1) Electric Charge and Electric Field

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(Chapter 22, Young and Freedman)

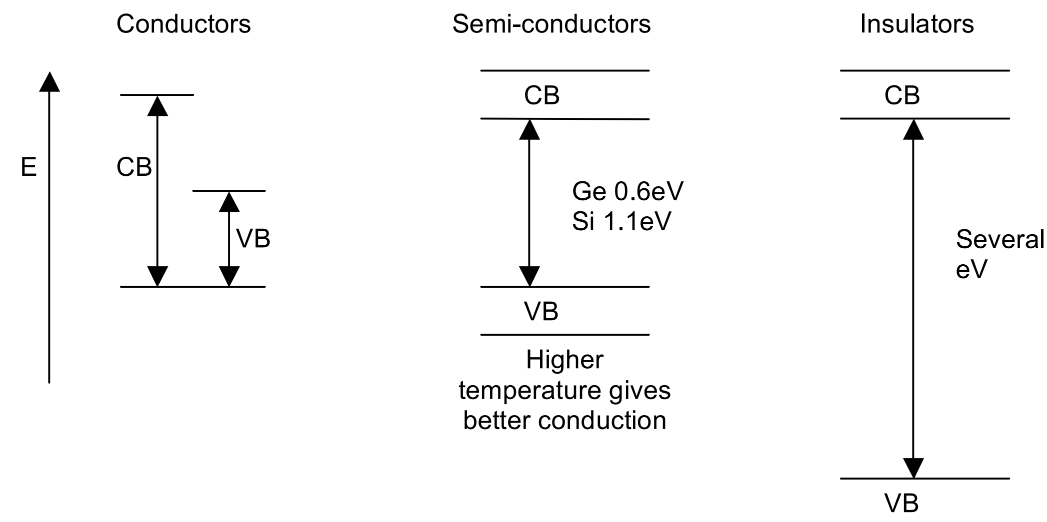
1.1 Matter

Simple!

1.2 Conductors, Semi-Conductors and Insulators

In solid state, the outer electron shells (the Valence electrons) are in bands of energy levels. If a band is not full, electrons can jump across energy levels and move around. This gives conduction.

Bands in materials:



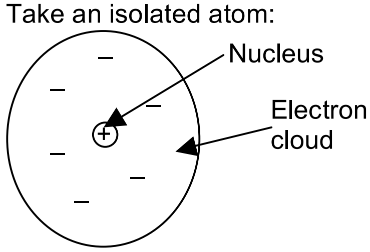
VB: Valence Band
CB: Conduction Band

In Insulators, the Valence band is full. There is nowhere free for an electron to move to. As it takes a lot of energy to get electrons into the conduction band (several eV) this doesn't happen very often, and most insulators simply melt when given the energy needed for the transition.

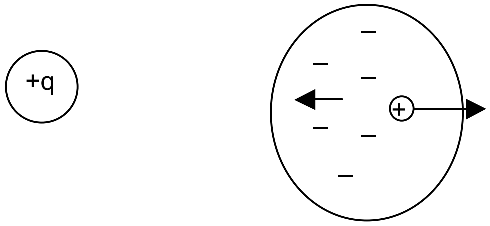
$$K_{T,room.temp} = {}^{1}\!/_{40}\, eV$$

1.3 Induced Charge

Take an isolated atom:

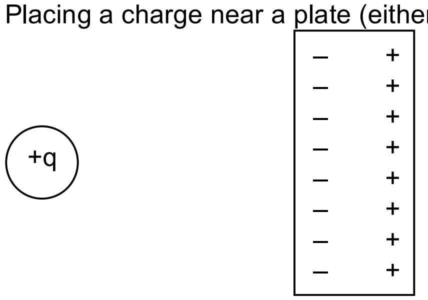


Placing a charge near the isolated atom causes the nucleus to be repelled, while the electron cloud is attracted.



The effect of charge separation is called polarization.

Placing a charge near a plate (either an insulator or metal):



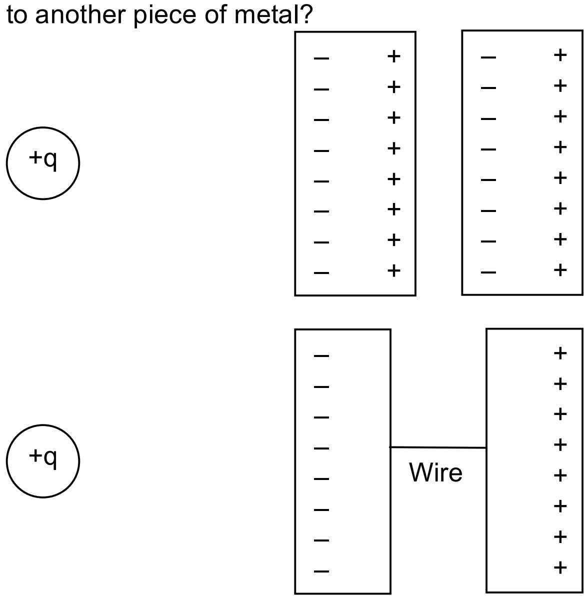
In an insulator, the net charge movement results in surface charges, while in a metal the electron cloud is affected. Bare ions are left on the back surface, and electrons on the front surface.

NB: charge densities $<< 1$ per surface lattice ion. Movement of charge is very small i.e. $<< 10^{-10}\,m$.

Example 1-1

Induced charge transfer

A charge is induced on a metal slab. What happens if this charged slab is then connected to another piece of metal?



It would be possible to break the connection, leaving both pieces of metal oppositely charged.

1.4 Coulomb's Law

The magnitude of the force between two point charges, q_1 and q_2, separated by r is given by:

$$|F| = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r^2}$$

ε_0 is the permittivity of free space, and is a constant.

The 4π appears here so it does not in other equations!

$$\frac{1}{4\pi\varepsilon_0} \approx 9.988x10^9\,Nm^2C^{-2}$$

$$\varepsilon_0 \approx 8.85x10^{-12}\,Fm^{-1}$$

Units are Newtons, Farads, Coulombs and Meters.

If q_1 and q_2 have the same sign, the force between them is repulsive. If they have opposite signs the force is attractive.

NB: Coulomb's law does not apply to distributive charges, just point charges.

Introduce the Electric field E.

Force on q_2 due to the presence of q_1:

$$|F_{21}| = \left(\frac{1}{4\pi\varepsilon_0}\frac{|q_1|}{r^2}\right)|q_2|$$
$$= |E_1||q_2|$$

where $|E_1| = \frac{1}{4\pi\varepsilon_0}\frac{|q_1|}{r^2}$

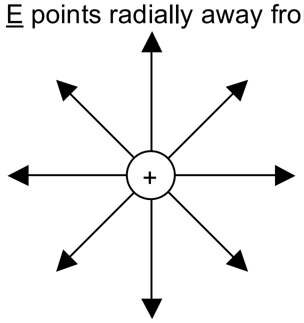
Units of E are $N.C^{-1}$ or Vm^{-1}.

Express E as a vector (drop subscripts):

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\underline{r}}$$
$$= \frac{\underline{F}_2}{q_2}$$

$\underline{E} \equiv$ Force on test charge q_2.

E points radially away from q if q is positive.



1.5 Electric Field Calculations

Principle of Superposition

For several point charges, the net force on a test charge is the vector sum of the forces experienced when the charges are taken one at a time.

For q_1 → E_1 and F_1 on test charge Q

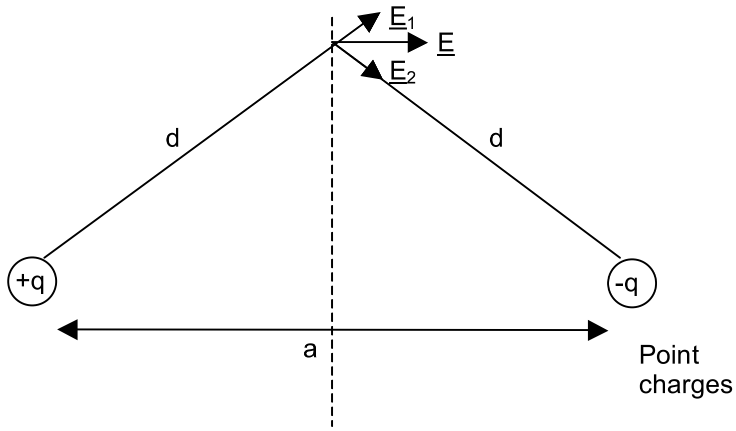
For q_2 → E_2 and F_2 on test charge Q

Etc…

Then $\underline{F} = \underline{F}_1 + \underline{F}_2 + \ldots = Q(\underline{E}_1 + \underline{E}_2 + \ldots)$

<u>Example 1-2</u>

Field of an electric dipole:



$|E_1| = |E_2|$

y components of E_1 and E_2 will cancel.

→ x-direction.

$$E_{1y} = |\underline{E_1}| \sin\alpha = -E_{2y} = -|\underline{E_2}| \sin\alpha$$

$$E_{1x} = |\underline{E_1}| \cos\alpha$$

$$E_{2x} = |\underline{E_2}| \cos\alpha$$

$$\underline{E} = (E_{1x} + E_{1y})\underline{\hat{i}} = E_{1x}\underline{\hat{i}} = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\cos\alpha\underline{\hat{i}}$$

$$\cos\alpha = \frac{a/2}{d}$$

$$\underline{E} = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\frac{a}{2d}\underline{\hat{i}}$$

$$\propto \frac{a}{d^3}$$

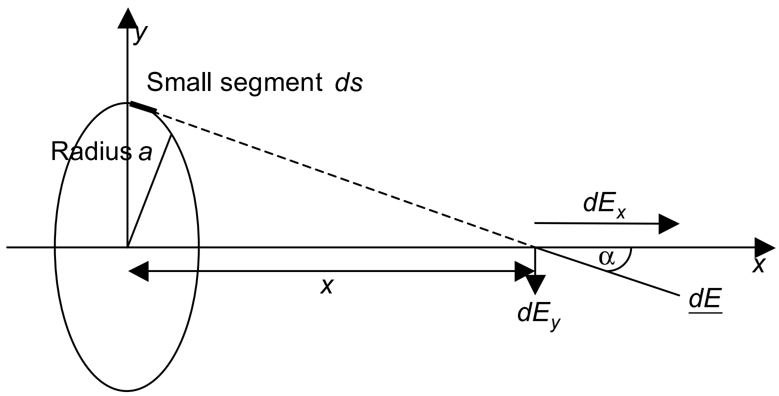
<u>Example 1-3</u>

A ring-shaped conductor of radius a has total charge Q uniformly distributed around the ring. What is the electric field on an axis a distance x away from the centre of the ring?
NB: there is a much simpler calculation like this in Workshop 2.
Remember to look for symmetry!

