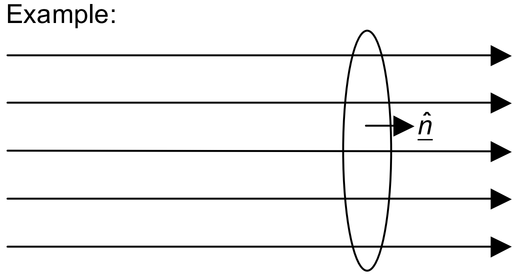
$\underline{E}(r) = E(r)\hat{\underline{r}}$

$$\int_{surface} \underline{E} \cdot \underline{dA} = \int_{surface} E(r)dA = E(r)\int_{surface} dA = E(r)4\pi r^2 = \frac{Q}{\varepsilon_o}$$

Therefore:

$E(r) = \frac{Q}{4\pi\varepsilon_o r^2}$ (Coulomb's Law)

Flux integrals need not be over closed surfaces. [closed surface has an inside and an outside e.g. a ball; an open surface doesn't]
We can calculate the flux of a vector field across an open surface.
Example:

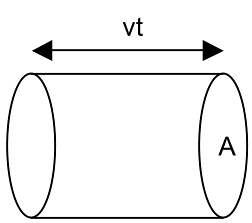


Fluid flow with velocity $\underline{v} = v\hat{\underline{i}}$

So the flux of $\underline{v}$ across the shaded disk $= \int_{disk} \underline{v} \cdot \underline{dA}$

Here, $\underline{dA} = dA\hat{\underline{i}}$. Therefore the flux $= \int_{disk} vdA = vA$ if v is a constant.

In this case, the flux = volume of fluid which crosses the disk in unit time.



All this fluid flows through the disk in time t i.e. a volume. vtA crosses the disk in time t. So vA crosses per unit time.
Generally, the flux of a field $\underline{V}$ through some surface S is defined to be

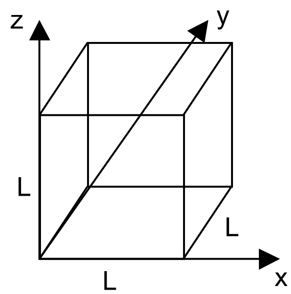
$$Flux = \int_S \underline{V} \cdot \hat{\underline{n}} dA = \int_S \underline{V} \cdot \underline{dA}$$

Note:
We need to specify $\hat{\underline{n}}$ (there is an ambiguity here as $-\hat{\underline{n}}$ is also a normal)
* For a closed surface, $\hat{\underline{n}}$ is the outward-pointing normal
* For an open surface, we need to specify explicitly the direction of $\hat{\underline{n}}$.

Examples of Flux Integrals

1) $\underline{V} = x^2\hat{\underline{i}} + y^2\hat{\underline{j}} + z^2\hat{\underline{k}}$

What is the flux of $\underline{V}$ out of this unit cube?



Do the six phases in turn.

1) Bottom face $(z=0)$:

$\underline{dA} = -\hat{\underline{k}}dxdy$.

$\underline{V}\cdot\underline{dA} = -z^2dxdy\Big|_{z=0} = 0$

2) Top face $(z=l)$

$\underline{dA} = \hat{\underline{k}}dxdy$.

$\underline{V}\cdot\underline{dA} = z^2dxdy\Big|_{z=L} = L^2dxdy$

3) Left face $(x=0)$

$\underline{dA} = -dydz\hat{\underline{i}}$

$\underline{V}\cdot\underline{dA} = -x^2dydz\Big|_{x=0} = 0$

4) Right face $(x=Lz)$

$\underline{dA} = \hat{\underline{i}}dxdy$.

$\underline{V}\cdot\underline{dA} = x^2dydz\Big|_{x=L} = L^2dydz$

5) Front face $(y=0)$

$\underline{dA} = -dxdz\hat{\underline{j}}$

$\underline{V}\cdot\underline{dA} = -y^2dxdz\Big|_{y=0} = 0$

6) Back face $(y=L)$

$\underline{dA} = dxdz\hat{\underline{j}}$

$\underline{V}\cdot\underline{dA} = y^2dxdz\Big|_{y=L} = L^2dxdz$

The total flux $= \int_{surface} \underline{V}\cdot\underline{dA} = \int_{top} dxdyL^2 + \int_{back} dydzL^2 + \int_{back} dxdzL^2 = 3L^4$

$$\int_S \underline{V}\cdot\underline{dA} = \int_V \underline{\nabla}\cdot\underline{V}dV = 6\int_0^L dx\int_0^L dy\int_0^L dz\left(x+y+z\right) = 3L^4 .$$

The flux of a field $\underline{V}$ through some surface S;

$$\int_S \underline{V}\cdot\underline{\hat{n}}dA = \int_S \underline{V}\cdot\underline{dA}$$

Example 2: Calculate the flux of $\underline{V} = z^3\underline{\hat{k}}$ out of a sphere of radius R centred on the origin.

$$\underline{dA} = \underline{\hat{r}}\underbrace{R^2 \overbrace{\sin\theta d\theta d\phi}^{d\Omega}}_{area}$$

$d\Omega$ = element of "solid angle"

$$\frac{dA}{4\pi r^2} = \frac{d\Omega}{4\pi}$$

There are 4π "steradians" in a spherical shell.

$$\underline{V}\cdot\underline{dA} = \underbrace{\underline{\hat{k}}\cdot\underline{\hat{r}}}_{\cos\phi} z^3 R^2 \sin\theta d\theta d\phi$$

Also $z = R\cos\theta$ (on the surface)

$$\underline{V}\cdot\underline{dA} = R^5\cos^4\theta\sin\theta d\theta d\phi$$

Integrate over the whole surface S, $0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$.