The approach to distributed-charge problem like this is to:

1. Break up the ring into tiny segments small enough to be regarded as point charges
2. Calculate the vector field from a segment
3. If possible from any symmetry, only calculate the component which is going to survive in the final result.
4. Sum or integrate over all possible components

In this problem, the field from the segment is going to be at an angle to the axis but the net field will be along the axis. The components perpendicular to the axis will integrate out. See the diagram.



Charge in element ds is dQ.

Ignore dE_y because the segment on the opposite side of the ring cancels it out.

$$dE_x = |\underline{dE}|\cos\alpha = |\underline{dE}|\frac{x}{\sqrt{a^2+x^2}}$$

but $|\underline{dE}| = \dfrac{1}{4\pi\varepsilon_0}\dfrac{dQ}{a^2+x^2}$

Therefore $dE_x = \dfrac{1}{4\pi\varepsilon_0}\dfrac{xdQ}{\sqrt{x^2+a^2}}$

$$\underline{E} = \hat{\underline{i}}E_x = \hat{\underline{i}}\int_0^Q dE_x$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{xQ}{\left(a^2+x^2\right)^{3/2}}\hat{\underline{i}}$$

At large distances, this appears to be a point charge.

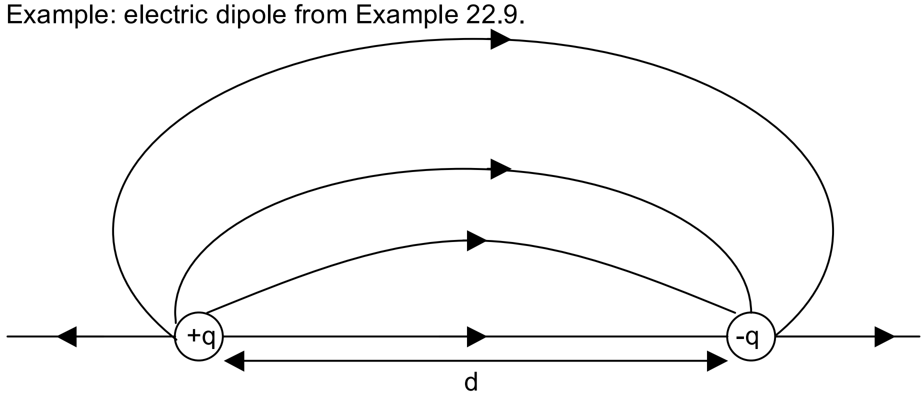
In the middle of the ring, $E_x = 0$.

1.6 Electric Field Lines

An electric field line is the curve drawn at a tangent to the direction of E. The line spacing tells you about the field strength $|E|$. Since E is unique at any point, the field lines never cross.

Caution: a released test charge does not move along an E line due to the inertia of the particle.

Example: electric dipole from Example 22.9.



1.7 Electric Dipoles

A dipole is two equal and opposite charges separated by a distance d. They are important in physical situations.

Examples:

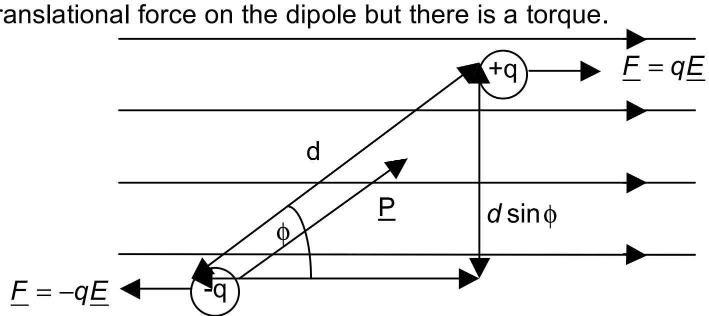
- Induced polarization due to E.
- Radiation from a "dipole" antenna.
- Ionic substances are natural dipoles e.g. NaCl.

Aside: water is a good solvent of ionic compounds (salts) because the H_2O molecule has a dipole moment.

Example 1-4

Force and Torque on a dipole in an electric field.

In a uniform E field (the field strength and direction are the same everywhere) there is no translational force on the dipole but there is a torque.



Torque $\tau = qEd\sin\phi$

Dipole moment $P = qd$ (definition)

Therefore $\tau = pE\sin\phi$

e.g. H_2O molecule has $p = 6.13x10^{-30}\,Cm$.

Take $q = 2|e|$ → $d = \frac{P}{q} = \frac{6.13x10^{-30}}{2x1.6x10^{-19}} \approx 2x10^{-11}\,m$.

$\tau = pE\sin\phi$

$\underline{\tau} = \underline{p}x\underline{E}$

Where p is drawn from the negative to the positive charge. As drawn, $\underline{\tau}$ is into the page.

If the dipole rotates in the field, work is done. $dW = \tau d\phi$. Because the torque is in the direction of decreasing φ we must write it as $\tau = -pE\sin\phi$ and $dW = -pE\sin\phi d\phi$.

For change from ϕ_1 to ϕ_2, the work done is:

$$W = \int_{\phi_1}^{\phi_2}(-pE\sin\phi)d\phi$$
$$= pE(\cos\phi_2 - \cos\phi_1)$$

Now the work done Is the negative of the change in E_P. So E_P of a dipole in a uniform field is:

$E_p(\phi) = -pE\cos\phi$ or $E_p = -\underline{P}\cdot\underline{E}$

Example 1-5

Field of a dipole on an axis at distance r>>d

We have seen from Example 1-2 that, along the bisector $\underline{E} = \hat{\underline{i}} E_x$.

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}$$

On an axis with r>>d:

The charge due to positive field $E_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$

$$E_- = -\frac{1}{4\pi\varepsilon_o}\frac{q}{(r+d)^2}$$

$$E_- = -\frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\left(1+\frac{d}{r}\right)^{-2}$$

But $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots \approx 1 + nx$ if nx<<1.

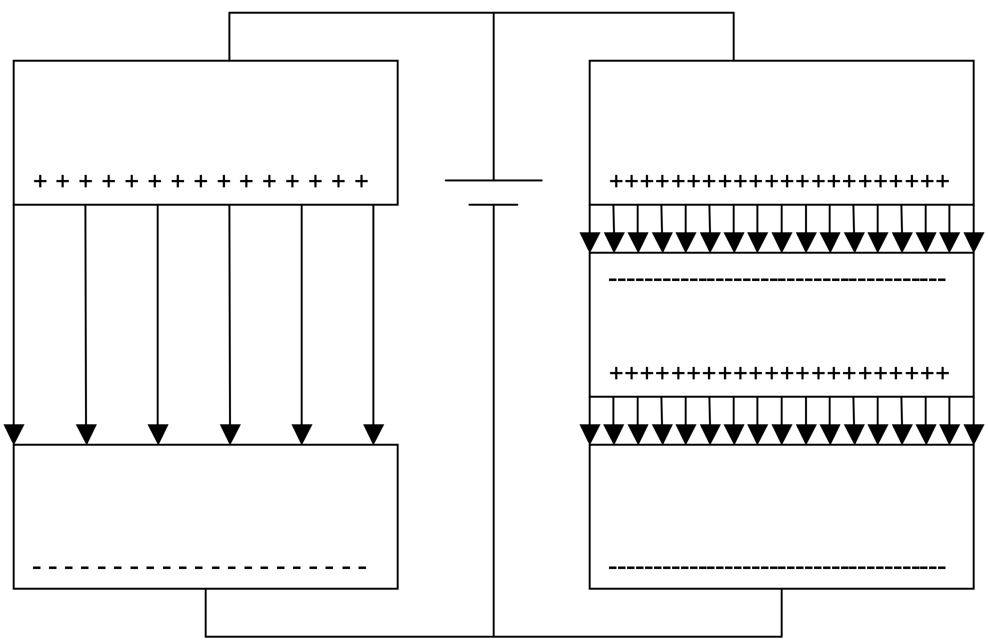
Therefore $\underline{E} = -\frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\left(1-\frac{2d}{r}\right)$

$$E = E_+ + E_- = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{2d}{r} = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3} \quad (p = qd)$$

1.8 Electric Fields inside Conductors

Static E is zero inside conductors.

Take a series of metal plates:



All charges are surface charges. Charge layers are very thin.

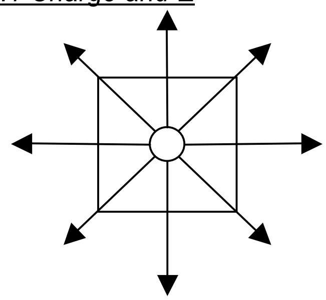
2. Gauss's Law

(Chapter 23, Young and Freedman)

Gauss's Law is useful for calculating E for distributive charges with some sort of symmetry. The basic idea is to enclose the charge distribution by a closed surface. The total flux of E out of the surface depends only on the total charge enclosed by the surface.

Flux means the flow of something. Analogy: consider a velocity vector V rather than E. The total flux of V = velocity x area = volume per second.

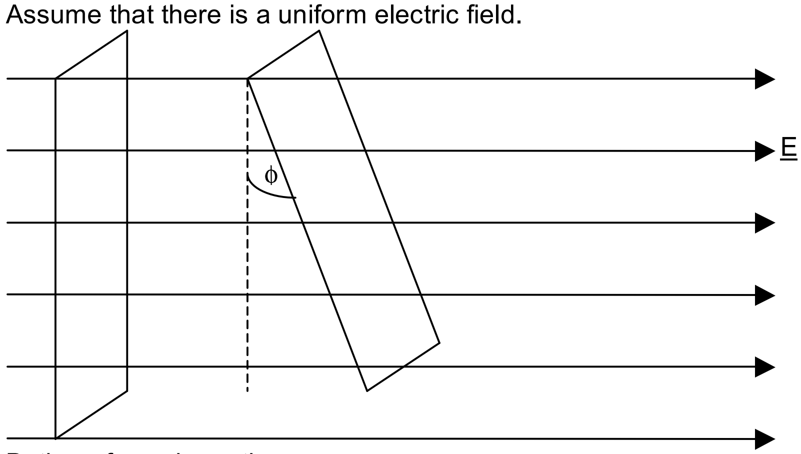
2.1 Charge and E



If the charge is not inside the box, then not all the flux lines will pass through the surfaces so the full charge will not be shown. Also, the flux entering the box will also leave the box on the opposite side, so a lot of the charge will cancel. Will equal 0 charge.