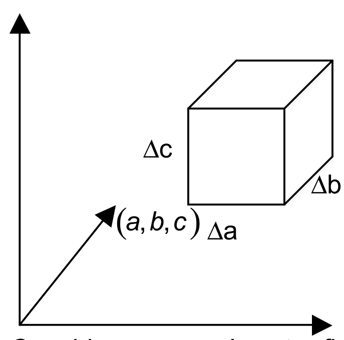
→ flux:

$$= \int_{surface} \underline{v}\cdot\underline{dA} = R^5\int_0^{2\pi} d\phi \underbrace{\int_0^{\pi} d\theta\cos^4\theta\sin\theta}_{=\left[\frac{1}{5}\cos^5\theta\right]_0^{\pi} = \frac{2}{5}}$$

$$= \frac{2}{5}x2\pi x R^5 = \frac{4}{5}\pi R^5$$

5.2 The Physical Meaning of Divergence:

Consider an infinitesimal box with one corner at (a,b,c) and sides $\Delta a, \Delta b, \Delta c$.



Consider a general vector field:

$$\underline{V}(x,y,z) = V_x(x,y,z)\underline{\hat{i}} + V_y(x,y,z)\underline{\hat{j}} + V_z(x,y,z)\underline{\hat{k}}$$

What is the flux of $\underline{V}$ out of the box?

Calculate the flux across each face. Look at the top face and the bottom face.

Top face $(z = c + \Delta c)$

$$\underline{dA} = dxdy\underline{\hat{k}}$$

$$\underline{V}\cdot\underline{dA} = V_z(x,y,c+\Delta c)dxdy$$

Total flux out of the top face:

$$= \int_a^{a+\Delta a} dx \int_b^{b+\Delta b} dy V_z\left(x, y, c+\Delta c\right) = \Delta a \Delta b V_z\left(a, b, c+\Delta c\right)$$

Aside: $\int_a^{a+\Delta a} f\left(x\right) dx = \Delta a f\left(a\right) + \text{terms of} \left(\Delta a\right)^2 \approx \Delta a f\left(a\right)$.

Formal argument:

$x = a + y$

Integral

$$= \int_0^{\Delta a} dy f\left(a+y\right) = \int_0^{\Delta a} dy \left(f\left(a\right) + y f'\left(a\right) + \frac{y^2 f''\left(a\right)}{2} \right) = \Delta a f\left(a\right) + \frac{\left(\Delta a\right)^2}{2} f'\left(a\right) + \frac{1}{6}\left(\Delta a^3\right) f''\left(a\right) + \ldots$$

Bottom face $\left(z = c\right)$:

$\underline{dA} = -\underline{\hat{k}} dxdy$

$\underline{V} \cdot \underline{dA} = -dxdy V_z\left(x, y, c\right)$

Therefore the flux out of the bottom face:

$$= -\int_a^{a+\Delta a} dx \int_b^{b+\Delta b} dy V_z\left(x, y, c\right) = -\Delta a \Delta b V_z\left(a, b, c\right)$$

Combine top and bottom faces:

Flux (top + bottom)

$$= \Delta a \Delta b \Big(V_z\left(a, b, c+\Delta c\right) - V_z\left(a, b, c\right) \Big)$$

$$= \Delta a \Delta b \Delta c \left(\frac{V_z\left(a, b, c+\Delta c\right) - V_z\left(a, b, c\right)}{\Delta c} \right)$$

Take $\Delta c \to 0$

$$= \Delta a \Delta b \Delta c \left. \frac{\partial V_z}{\partial z} \right|_{\substack{x=a \\ y=b \\ z=c}}$$

Similarly:

Flux (left + right) $= \Delta a \Delta b \Delta c \left. \frac{\partial V_x}{\partial x} \right|_{\substack{x=a \\ y=b \\ z=c}}$

Flux (front + back) $= \Delta a \Delta b \Delta c \left. \frac{\partial V_y}{\partial y} \right|_{\substack{x=a \\ y=b \\ z=c}}$

Therefore the total flux out of the box:

$$= \Delta a \Delta b \Delta c \left. \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) \right|_{\substack{x=a \\ y=b \\ z=c}}$$

$$= \Delta a \Delta b \Delta c \, \underline{\nabla} \cdot \underline{V} \Big|_{\substack{x=a \\ y=b \\ z=c}}$$

$$\underline{\nabla} \cdot \underline{V} = \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \int_{surface(\delta\tau)} \underline{V} \cdot \underline{dA}$$

where $\delta\tau = dxdydz =$ volume of a box.

The divergence of a vector field at a point P is equal to the flux “outflow” per unit volume at the point P.

5.2 “Continuity Equation”

Take a compressible fluid, e.g. a gas, Consider a volume element $d\tau = dxdydz$. Calculate the rate of change of mass of the gas in $d\tau$ in two ways.