2.2 Calculating E flux

Assume that there is a uniform electric field.



Both surfaces have the same area.

The flux of E through the surface normal to E is $\Phi_n = EA$.

For the other surface: $\Phi_i = EA\cos\phi$

If you represent the surface by a vector $\underline{A} = A\underline{\hat{n}}$ where $\underline{\hat{n}}$ is a unit vector normal to the surface.

In the general case, $\Phi_E = \underline{E} \cdot \underline{A}$ or $d\Phi_E = \underline{E} \cdot \underline{dA}$.

In general, the total flux out of a closed surface is $\Phi_E = \int \underline{E}.\underline{dA}$. The integral needs to be applied over the whole of the surface.

2.3 Gauss's Law

Start with a point charge.

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\underline{\hat{r}}$$

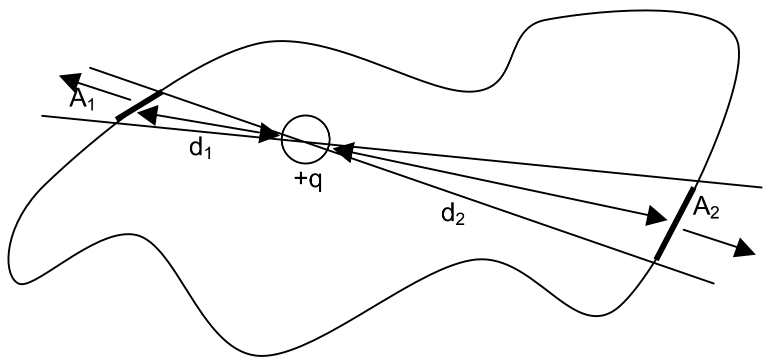
Choose a surface of radius R about the charge.

$$\Phi_E = \int \underline{E}.\underline{dA} = \left(\frac{1}{4\pi\varepsilon_o}\frac{q}{R^2}\right)\left(4\pi R^2\right) = \frac{q}{\varepsilon_0}$$

where $\left(4\pi R^2\right)$ is the surface area of the sphere, and $\left(\frac{1}{4\pi\varepsilon_o}\frac{q}{R^2}\right)$ is the E at the surface of the sphere. Φ_E is actually independent of the position of the charge in the sphere and the size of the sphere.

Gauss's Law states that the total flux of E out of the whole of a surface is equal to the net charge enclosed by the surface divided by ε_0.

$$\Phi_E = \int \underline{E}.\underline{dA} = \frac{\sum q}{\varepsilon_o}.$$



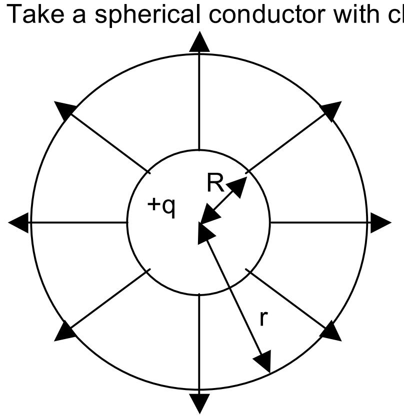
$$E_1 \propto \frac{1}{{d_1}^2}$$
$$A_1 \propto {d_1}^2$$
$$E_2 \propto \frac{1}{{d_2}^2}$$
$$A_2 \propto {d_2}^2$$
$$E_1 A_1 = E_2 A_2$$

Examples include a Parallel plate capacitor.

$$\Phi_E = \int_{whole.surface} \underline{E}.\underline{dA} = \frac{Q_{enclosed}}{\varepsilon_o}$$

Example 2-1

Take a spherical conductor with charge Q and radius R.

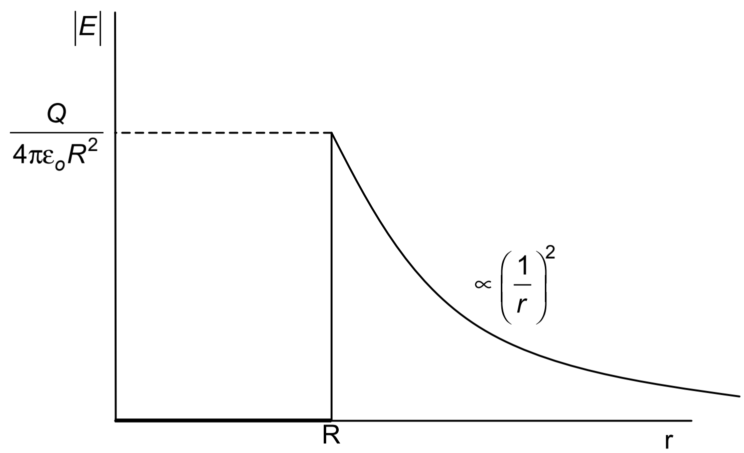


Take a small area of the surface of the outer sphere, dA.

$$\Phi_E = \int \underline{E}.\underline{dA} = E(r)\int dA = 4\pi r^2 E(r)$$

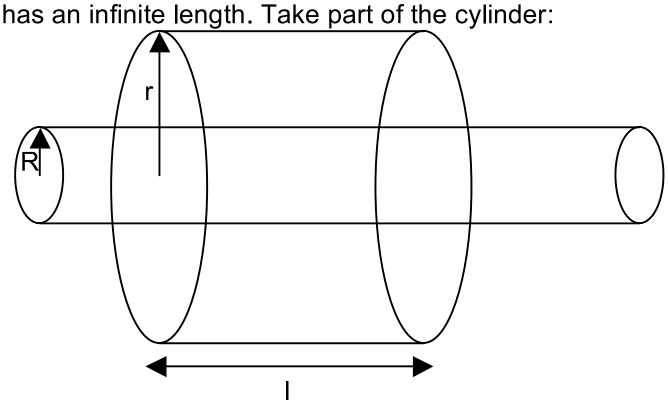
If R<r, $Q_{enclosed} = Q \rightarrow E(r) = \frac{Q}{4\pi\varepsilon_o r^2}$.

If r<R, $Q_{enclosed} = 0 \rightarrow E(r) = 0$.



Example 2-2

Take a long, cylindrical conductor of radius R and charge λ per unit length (Cm^{-1}), which has an infinite length. Take part of the cylinder:



Field lines must be normal to the cylinder, or there would be an electric force along the cylinder causing electrons to move. This would then set up an electric force in the opposite direction; these forces would then balance themselves out, ensuring the field lines have no horizontal component.
There is no $\underline{E}$ out of the ends of the surface, only the sides. At radius r the area of the surface of the cylinder (not counting the ends) is $2\pi rl$.

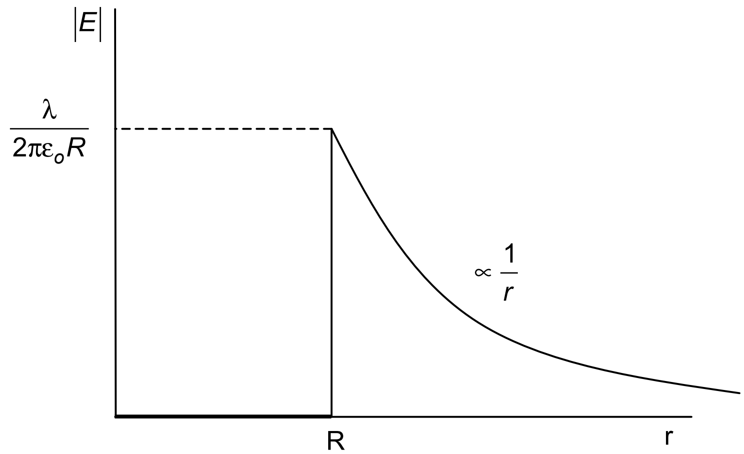
$$\Phi_E = E(r > R)2\pi rl$$

$$= \frac{Q_{enclosed}}{\varepsilon_o}$$

$$= \frac{\lambda l}{\varepsilon_o}$$

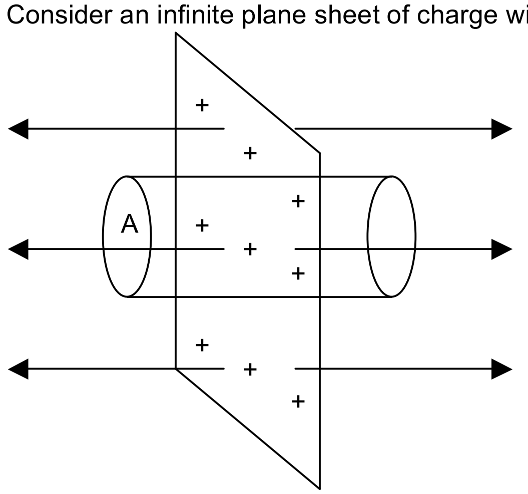
$$E(r > R) = \frac{1}{2\pi\varepsilon_o}\frac{\lambda}{r}$$

$$E(R > r) = 0$$



Example 2-3

Consider an infinite plane sheet of charge with a charge density of σ cm^{-2}.



There is only an E flux out of the ends of the cylinder.

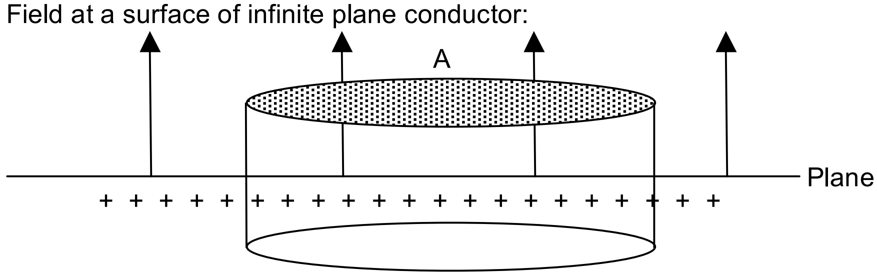
$$\Phi_E = 2EA$$

$$= \frac{\sigma A}{\varepsilon_o}$$

$$E = \frac{\sigma}{2\varepsilon_o}$$

Example 2-4

Field at a surface of infinite plane conductor:



The only place where E flux comes out is at the top end.

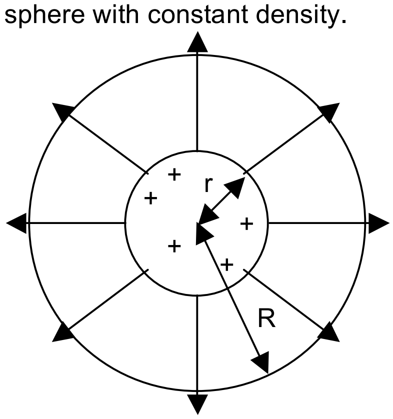
$$\Phi_E = EA = \frac{\sigma A}{\varepsilon_o}$$

σ is the surface charge density.

$$E = \frac{\sigma}{\varepsilon_o}.$$

Example 2-5

Uniformly charged sphere with radius R and charge Q. Charge is distributed throughout the sphere with constant density.



For r>R

$$E(r > R) = E(r)4\pi r^2 = \frac{Q}{\varepsilon_o}$$

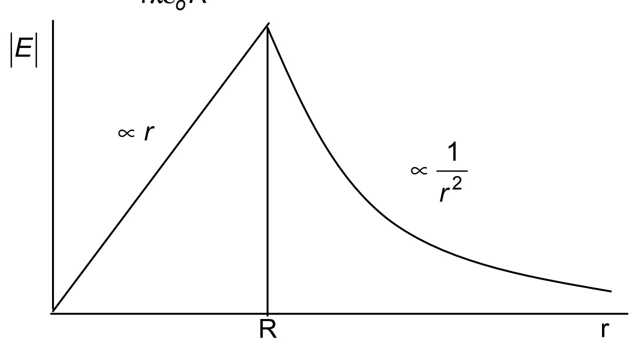
$$= \frac{Q}{4\pi r^2 \varepsilon_o}$$

For R>r

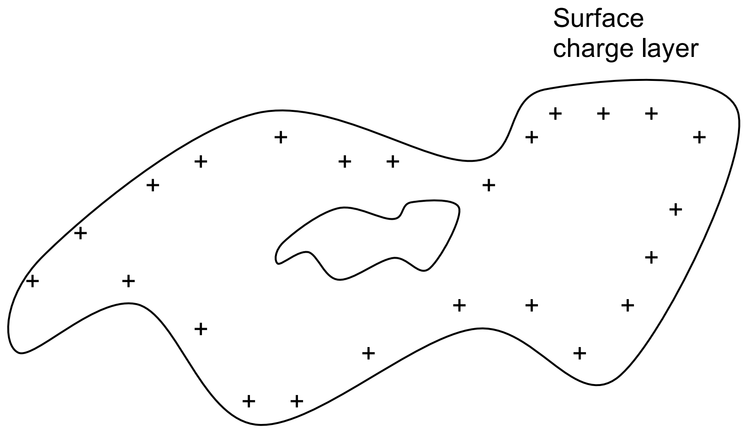
$$E(r < R) = E(r < R)4\pi r^2 = \frac{Q}{\varepsilon_o}$$

$$= \frac{Q}{\varepsilon_o}\left(\frac{r}{R}\right)^3$$

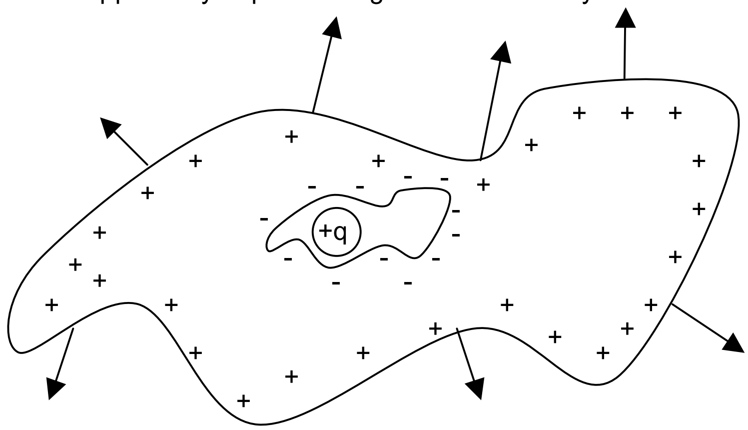
$$= \frac{Qr}{4\pi\varepsilon_o R^3}$$



Cavities in a conductor. Put a charge on the conductor.



As there can be no electric fields within the conductor, there is no electric field in the cavity. What happens if you put a charge within the cavity?



Field lines come out from every part of the surface.

3. Electric Potential

(Chapter 24, Young and Freedman)

3.1 Electric Potential and Energy

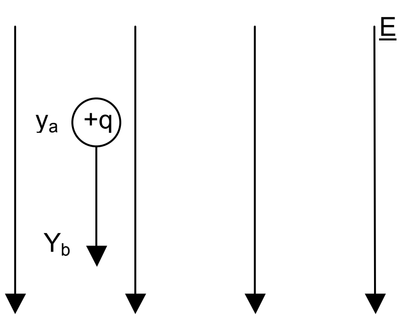
Work done = force x distance.

$dW = \underline{F} \cdot \underline{dl} = Fdl\cos\phi$

If $dW > 0$ the force does work on the outside world and the system creating the force looses energy. The potential energy change $dU = -dW = -\underline{F} \cdot \underline{dl}$.

In electrostatics $\underline{F} = q\underline{E}$ therefore $dU = -q\underline{E} \cdot \underline{dl}$

Take a uniform field.



$W_{a\to b} = F_d = qEd$.

For gravity: For electrostatics:

$F_y = -mg$ $F_y = -qE$

$U = -mgy$ $U = qEy$

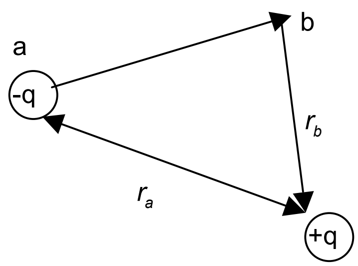
When the test charge moves from a → b then:

$W_{a\to b} = -\Delta U = -(U_a - U_b) = qE(y_a - y_b)$

Just like gravity except you can't have a negative mass.

3.2 Electric potential of two point charges:

There is a force between the charges $F = \frac{1}{4\pi\varepsilon_o}\frac{qq_o}{r^2}$. q_o is a test charge which will move from position a to position b.



$$W_{a\to b} = \int_{r_a}^{r_b} F_r dr = \frac{qq_o}{4\pi\varepsilon_o}\int_{r_a}^{r_b}\frac{dr}{r^2} = \frac{qq_o}{4\pi\varepsilon_o}\left|-\frac{1}{r}\right|_{r_a}^{r_b}$$

$W_{a\to b}$ depends only on the end points not on the path between. Justification:

$dW = q\underline{E}\underline{dl} = qEdl\cos\phi = qEdr$

The static E-field is conservative.

So the potential energy of our test charge q_o is $U = \frac{1}{4\pi\varepsilon}\frac{q}{r}q_o$ (with respect to the energy at infinity)

For several charges $q_1, q_2, q_3, ...$

Test charge at r_1 from q_1, r_2 from q_2 …

$$U = \frac{q_o}{4\pi\varepsilon_o}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + ...\right)$$

$$U = \frac{q_o}{4\pi\varepsilon_o}\sum_i\left(\frac{q_i}{r_i}\right)$$

Using this, we can find U for test charge for any distribution of charge.

Total energy of the system of charges:

$$U = \frac{1}{4\pi\varepsilon}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}}$$

3.3 Electric Potential

Potential V is the potential energy per unit (test) charge.

So

$$V = \frac{U}{q_o}$$

$$U = q_o V$$

Unit of V is the volt. $1V=1JC^{-1}$.

For U_a and U_b as defined previously

$$V_a = \frac{U_a}{q_o}$$

$$V_b = \frac{U_b}{q_o}$$

Potential due to a single point charge is $V = \frac{U}{q_o} = \left(\frac{1}{4\pi\varepsilon_o}\frac{qq_o}{r}\right)\frac{1}{q_o} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r}$

This is the potential at a distance r from charge q.
V and U are scalar properties. For an assembly of charges

$$V = \frac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{r_i}$$

For a charge distribution:

$$V = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{r}$$

We can calculate V from E where E is known.

$$W_{a\to b} = \int_a^b \underline{F}\cdot \underline{dl} = q\int_a^b \underline{E}\cdot \underline{dl}$$

Thus

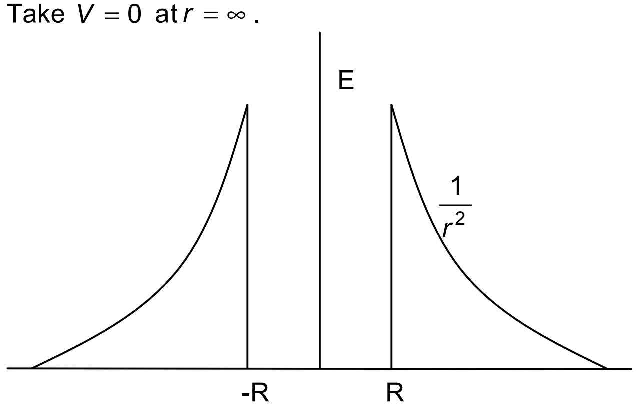
$$V_a - V_b = \int_a^b \underline{E}\cdot \underline{dl} = \int_a^b E\cos\phi dl$$

Can see that $V =$ field x distance, therefore the unit of E can be expressed as Vm^{-1}.
TV signal $\approx 10^{-3} Vm^{-1}$.

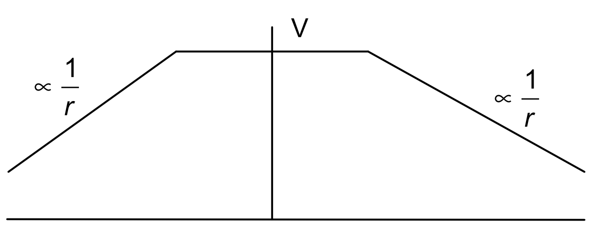
3.4 Examples

Example 3-1

There is a charge q in a conducting sphere of radius R. We have showed that, for $r > R$, field is the same as for a point charge.
Take $V = 0$ at $r = \infty$.