Method 1:

Mass of gas in the box $dm = \rho d\tau$ [here $\rho = \rho(x, y, z, t)$]

So $\frac{dm}{dt} = \frac{d\rho}{dt} d\tau$ (1)

Method 2:

Mass current (density) (= mass crossing unit area per unit time)

$\underline{j} = \rho \underline{v} = \frac{1}{A} \frac{dm}{dt}$

The mass flowing out of the element per unit time:

$$\int_{surface\tau} \underline{j} \cdot \underline{dA} = -\frac{dm}{dt}$$

$$\Rightarrow \frac{dm}{dt} = -div\, \underline{j} d\tau \ (2)$$

Equating (1) and (2)

$$\frac{d\rho}{dt} d\tau = -\underline{\nabla} \cdot \underline{j} d\tau$$

$$\boxed{\frac{d\rho}{dt} + \underline{\nabla} \cdot \underline{j} = 0}$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0}$$

The continuity equation expresses the fact that some quantity is conserved (in this case mass) is conserved.
Could be energy, heat, electric charge, particle number, ...
e.g. electric charge

$\rho =$ Charge density $\left(cm^{-1}\right)$

$\underline{j} =$ Current density $\left(Cm^{-2}s^{-1}\right)$

Special case: incompressible fluid

$\rho = co\, nst.$

$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \underline{\nabla} \cdot \underline{v} = 0$

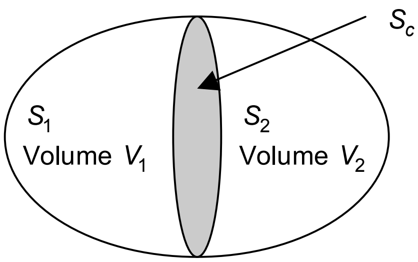
Note:

$\underline{\nabla} \cdot \underline{v} > 0$ at some point p net “outflow” of $\underline{v}$ at p.

$\underline{\nabla} \cdot \underline{v} < 0$ net “inflow”

7. The Divergence Theorem

Consider a closed surface S, and divide into 2 parts S_1 and S_2.



Total outflow / flux of $\underline{v}$ out of S

$$Flux = \int_{S_1+S_2} \underline{v}\cdot\underline{dA} = \int_{S_1} \underline{v}\cdot\underline{dA} + \int_{S_2} \underline{v}\cdot\underline{dA}$$

Define the direction of the normal to S_c to be <u>out of</u> V_1 and <u>into</u> $\underline{V_2}$.

→ flux out of the total volume $\left(V_1+V_2\right)$ is

$$\int_{S_1+S_2} \underline{v}\cdot\underline{dA} = \int_{S_1} \underline{v}\cdot\underline{dA} + \int_{S_c} \underline{v}\cdot\underline{dA} + \int_{S_2} \underline{v}\cdot\underline{dA} - \int_{S_c} \underline{v}\cdot\underline{dA} = \int_{S_1+S_c} \underline{v}\cdot\underline{dA} + \int_{S_1+S_c} \underline{v}\cdot\underline{dA}$$

where now $\underline{dA}$ is the outward normal everywhere.

Summery:

Flux out of V_1 and V_2 = flux out of V_1 + flux out of V_2.

(Flux across internal surface cancels)

Take a region of volume V, and split it up into elementary boxes.

Flux out of volume $= \int_S \underline{v}\cdot\underline{dA} = \sum_i \int_{S_i} \underline{v}\cdot\underline{dA}$ (fluxes across internal surfaces cancel)

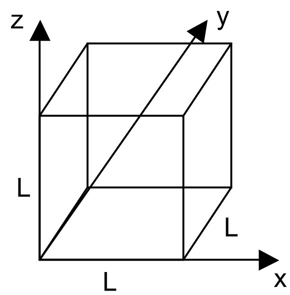
$$= \sum_i \left(\underline{\nabla}\cdot\underline{v}\right)_i dV_i$$

In the limit when all $dV_i \to 0$, replace sum b integral.

$$\int_S \underline{v}\cdot\underline{dA} = \int_V \underline{\nabla}\cdot\underline{v}dV$$

This is the divergence theorem.

<u>Example (i):</u>



Earlier we worked out the flux of $\underline{v} = x^2\hat{\underline{i}} + y^2\hat{\underline{j}} + z^2\hat{\underline{k}}$

Out of the cube recall $\int \underline{v} \cdot \underline{dA} = 3l^4$

Let's repeat this using the divergence theorem.

$$Flux = \int_S \underline{v} \cdot \underline{da} = \int_V \left(\underline{\nabla} \cdot \underline{v}\right) dV$$

$$\underline{\nabla} \cdot \underline{V} = 2\left(x + y + z\right)$$

$$\Rightarrow Flux = 2\int_0^l dx \int_0^l dy \int_0^l dz \left(x + y + z\right) = 6\int_0^l xdx \int_0^l dy \int_0^l dz = 6\left[\frac{x^2}{2}\right]_0^l \left[y\right]_0^l \left[z\right]_0^l = 3l^4$$

Example (ii)

We also calculated the flux of $\underline{v} = z^3\hat{\underline{k}}$ out of a sphere of radius R centred on the origin. We got $Flux = \int_{sphere} \underline{v} \cdot \underline{dA} = \frac{4\pi R^5}{5}$

Let's check this using the Divergence theorem.

$$Flux = \int_S \underline{v} \cdot \underline{da} = \int_V \left(\underline{\nabla} \cdot \underline{v}\right) dV$$

$$\underline{\nabla} \cdot \underline{v} = 3k^2$$

$$\Rightarrow Flux = \int_V 2z^2 dV$$

$$z = r\cos\theta$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

$$Flux = 3\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi r^2 \sin\theta \underbrace{\left(r^2\cos^2\theta\right)}_{z}$$

$$= 3\int_0^R dr r^4 \int_0^\pi d\theta \sin\theta \cos^2\theta \int_0^{2\pi} d\phi$$

$$= \frac{R^5}{5}\left[-\frac{1}{3}\cos^3\theta\right]_0^\pi 2\pi = \frac{4\pi R^5}{5}$$

as before.

Alternatively we can exploit the spherical symmetry:

$$\int_{\text{interior}} 3z^3 dV = \int_{\text{interior}} \left(x^2+y^2+z^2\right) dV = \int_{\text{interior}} r^2 dV = \int_0^R dr 4\pi r^2 r^2 = \frac{4}{5}\pi R^5$$

$$\left(\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 2\pi\left[-\cos\theta\right]_0^{\pi} = 4\pi\right)$$

Example (iii)

Verify the divergence theorem for $\underline{F} = r^2\underline{\hat{r}}$ (spherical polar coordinates) for a sphere of radius R centred on the origin.

Divergence theorem $\int_S \underline{F}\cdot d A = \int_V \left(\underline{\nabla}\cdot\underline{F}\right)\cdot dV$

Left side:

$$\underline{dA} = \underline{\hat{r}} dA \ \left(dA = R^2 \sin\theta d\theta d\phi\right)$$

$$\underline{F}\cdot\underline{dA}\Big|_{surface} = R^2 dA$$

$$\int_S \underline{F}\cdot\underline{dA} = R^2 \int_S dA - R^2 4\pi R^2 = 4\pi R^4$$

Right side:

$$\underline{\nabla}\cdot\underline{F} = \underline{\nabla}\cdot\left(r^2\underline{\hat{r}}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right)\left[F_\theta = 0, F_\phi = 0\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4\right) = 4r$$

[Alternative method: $\underline{\nabla}\cdot\left(r^2\underline{\hat{r}}\right) = \underline{\nabla}\left(r\underline{r}\right) = r\underline{\nabla}\cdot\underline{r} + \underline{r}\underline{\nabla}r = 3r + r = 4r$]

$$\int_V \left(\underline{\nabla}\cdot\underline{F}\right) dV = \int_0^R dr 4\pi r^2 .4r = 16\pi\frac{R^4}{4} = 4\pi R^4$$

Applications of Electromagnetism

Maxwell 1:

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_o}$$

$\underline{E}$ – Electric Field

ρ - charge density

Integrate both sides over the same volume V

$$\int_V \underline{\nabla}\cdot\underline{E} dV = \frac{1}{\varepsilon_o}\underbrace{\int_V \rho dV}_{Q}$$

$$\int_S \underline{E}\cdot\underline{dA} = \frac{Q}{\varepsilon_o}$$

This is Gauss' law

Maxwell 2:

$\underline{\nabla} \cdot \underline{B} = 0$

$\int_V \underline{\nabla} \cdot \underline{B} dV = 0$

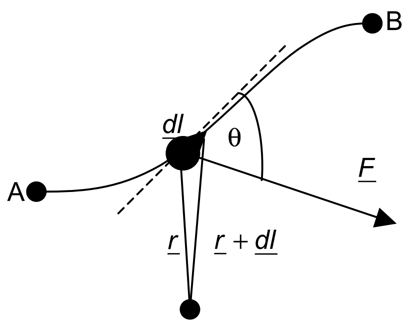
$\int_S \underline{B} \cdot \underline{dA} = 0$

→ flux of B on A of any volume V is zero
("what goes in must come out")

8. Line Integrals & the Meaning Of Curl

Line integrals are:
– useful in their own right
– Allow us to uncover the meaning of curl

Example:
Work done by a force acting on a particle moving along a particular path.



$\underline{dl}$ is the element of the path. The direction of $\underline{dl}$ gives the direction of the tangent of the path.

The magnitude of $\underline{dl}$ gives the length of the element.

The work done on the particle by the force F in moving it from $\underline{r}$ to $\underline{r} + \underline{dl}$ is:

$$dW = |\underline{F}| \cos\theta \underline{dl}$$
$$= \underline{F} \cdot \underline{dl}$$

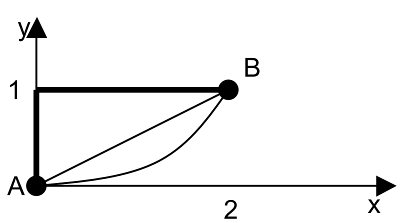
The total work done by $\underline{F}$ moving the particle from A to B

$$W = \int_A^B \underline{F} \cdot \underline{dl}$$

This is the definition of a line integral.
Note: the work done may depend on the path taken. If it does not depend on the force F then it is said to be a "conservative force".
Example: Given the force field:

$$\underline{F} = xy\hat{\underline{i}} - y^2\hat{\underline{j}}$$

find the work done by F for $A = (0,0)$, $B = (2,1)$ for the following paths:



Path 1: $y = \frac{1}{2}x$

Path 2: $y=\frac{1}{4}x^2$

Path 3: $x=0$ for $0\le y\le 1$. $y=1$ for $0\le x\le 2$

In each case, $\underline{dl}=dx\hat{\underline{i}}+dy\hat{\underline{j}}$. There is no $dz\hat{\underline{k}}$ term here because z is a constant for these three paths.