$V = \frac{1}{4\pi\varepsilon_o}\frac{q}{r}$ (for $r > R$)



V is constant inside the conductor since static $\underline{E} = 0$ inside a conductor.

$E_{surface} = \frac{1}{4\pi\varepsilon_o}\frac{q}{R^2} < 2x10^6 Vm^{-1}$ in air.

$V_{surface} = RE_{surface} \rightarrow 3MV$ for $R = 1m$, $30KV$ for $R = 1cm$, $300V$ for $R = 0.1mm$.

Example 3-2

Take an infinite line of charge density λcm^{-1}.

Found that $E_{radial} = \frac{1}{2\pi\varepsilon_o}\frac{\lambda}{r}$

$$V_a - V_b = \int_a^b \underline{E}\cdot\underline{dl} = \int_a^b E_r dr = \frac{\lambda}{2\pi\varepsilon_o}\int\frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(r_b/r_a\right)$$

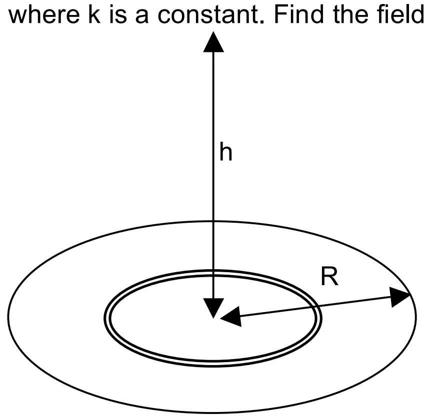
We could define V to be zero at say r_o. $V = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{r_o}{r}\right)$.

Example 3-3

Take a long conducting cylinder of radius r, with charge density λcm^{-1}.

$V(r > R) = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{R}{r}\right)$. $V(r \le R) = 0$.

E.g. remember $V = \frac{1}{4\pi\varepsilon_o}\int\frac{dq}{r}$. A disk of radius R and charge density $\sigma = \sigma_o e^{-kr} Cm^{-2}$ where k is a constant. Find the field at distance h above the disc on the axis point.



The inner circle has a radius a, with the outer radius $a + da$.
The area of the annulus is $\approx 2\pi a da$ if $da << a$. Therefore the charge on the annulus is $= 2\pi a\sigma_o e^{-kr} da$.

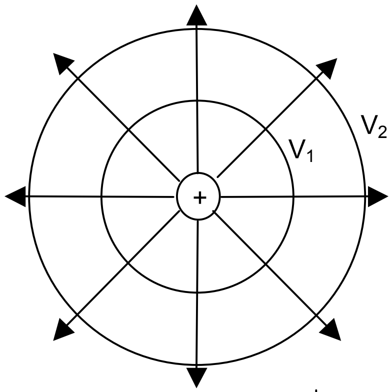
The distance to an axis point is just $\sqrt{h^2 + a^2}$.

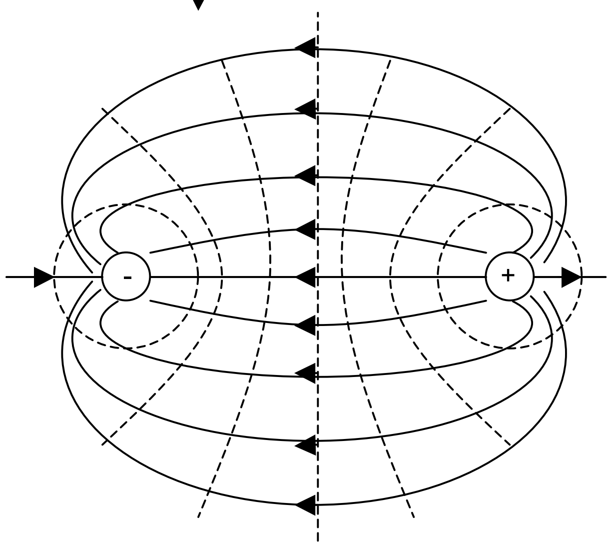
Therefore $V(h) = \frac{1}{4\pi\varepsilon_o}\int_0^h \frac{2\pi a\sigma_o e^{-ka} da}{\sqrt{h^2 + a^2}} = \frac{\sigma_o}{2\varepsilon_o}\int_0^h \frac{ae^{-ka} da}{\sqrt{h^2 + a^2}}$.

$V = 0$ at $h = 0$.

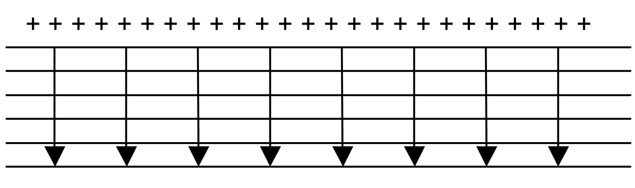
3.5 Equipotential Surfaces

These are three-dimensional surfaces on which V is the same everywhere.





Plates of a capacitor.



1) E is always perpendicular to the EP surface.
2) Equipotential surfaces cannot touch or cross (c.f. E field lines)
3) The whole of a conductor (static fields) is an equipotential surface.
4) Since static E is always perpendicular to the surface of a conductor, equipotential surfaces are locally parallel to the surfaces of conductors.

3.6 Potential Gradient

$$V_a - V_b = \int_a^b \underline{E}\cdot \underline{dl} = \int_b^a dV = -\int_a^b dV$$

$$\therefore -\int_a^b dV = \int_a^b \underline{E}\cdot \underline{dl}$$

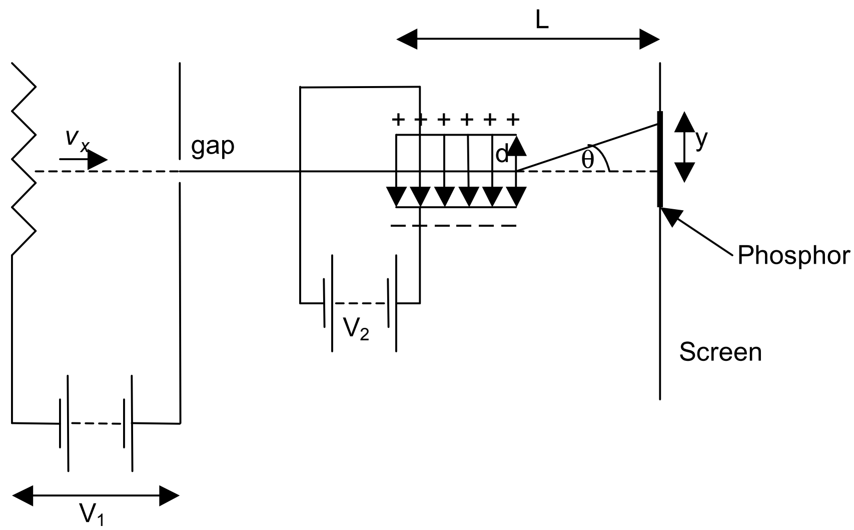
$$-dV = \underline{E}\cdot \underline{dl}$$

$$= E_x dx + E_y dy + E_z dz$$

$$E_x = \frac{\partial V}{\partial x}, E_y = \frac{\partial V}{\partial y}, E_z = \frac{\partial V}{\partial z},$$

$$\therefore \underline{E} = -\left(\underline{\hat{i}}\frac{\partial V}{\partial x} + \underline{\hat{j}}\frac{\partial V}{\partial y} + \underline{\hat{k}}\frac{\partial V}{\partial z}\right) = -\underline{\nabla}V$$

3.7 Cathode Ray Tube



Working out the speed of the electrons $\frac{1}{2}mv_x^2 = eV_1$. $v_x = \sqrt{\frac{2eV_1}{m}}$. If $V_1 = 2KV$,

$v_x = 2.65x10^7 ms^{-1}$.

Between the plates the acceleration upwards is $a = \frac{eE}{m}$. $E = \frac{V_2}{d}$.

The electrons spend a time $t = \frac{L}{v_x}$ within the plates. Therefore $v_y = at = \frac{eV_2}{md}\frac{L}{v_x}$ when they exit the plates. $\tan\theta = \frac{V_y}{V_x} \approx \frac{y}{D}$. Therefore $y = \frac{DeV_2L}{mdv_x^2} = \left(\frac{LD}{2d}\right)\left(\frac{V_2}{V_1}\right)$.

4. Capacitors and Capacitance

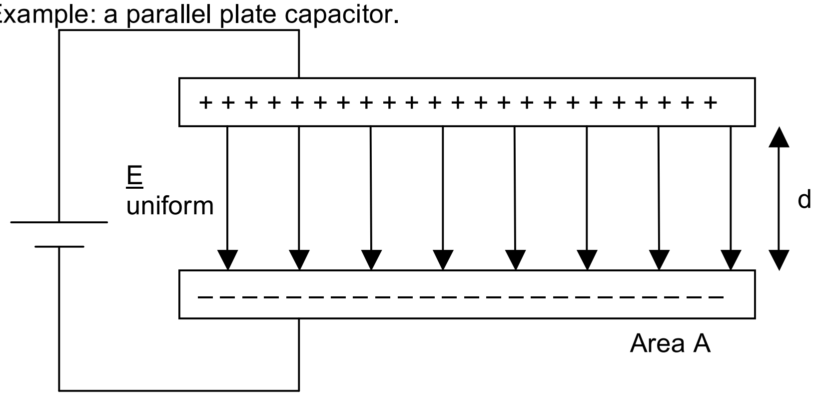
(Chapter 25, Young and Freedman)

4.1 Capacitors

A capacitor is a device for storing charge. Different capacitors can have different capacitance; the capacitor's ability to store charge. Different arrangements of pairs of electrodes or metal surfaces give different capacitors.

We define capacitance as $C =$ (Charge stored on electrodes) / (Potential difference between electrodes). The unit of capacitance is the Farad (F). $1F = 1CV^{-1}$.

Example: a parallel plate capacitor.



Assume d is small (comparable to $\sqrt{A}$)
Assume E is uniform, and that we can neglect edge effects.
Assume there is a charge Q on the plates.

Charge density $\sigma = \frac{Q}{A} Cm^{-1}$.

Gauss's Law: $E = \frac{\sigma}{\varepsilon_o}$.

As E is uniform, $V_{AB} = Ed$ =battery voltage $= \frac{1}{\varepsilon_o}\frac{Qd}{A}$.

Therefore $C = \frac{Q}{V_{ab}} = \varepsilon_o \frac{A}{d}$.

Take $A = 10^{-4} m^2$, $d = 10^{-4} m$. $C = 8.85x10^{-12} F = 8.85pF$.

Example 4-1

Isolated metal sphere
The second electrode in this case is the rest of the universe.

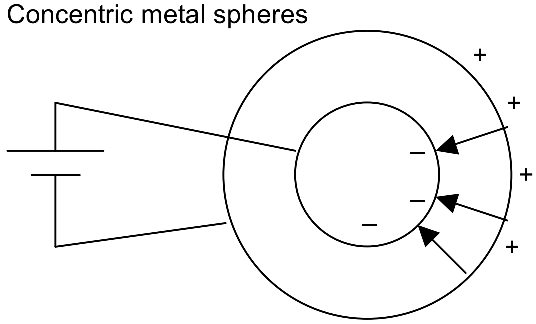
We previously found $V = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R}$.

Therefore $C = \frac{Q}{V} = 4\pi\varepsilon_o R$.

Take $R = 10cm$ $C = 4\pi x 8.5x10^{-12} x10^{-1} \approx 10pF$.

Example 4-2

Concentric metal spheres



$$E\left(r_a < r < r_b\right) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$

This is from considering a third sphere midway between the two above, and applying Gauss's law.
Electric field is 0 outside the capacitor.

$$\left.\begin{aligned} V_a &= \frac{1}{4\pi\varepsilon_o}\frac{Q}{r_a} \\ V_b &= \frac{1}{4\pi\varepsilon_o}\frac{Q}{r_b} \end{aligned}\right\} \text{Alternatively } -\int \underline{E}\cdot \underline{dl}$$

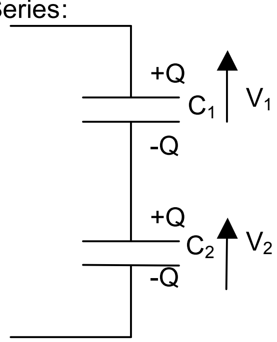
$$V_{ab} = \frac{1}{4\pi\varepsilon_o}Q\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$C = \frac{Q}{V_{ab}} = 4\pi\varepsilon_o\left(\frac{r_a r_b}{r_b - r_a}\right)$$

Take $r_a = 9.5cm, r_b = 10.5cm \;\; C \approx 110pF$.

4.2 Combinations of Capacitors in Series and Parallel

Series:



The Q's are the same since same charging current and $q = \int idt$.

$$V_1 = \frac{Q}{C_1}$$

$$V_2 = \frac{Q}{C_2}$$

Therefore

$$V_{ab} = V_1 + V_2 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

$$\frac{V}{Q} = \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

C_{eq} is the equivalent capacitance of two capacitors in series. This is comparable with resistors in parallel.

Parallel:

$V_1 = V_2$

$Q_1 = C_1 V$

$Q_2 = C_2 V$

$C_{eq} = \frac{Q_{total}}{V} = C_1 + C_2$

This is comparable to resistors in series.

4.3 Energy Storage in Capacitors and Electric Fields

Capacitor $V = \frac{Q}{C}$. If we move charge dq from the negative plate to the positive plate then the work done $dW = Vdq \left(-\underline{E}.\underline{dl}dq\right)$

Therefore the total work done in charging the capacitor is

$$W = \int dW = \frac{1}{C}\int Qdq = \frac{Q^2}{2x} = \frac{1}{2}CV^2$$

If we define the potential energy of a capacitor to be 0 when $Q = 0$ then;

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV .$$

E.g. camera flash

$C \approx 1\mu F$

$V \approx 300V$

$U = \frac{1}{2}CV^2 \approx 4.5x10^{-2} J$

If the flash lasts in the order of 10^{-3} seconds, power in flash $\approx 45W$.

The energy stored in fields (alternative view)

For parallel plates $E = \frac{\sigma}{\varepsilon_o}$.

$V = Ed$

Volume of space occupied by $\underline{E}$ is Ad

Energy density in a capacitor is

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad}$$

$$= \frac{1}{2}\frac{\varepsilon_o A}{d}\frac{E^2 d^2}{Ad}$$

$$= \frac{1}{2}\varepsilon_o E^2$$

$\left(Jm^{-3}\right)$

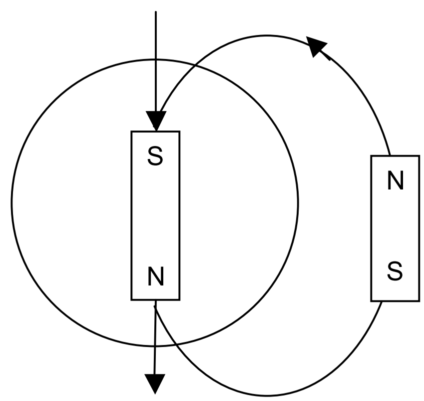
5. Magnetic Fields and Forces

(Chapter 28, Young and Freedman)

5.1 Magnetism

Magnetism has been known for a long time. The oldest use is the magnetic compass used in navigation.

The Earth is a big magnet:



The geomagnetic North Pole is the S side.
The compass needle will point NS while the magnet in the Earth is SN as like poles repel, while unlike poles attract.
All magnetism is associated with currents – i.e. with moving charges. Even permanent magnets can be explained by this – they have permanent atomic currents.
You can induce magnetism into "soft" magnetic materials, e.g. soft iron. These materials become magnetized in the presence of an external field. If you remove the external field, this magnetism disappears.

<u>Applications of magnets:</u>
Permanent magnets:
- Compass needle
- Fridge door
- Door seal
- Electricity meters
- Small motors
- Tool holder

Induced magnetism:
- Motors
- Transformers
- Inductors e.g. the ignition coil in a car, tuned circuits LC, filters to pass or reject certain frequencies (50Hz in power supplies)

5.2 Magnetic Field

A moving charge or current creates a magnetic field. A magnetic field exerts a force on moving charges or currents.
Represent a magnetic field by $\underline{B}$.
Force on a charge q moving at a velocity $\underline{v}$ in a field $\underline{B}$:

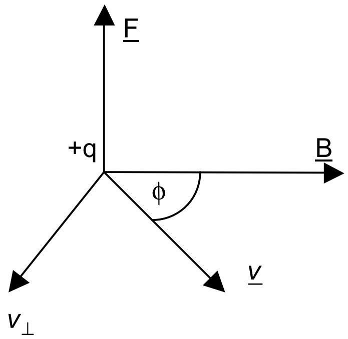
$\propto |\underline{B}|$

$\propto |\underline{v}|$

$\propto q$

$|F| = |q| v_{\perp} B$

$\underline{F} = q\underline{v}x\underline{B}$



$|F| = qvB \sin\phi$

$\underline{F} = q\left(\underline{E} + \underline{v} x \underline{B}\right)$.

(qE is the electric force).

The unit of B is the Tesla T.

$1T = 1NsC^{-1}m^{-1}$

$= 1NA^{-1}m^{-1}$

(Newton second) / (Coulomb Meter)
(Newton) / (Ampere Meter)

Current $I = \dfrac{dq}{dt}$ Amperes

$1A = Cs^{-1}$

Centimetre Gram Second (cgs) unit of B is Gauss (G)

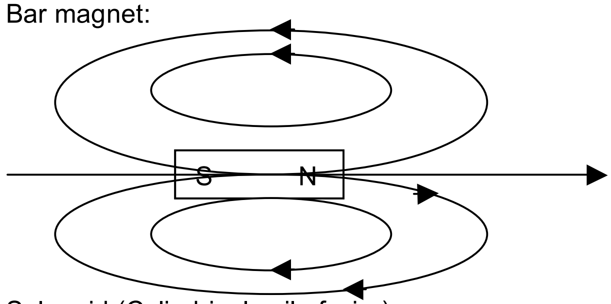
$1G = 10^{-4}T$.

Typical sizes of magnetic fields:

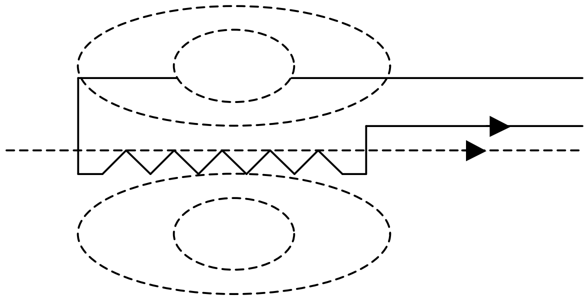
- The Earth's magnetic field $\approx 1G$.
- A bar magnet $\approx 5000G = 0.5T$
- Atomic fields $\approx 10T$.
- Electromagnets – few T.
- Steady fields in the lab $\leq 50T$.
- Pulsed field in the lab $\leq 120T$
- Neutron star (surface) $\sim 10^8 T$.

5.3 Magnetic Field Lines

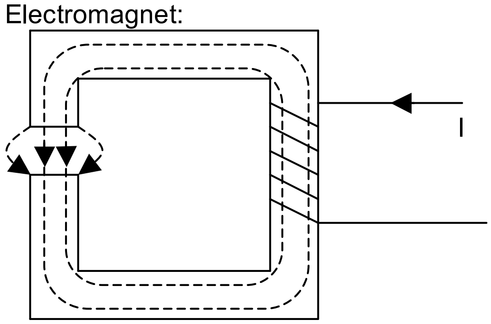
Bar magnet:



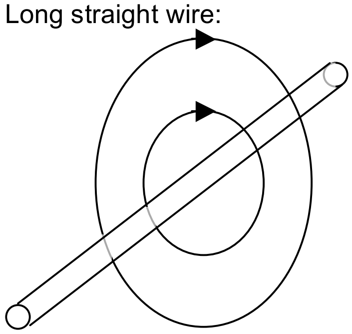
Solenoid (Cylindrical coil of wire):



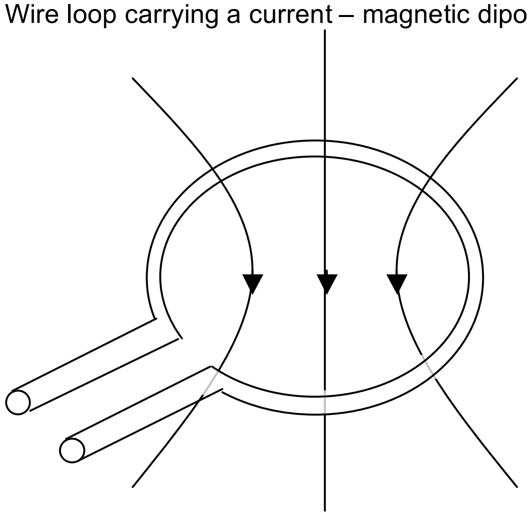
Electromagnet:



Long straight wire:



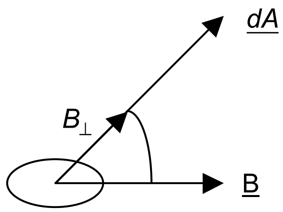
Wire loop carrying a current – magnetic dipole:



5.4 Magnetic Flux, Gauss's Law for B

Define the magnetic flux:

$$
\begin{aligned}
d\Phi_B &= B_\perp dA \\
&= B\cos\phi dA \\
&= \underline{B}\cdot\underline{dA}
\end{aligned}
$$



Total flux through a surface is:

$$\Phi = \int B_{\perp} \cdot dA = \int B \cos\phi dA$$

$$= \int \underline{B} \cdot \underline{dA}$$

SI unit of flux is the Weber (Wb).