For path 1;

$$W=\int_{(0,0)}^{(2,1)}\left(xy\hat{\underline{i}}-y^2\hat{\underline{j}}\right)\cdot\left(dx\hat{\underline{i}}+dy\hat{\underline{j}}\right)$$

$$=\int_?^? xydx-\int_?^? y^2dy$$

There are many ways to proceed. Along the path x and y are related i.e. y is a function of x or vice-versa. For path (1), $y=\frac{1}{2}x$ so $dy=\frac{1}{2}dx$. So we can integrate over x.

$$W=\int_0^2 x\left(\frac{1}{2}x\right)dx-\int_0^2\left(\frac{1}{2}x\right)^2\frac{1}{2}dx$$

$$=\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2-\frac{1}{8}\left[\frac{x^3}{3}\right]_0^2=\frac{1}{2}\frac{8}{3}-\frac{1}{8}\frac{8}{3}=1$$

Alternatively we can write $x=2y$, $dx=2dy$, and integrate over y.

$$W=\int_0^1(2y)y(2dy)-\int_0^1 y^2dy=4\left[\frac{y^3}{3}\right]_0^1-\left[\frac{y^3}{3}\right]_0^1=\frac{4}{3}-\frac{1}{3}=1$$

Finally we could do the first integral over x, and the second over y.

$$W=\int_0^2 x\left(\frac{1}{2}x\right)dx-\int_0^1 y^2dy=\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2-\left[\frac{y^3}{3}\right]_0^1=\frac{4}{3}-\frac{1}{3}=1$$

→ Chose the most convenient integration variable.

For path 2:

$$y=\frac{1}{4}x^2$$

$$\underline{F}\cdot\underline{dl}=\left(xy\hat{\underline{i}}-y^2\hat{\underline{j}}\right)\cdot\left(\hat{\underline{i}}dx+\hat{\underline{j}}dy\right)$$

$$=xy-y^2$$

$$\int_{path2}\underline{F}\cdot\underline{dl}=\int xydx-\int y^2dy$$

The easiest way is $\int\underline{F}\cdot\underline{dl}=\int_0^2 x\left(\frac{1}{4}x^2\right)dx-\int_0^1 y^2dy=\frac{1}{4}\left[\frac{x^4}{4}\right]_0^2-\left[\frac{y^3}{3}\right]_0^1=1-\frac{1}{3}=\frac{2}{3}$

This is different for path 1.

For path 3:

Path has two parts: $(0,0)\to(0,1)$, $(0,1)\to(2,1)$

For $(0,0)\to(0,1)$, $dx=0$.

$$\int_{(0,0)}^{(0,1)}\underline{F}\cdot\underline{dl}=-\int_0^1 y^2dy=-\frac{1}{3}$$

For $(0,1) \to (2,1)$, $y = 1$ and $dy = 0$.

$$\int_{(0,1)}^{(2,1)} \underline{F} \cdot \underline{dl} = \int_0^2 x dx = \left[\frac{x^2}{2} \right]_0^2 = 2$$

Total integral is $2 - \frac{1}{3} = \frac{5}{3}$. (again different)

We can also ask how much work is done by F in moving a particle along the closed path $(0,0) \to (2,1)$ via path 1 then $(2,1) \to (0,0)$ via path 2. Since we have reversed the direction of path 2, the work for this part changes sign since $\underline{dl} \to -\underline{dl}$ everywhere.

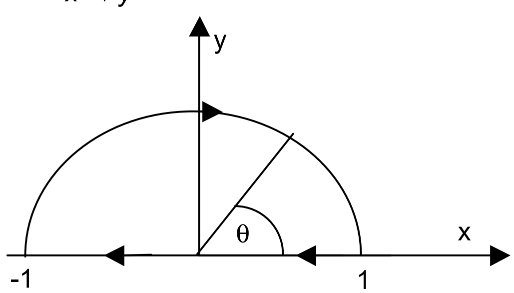
Net work $= 1 - \frac{2}{3} = \frac{1}{3}$.

<u>Circulation:</u>

Line integrals around closed paths have a special name, “circulation”, and are usually written $\oint_C \underline{F} \cdot \underline{dl}$ (C tells us the name of the path. $\oint$ denotes the <u>closed</u> path.)

Example:

$$\underline{F} = -\frac{y\underline{\hat{i}} + x\underline{\hat{j}}}{x^2 + y^2}$$



What is the circulation of $\underline{F}$ around path shown?

Do the integral along the base first:

$$\int_{x-axis} \underline{F} \cdot \underline{dl} = \int_{x-axis} -\frac{ydx + xdy}{x^2 + y^2} = 0 \text{ because } y = dy = 0 \text{ along this path.}$$

Do the semi-circle using polar coordinates:

$x = \cos\theta \quad dx = -\sin\theta d\theta$

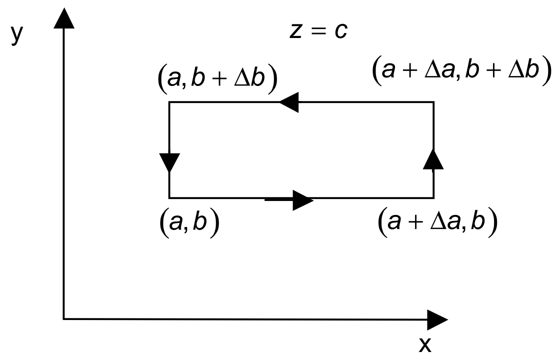
$y = \sin\theta \quad dy = \cos\theta d\theta$

Therefore:

$$\int_{semi-circle} \underline{F} \cdot \underline{dl} = \int \frac{-\sin\theta(-\sin\theta d\theta) + \cos\theta(\cos\theta d\theta)}{\cos^2\theta + \sin^2\theta} = \int_\pi^0 d\theta = -\pi$$

<u>The meaning of curl</u>

Calculate the circulation of some vector field $\underline{V}$ around a small rectangle lying in the plane $z = c$.