$1Wb = 1Tm^2 = 1NmA^{-1}$

Aside: Old books sometimes define B in terms of Φ_B i.e. $B = \frac{d\Phi_{\mathbf{B}}}{dA}$. Units of Wbm^{-2}. B is called the magnetic flux density.

If the surface completely encloses the volume then:

$$\Phi_B = \int \underline{B} \cdot \underline{dA} = 0 .$$

This is Gauss's law for B.
This reflects the fact that B never starts or ends.

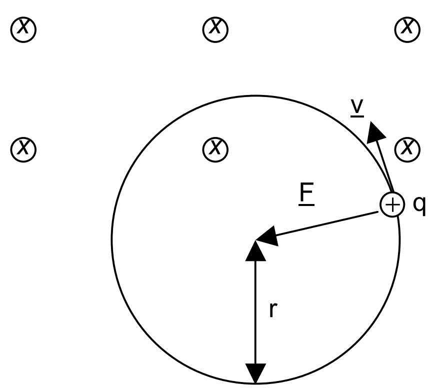
5.5 Motion of Charged Particle in B

$\underline{F} = q\left(\underline{v} x \underline{B}\right)$

$\underline{F} = q\left(\underline{E} + \underline{v} x \underline{B}\right)$

B is static and uniform in all but one of the following examples. Speed of the particle is not affected.

$\underline{B} \propto \underline{v}$



Motion of particle is a circle with radius r.

$$F = |q| vB = \frac{mv^2}{R}$$

$$R = \frac{mv}{|q| B}$$

The angular frequency of the motion $\omega = \dfrac{v}{R}$

$$\omega = |q|\frac{B}{m}$$

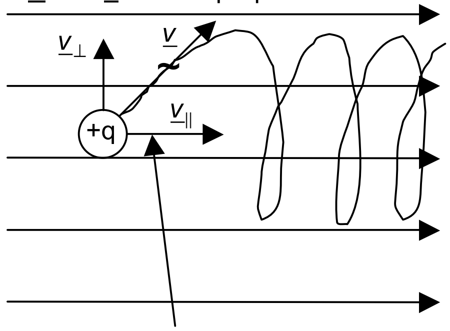
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}|q|\frac{B}{m}$$

f is called the "cyclotron" frequency. For an electron $f \simeq 2.7x10^{10}B$
e.g. microwave oven

$f = 2.45x10^{9}Hz$

$B = 0.09T$

If B and v are not perpendicular:

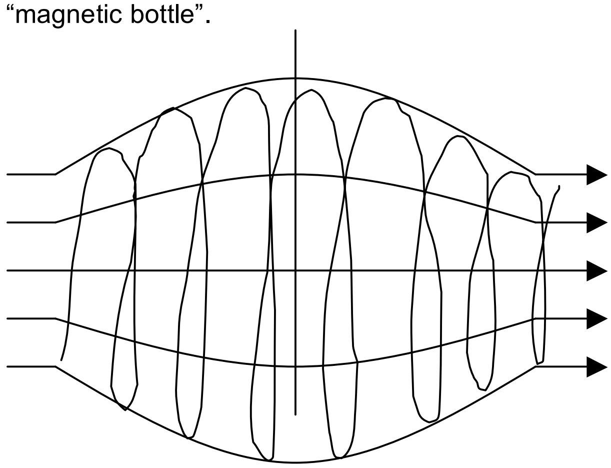


No force as $v_{||}xB = 0$

$v_{//}$ is not affected by B.

$v_\perp$ causes circular motion in that plane. Therefore the path is helical.

In a non-uniform field, the particle can be trapped in the magnetic field. This is called a "magnetic bottle".

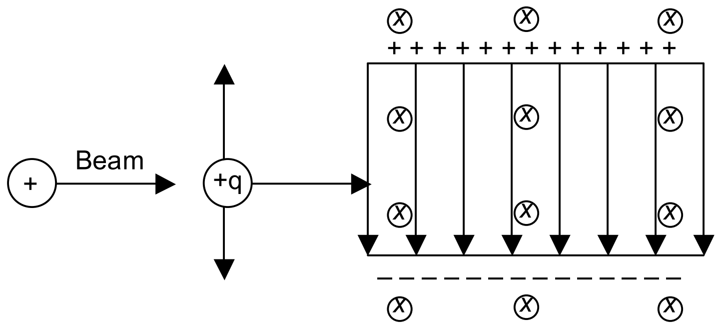


It is possible for the particle to bounce back from the right-hand-side, and then bounce backward and forward in the "bottle".
The component of the force on the right-hand-side is to the left.
The component of the force on the left-hand-side is to the right.
This can be observed in the Van Allen radiation belts around the Earth.

5.6 Applications

Velocity selector

Select mono-velocity particles from a beam.



Uniform B field into the paper; uniform E field.

For no variation, $F = -qE + qvB = 0$

Therefore $v = \frac{E}{B}$.

This works whether q is positive or negative.

Thompson's determination of the $\frac{e}{m}$ of the electron used this. His apparatus consisted of an electron gun accelerating the electrons through a potential difference V. He used the velocity accelerator and a florescent screen to view the beam of electrons.

We can equate the kinetic energy of the electron to eV.

$$\frac{1}{2}mv^2 = eV.$$

Therefore $v = \sqrt{\frac{2eV}{m}}$.

For no deflection, $\frac{E}{B} = \sqrt{\frac{2eV}{m}}$

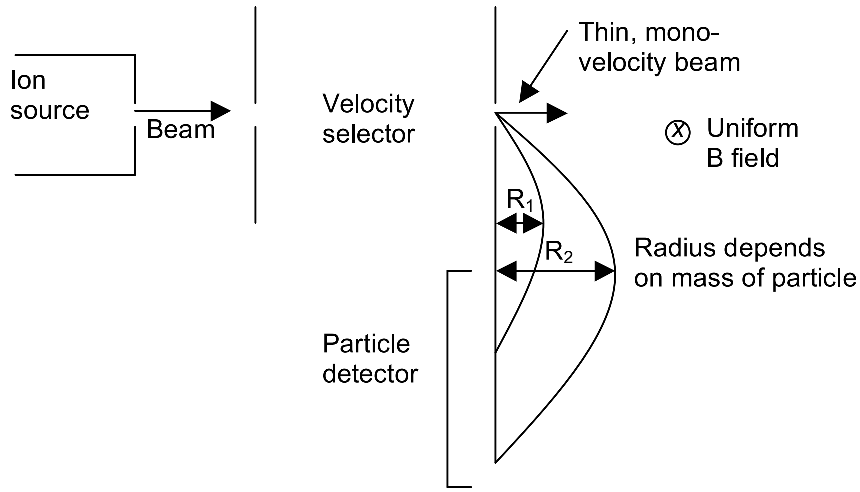
Therefore $\frac{e}{m} = \frac{1}{2V}\left(\frac{E}{B}\right)^2$.

He found only a single value for $\frac{e}{m}$ independent of the source of the electrons. This shows that there is only one sort of electrons. This proves that they are basic constituents of matter.

$$\frac{e}{m} = 1.75881962(53)x10^{11}Ckg^{-1}.$$

Mass Spectrometer

(Third year lab)



Velocity $v = \frac{E}{B_1}$.

E and B_1 are the fields in the velocity selector.

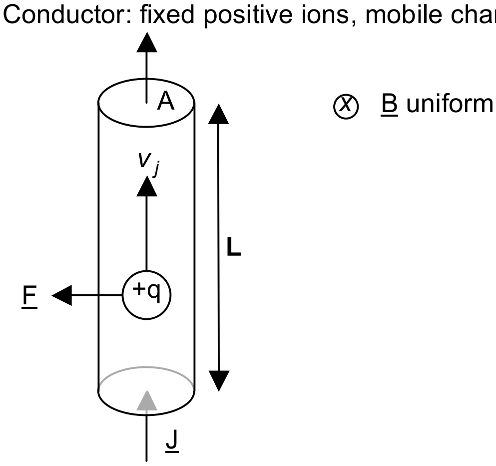
$$R = \frac{mv}{qB_2} = \frac{m}{q}\frac{E}{B_1 B_2}$$

→ Discovery of isotopes (Same chemical species, different masses) → part of the evidence for the existence of a neutron.

For helium, masses of 20 and 22 come out.

5.7 Magnetic Force on a conductor carrying a current

Conductor: fixed positive ions, mobile charge carriers (e.g. electrons)



J is the current density.

Let the carrier drift velocity be $\underline{v_d}$ upwards.

$F = qv_d B$

Let n = number of carriers per unit volume.