Circulation is $\oint_{rectangle} \underline{V} \cdot \underline{dl} = \oint_{rectangle} \left[V_x(x,y,c)\,dx + V_y(x,y,c)\,dy + \cancel{V_z(x,y,c)\,dz} \right]$

$(dz = 0)$

Bottom edge: $y = b, dy = 0$

$$\int_a^{a+\Delta a} V_x(x,b,c)\,dx = \Delta a V_x(a,b,c) + \cancel{0(\Delta a)^2}$$

Top edge: $y = b + \Delta b, dy = 0$

$$\int_{a+\Delta a}^{a} V_x(x,b+\Delta b,c) = -\Delta a V_x(a,b+\Delta b,c) + \cancel{0(\Delta a)^2}$$

Combining top and bottom edges gives:

$$\Delta a\left(V_x(a,b,c) - V_x(a,b+\Delta b,c)\right)$$

$$-\Delta a \Delta b \left(\frac{V_x(a,b+\Delta b,c) - V_x(a,b,c)}{\Delta b} \right)$$

$$-\lim_{\Delta b \to 0} \Delta a \Delta b \left.\frac{\partial V_x}{\partial y}\right|_{\substack{x=a\\y=b\\z=c}}$$

Left edge: $x = a, dx = 0$

$$\int_{b+\Delta b}^{b} dy V_y(a,y,c) = -\Delta b V_y(a,b,c) + O\left((\Delta b)^2\right)$$

Right edge: $(x = a + \Delta a, dx = 0)$

$$\int_b^{b+\Delta b} dy V_{y(a+\Delta a,y,c)} = \Delta b V_y(a+\Delta a,b,c) + O\left((\Delta b)^2\right)$$

(O = of order of)

Combining left and right edges gives:

$$\Delta a \Delta b \left(\frac{V_y(a+\Delta a,b,c) - V_y(a,b,c)}{\Delta a} \right) = \Delta a \Delta b \left.\frac{\partial V_y}{\partial x}\right|_{\substack{x=a\\y=b\\z=c}}$$

Total circulation (Top + Bottom + Left + Right)

$$\oint \underline{V} \cdot \underline{dl} = -\Delta a \Delta b \left.\frac{\partial V_x}{\partial y}\right|_{\substack{x=a \\ y=b \\ z=c}} + \Delta a \Delta b \left.\frac{\partial V_y}{\partial x}\right|_{\substack{x=a \\ y=b \\ x=c}}$$

$$= \Delta a \Delta b \left.\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)\right|_{\substack{x=a \\ y=b \\ z=c}} = \Delta a \Delta b \left(\underline{\nabla} x \underline{V} \right)_z$$

This is the area of the loop times $\left(\underline{\nabla} x \underline{V}\right)_z = \left(\underline{\nabla} x \underline{V}\right)_z \cdot \underline{\hat{k}}$.

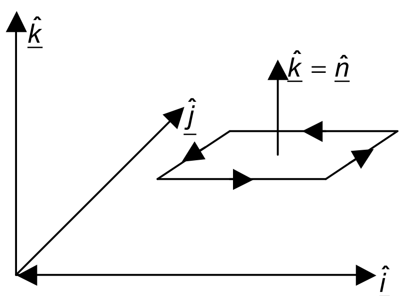
Aside: $\underline{\nabla} x \underline{V} = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$

$\underline{k}$ component (z component) $= \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$

Here the "z component" becomes in general the component in the direction normal to the plane of the loop. Let this normal be $\underline{\hat{n}}$ in general.
The circulation around any infinitesimal loop is:

$$\oint \underline{V} \cdot \underline{dl} = \left(\underline{\nabla} x \underline{V}\right) \cdot \underline{\hat{n}} dA$$

As usual, there is potentially an ambiguity in the direction of $\underline{\hat{n}}$ ($-\underline{\hat{n}}$ is also a normal). Recall the loop in the $z = c$ plane.



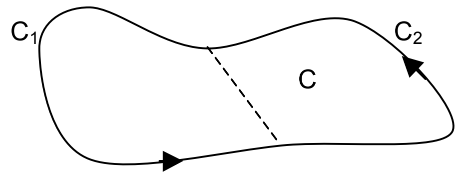
We determine the direction of $\underline{\hat{n}}$ using the right-hand screw rule.
Final result:

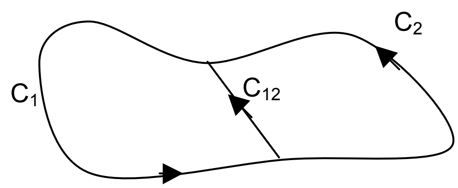
$$\left(\underline{\nabla} x \underline{V}\right) \cdot \underline{\hat{n}} = \lim_{dA \to 0} \frac{1}{dA} \oint \underline{V} \cdot \underline{dl}$$

The component of the curl of $\underline{V}$ in some direction $\underline{\hat{n}}$ is equal to the circulation per unit area of $\underline{V}$ around a loop to which $\underline{\hat{n}}$ is the unit normal.

9. Stoke's Theorem

Consider any closed loop C.





Circulation of F around C

$$\oint_C \underline{F}\cdot\underline{dl} = \int_{C_1} \underline{F}\cdot\underline{dl} + \int_{C_2} \underline{F}\cdot\underline{dl}$$

$$= \left(\int_{C_1} \underline{F}\cdot\underline{dl} + \int_{C_{12}} \underline{F}\cdot\underline{dl} \right) + \left(\int_{C_2} \underline{F}\cdot\underline{dl} - \underbrace{\int_{C_{12}} \underline{F}\cdot\underline{dl}}_{=+\int_{\overline{C_1}} \underline{F}\cdot\underline{dl}} \right)$$

$$= \oint_{C_1+C_{12}} \underline{F}\cdot\underline{dl} + \oint_{C_2+\overline{C_{12}}} \underline{F}\cdot\underline{dl}$$

$\overline{C_{12}}$ is the current in the opposite direction along the line C_{12}

Circulation around an arbitrary shape is equal to the circulation around the two halves of the arbitrary shape added together.

Circulation is an additive property, just like Flux. Any loop can be divided into sub-loops and the circulations add.

Now consider a surface S (needn't be flat) bounded by a contour C (needn't be planar). Divide the surface into an infinite number of infinitesimal rectangles. (Circulation anti-clockwise around each rectangle, indicated by arrows).

$$\oint_C \underline{V}\cdot\underline{dl} = \sum_i \oint_{loop\ i} \underline{v}\cdot\underline{dl} .$$

But $\oint_{loop\ i} \underline{V}\cdot\underline{dl} = \left(\underline{\nabla} x \underline{V}\right)_i \cdot \underline{\hat{n}}_i .dA_i$.

Therefore:

$$\oint_C \underline{V}\cdot\underline{dl} = \sum_i \left(\underline{\nabla} x \underline{V}\right)_i \cdot \underline{\hat{n}} dA_i$$

$$= \int_S \left(\underline{\nabla} x \underline{V}\right)\cdot \underline{\hat{n}} dA$$

Or more compact:

$$\oint_C \underline{V}\cdot\underline{dl} = \int_S \left(\underline{\nabla} x \underline{V}\right)\cdot \underline{dA}$$

This is Stoke's Theorem.

"Circulation of V around C is equal to the flux of $\underline{\nabla} x \underline{V}$ through S".

Example:

Compute the circulation of $\underline{F} = -3y\hat{\underline{i}}$ around the circle $x^2 + y^2 = a^2$, $z = 0$.

If no direction is specified assume anticlockwise.

By Stoke's;

$$\oint_C \underline{F} \cdot \underline{dl} = \int_S \left(\underline{\nabla} x \underline{F}\right) \cdot \underline{dA}$$

$$\underline{\nabla} x \underline{F} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y & 0 & 0 \end{vmatrix} = 3\hat{\underline{k}}$$

$$\oint_C \underline{F} \cdot \underline{dl} = 2\int_S \hat{\underline{k}} \cdot \underline{dA}$$

The most convenient surface is just the interior of the circle. Then $\underline{dA} = \hat{\underline{k}} dA$

$$= 3\int_S dA = 3\pi a^2$$

Alternatively, we could choose the hemispherical shell $x^2 + y^2 + z^2 \leq a^2$, z positive.

$$\underline{dA} = a^2 \sin\theta d\theta d\phi \hat{\underline{r}}$$

$$\hat{\underline{k}} \cdot \underline{dA} = a^2 \sin\theta d\theta d\phi \underbrace{\hat{\underline{k}} \cdot \hat{\underline{r}}}_{=\cos\theta} = a^2 \sin\theta \cos\theta d\theta d\phi$$

$$\oint_C \underline{F} \cdot \underline{dl} = 3\int_S \hat{\underline{k}} \cdot \underline{dA} = 3a^2 \underbrace{\int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta d\theta}_{\frac{1}{2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} = 3\pi a^2$$

Check:

Direct evaluation of the line integral:

$$\oint_C \underline{F} \cdot \underline{dl} = -3\oint_C y\hat{\underline{i}} \cdot \underline{dl} = -3\int \ ydx$$

$\int \ ydx$ is the area under the curve $= \underbrace{-\frac{1}{2}\pi a^2}_{top} \underbrace{-\frac{1}{2}\pi a^2}_{bottom}$ (Minus for the first half as $y > 0, dx < 0$, for the second half as $y < 0, dx > 0$)

$$\oint_C \underline{F} \cdot \underline{dl} = -3\left(-\frac{2}{2}\pi a^2\right) = 3\pi a^2 \text{ (as before)}$$

Exam paper:

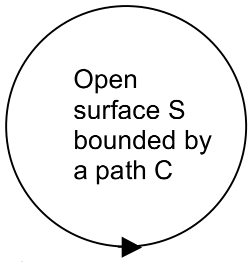
3 questions, no choice. Need flux integrals and divergence theorem, line integrals and Stoke's theorem.

Stoke's Theorum in Electromagnetism

Maxwell 4:

$$\underline{\nabla} x \underline{B} = \mu_o \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \text{ (1)}$$

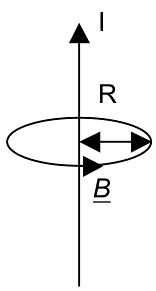
The last part of this equation is discarded of there is no time dependence ("magnetostatics"). Calculate the flux of both sides of equation (1) through surface S (flux out of the paper)



$$\underbrace{\int_S \left(\underline{\nabla} x \underline{B}\right)\cdot \underline{dA}}_{=\oint_C \underline{B}\cdot\underline{dl}} = \mu_o \underbrace{\int_S \underline{J}\cdot\underline{dA}}_{=I,\ \text{total current through loop } C}$$

$$\Rightarrow \oint_C \underline{B}\cdot\underline{dl} = \mu_o I_{enclosed}$$

This is Ampere's Law.
e.g. for an infinite straight wire:

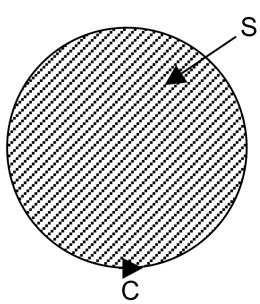


$$\oint_C \underline{B}\cdot\underline{dl} = \underline{B}.2\pi r = I\mu_o$$

$$B = \frac{\mu_o I}{2\pi r}$$

Maxwell 3:

$$\underline{\nabla} x \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$



Calculate the flux of both sides through S.

$$\int_S \left(\underline{\nabla} x \underline{E}\right)\cdot\underline{dA} = -\frac{d}{dt}\left(\int \underline{B}\cdot\underline{dA}\right)$$

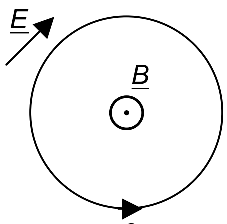
$$\oint_C \underline{E}\cdot\underline{dl} = -\frac{d\Phi}{dt}$$

This is Faraday's Law.

$$E = -\underline{\nabla}V - \frac{\partial \underline{A}}{\partial t}$$

The first part of this equation is for electrostatics; the last part is for time dependence.

$$\oint \underline{E} \cdot \underline{dl} = -A\frac{d\underline{B}}{dt}$$



Creates a clockwise current around the wire.

$$I = \frac{\oint \underline{E} \cdot \underline{dl}}{R}\ \frac{A}{R}\frac{dB}{dt}$$

10. Conservative Fields, Potentials and Exact Differentials

In deriving Stoke's theorem we implicitly assumed that the vector field is "smooth" (Continuous, differentiable, continuous first differential) everywhere we needed, i.e. on the path C and the surface S.

If the field is smooth in domain D, we say that D is "simply connected" if any closed surface in D can be shrunk to a point without leaving D. If the surface has a whole in it, then the field around that point can't be shrunk without leaving D, therefore it is not simply connected but "multiply connected".

If $\underline{\nabla} x \underline{F} = 0$ in some domain D <u>and</u> D is simply connected, then $\oint_C \underline{F} \cdot \underline{dl} = 0$ for <u>any</u> closed path in D (by Stoke's theorem).

→ integral between any two points is independent of any path in D.

Proof: for any points $\underset{(1)}{\int_A^B \underline{F} \cdot \underline{dl}} + \underset{(2)}{\int_B^A \underline{F} \cdot \underline{dl}} = 0$

$$\Rightarrow \underset{(1)}{\int_A^B \underline{F} \cdot \underline{dl}} - \underset{(2)}{\int_A^B \underline{F} \cdot \underline{dl}} = 0 \text{ QED}$$

$\underline{\nabla} x \underline{F} = 0$ in D

$\Rightarrow \oint_C \underline{F} \cdot \underline{dl} = 0$ for any closed path in D.

$\underline{F}$ is said to be a <u>conservative field</u> and $\underline{F} \cdot \underline{dl}$ is said to be an <u>exact differential</u>.

$$\underline{F} \cdot \underline{dl} = -dV = -\left(\frac{\partial V}{\partial x}\right)dx - \left(\frac{\partial V}{\partial y}\right)dy - \left(\frac{\partial V}{\partial z}\right)dz = -\underline{\nabla}V \cdot \underline{dl} \Rightarrow \boxed{\underline{F} = -\underline{\nabla}V}$$

for some function $V(x, y, z)$.

$V(x, y, z)$ is the scalar potential associated with $\underline{F}$ and is single-valued.

$$\int_A^B \underline{F} \cdot \underline{dl} = -\int_A^B dV = -\left[V_B - V_A\right] = V_A - V_B.$$

Recap:
So $\underline{\nabla} x \underline{F} = 0 \Rightarrow$ there exists a $V(x,y,z)$ such that $F = -\underline{\nabla} V$. Recall that $\underline{\nabla} x(\underline{\nabla} V) = 0$ for any V.
How do we calculate V for a given F?
Example:

$$\underline{F} = (2xy - z^3)\underline{\hat{i}} + x^2 \underline{\hat{j}} - (3xz^2 + 1)\underline{\hat{k}}$$

Is $\underline{\nabla} x \underline{F} = 0$?

$$\underline{\nabla} x \underline{F} = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^3 & x^2 & -(3xz^2 + 1) \end{vmatrix} = \underline{\hat{i}}(0) - \underline{\hat{j}}(-3z^2 + 3z^2) + \underline{\hat{k}}(2x - 2x)$$

So $\underline{F}$ is conservative $\Rightarrow \underline{F} = -\underline{\nabla} V$

$$F_x = 2xy - z^3 = -\frac{\partial V}{\partial x} \Rightarrow V(x,y,z) = -x^2 y + z^3 x + \underbrace{f(y,z)}_{\text{int egration function}}$$

$$F_y = x^2 = -\frac{\partial V}{\partial y} = x^2 - \frac{\partial f}{\partial y} \Rightarrow f(y,z) = g(z)$$

$$F_z = -\frac{\partial V}{\partial z} = -3z^2 x - \frac{\partial g}{\partial z} = -(3xz^2 + 1) \Rightarrow \frac{dg}{dz} = 1 \Rightarrow g(z) = z$$

$$V(x,y,z) = -x^2 y + z^3 x + z + c$$

c is a constant.

<u>BUT</u> need to be careful.
e.g. Maxwell 4 (Time-independent)

$\underline{\nabla} x \underline{B} = \mu_o \underline{j}$

Infinite straight wire carrying current I.
$\underline{\nabla} x \underline{B} = 0$ everywhere except in the wire.
$\Rightarrow \underline{V} = -\underline{\nabla}\psi$ (ψ is the magnetic scalar potential)
So $\oint_C \underline{B} \cdot \underline{dl} = 0$? Not necessarily.
The domain D where $\underline{\nabla} x \underline{B} = 0$ has a hold in it where the wire is.