Volume of cylinder $= AL$

Therefore the number of carriers in the cylinder $= nAL$. Therefore the total force on all the carriers is

$$F = nALqv_d B$$
$$= (nqv_d A)LB$$

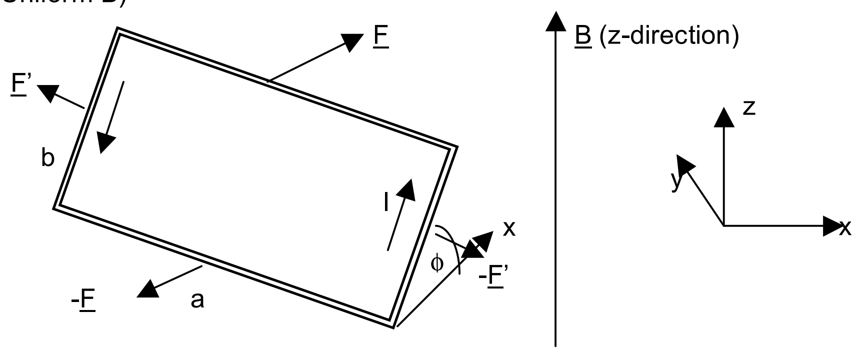
But $J = nqv_d$

Therefore:
$I = JA$
$F = ILB$
$\underline{F} = I\underline{L}x\underline{B}$

For a non-straight conductor:
Consider a little segment.
$\underline{dF} = I\underline{dL}x\underline{B}$.

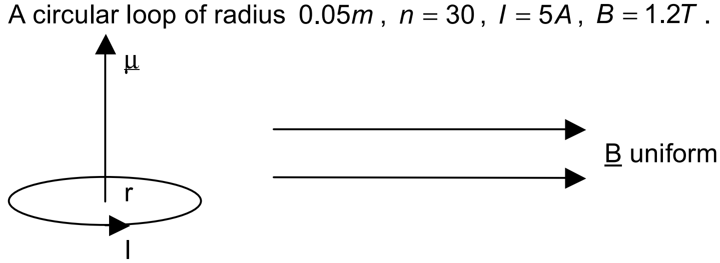
What if the charge is negative?
Upwards current is a downward drift of negative charge carriers, and q is negative. These two cancel each other out. Therefore this derivation is not sensitive to the charge polarity.

5.8 Force and Torque on a current loop
(Uniform B)



Net force $F_y = 0$
No translational force on loop.
$|\underline{F}| = IaB$
$|\underline{F}'| = IbB\sin(90-\phi) = IbB\cos\phi$
Torque about y direction $\tau = 2F\frac{b}{2}\sin\phi = IAB\sin\phi$ where A is the area of the loop $A = ab$.
IA is called the "magnetic dipole moment", denoted by $\underline{\mu} = I\underline{A}$. The sign of $\underline{\mu}$ is positive if, looking along $\underline{\mu}$ the current circulates clockwise.
Thus $\underline{\tau} = \underline{\mu}x\underline{B}$
Torque on a magnetic dipole $\underline{\mu}$ in a B field.
Compare with the electric dipole, $\underline{\tau} = \underline{P}x\underline{E}$ and $E_P = U = -\underline{P}\cdot\underline{E}$
By analogy, the potential energy of a magnetic dipole in a magnetic field is $U = -\underline{\mu}\cdot\underline{B}$.

Example 5-1
A circular loop of radius $0.05m$, $n = 30$, $I = 5A$, $B = 1.2T$.



The area of the loop $A = \pi r^2$

For one turn $\mu_1 = IA = I\pi r^2$

$\mu_{total} = n\mu = nI\pi r^2$

Torque on one turn $\tau_1 = IAB\sin\phi = 0.0471Nm$

Torque on n turns $\tau = n\tau_1 = 1.41Nm$.

Example 5-2

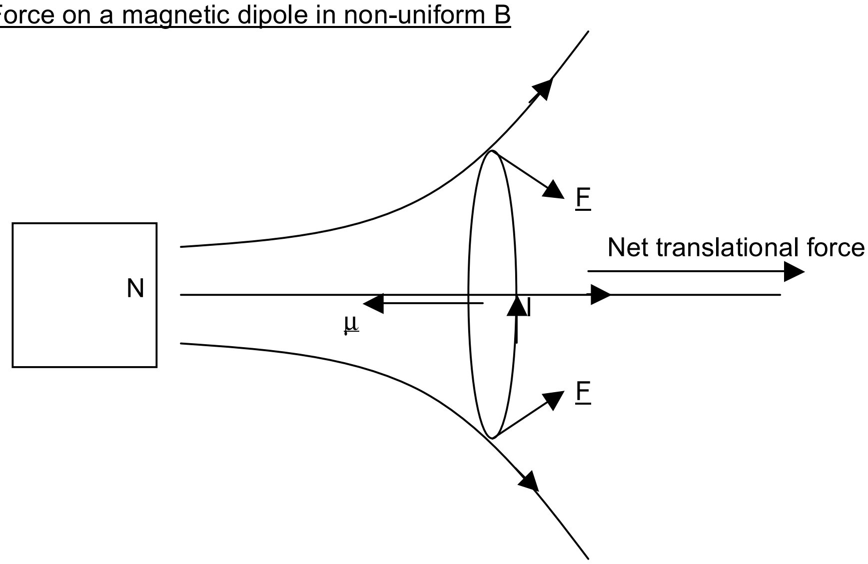
Change the angle in the previous example.

Initial, $\phi = 90^o$ $E_p = U_1 = -\mu_{total}B\cos\phi_1 = 0$

Final, $\phi = 0$, $E_p = U_2 = -\mu_{total}B\cos\phi_2 = -1.41J$

This is negative as work is done on the outside world.

Force on a magnetic dipole in non-uniform B



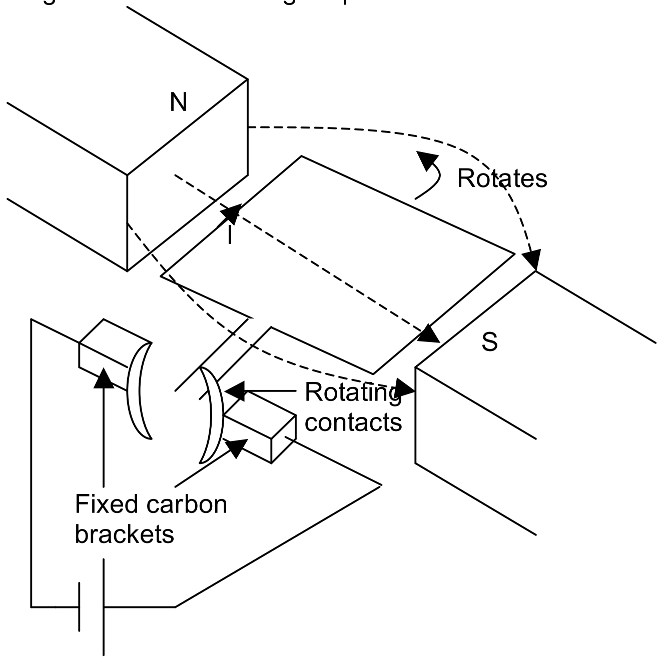
Magnetic materials

Many elements have no permanent magnetic dipole moment.

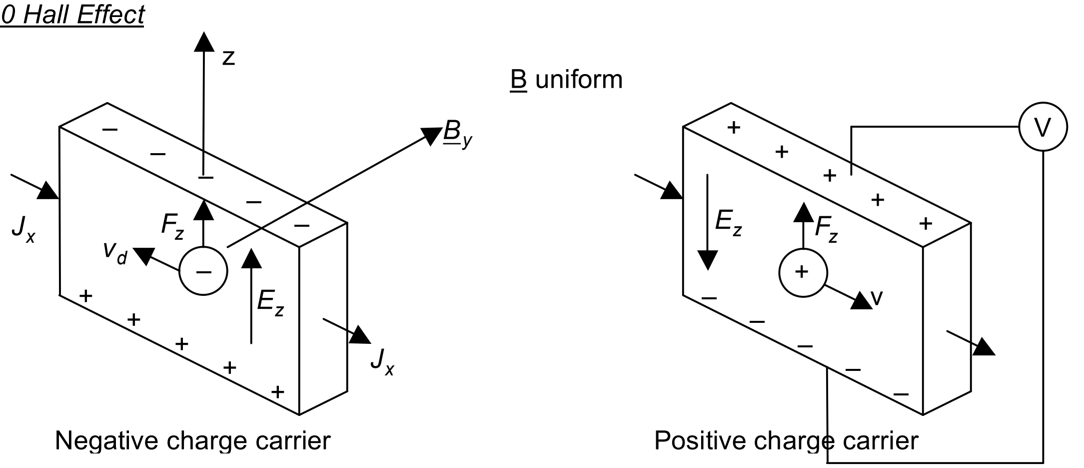
Iron has several permanent magnetic moments aligned. It can exist in many different states. In the unmagnetised state, there is random orientation of the dipoles. In the magnetized state, there is alignment of the dipoles.

5.9 Direct Current Motor

Most motors >50W, e.g. vacuum cleaner, drill, mixer, … These all work off AC because the magnet is an electromagnet powered off the same AC.



5.10 Hall Effect



$F_z = qv_d B$

Net transverse force:

$F = qE_z + qv_d B_y = 0$

$\therefore E_z = -v_d B_y$

q is positive, E_z is negative.

The current density $J = nqv_d$

Therefore $nq = -\dfrac{J_x B_y}{E_z}$.

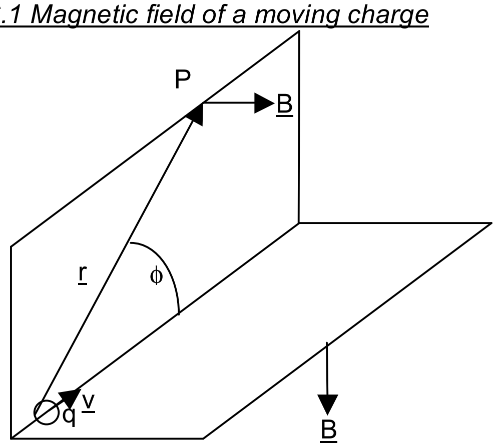
For a metal, n is large and $q \simeq e$. v_d is very small, and the Hall effect is small. For semi-conductors, n is relatively small and v_d is large, and the Hall Effect is relatively large.

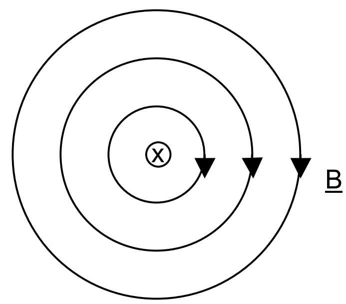
6. Sources of B

(Chapter 29, Young and Freedman)

- Moving charges or a current-carrying conductor experience a force in a magnetic field.
- If the magnetic field is static, it alters the velocity but not the speed or energy of the moving charge. Due to the conservation of momentum, there must be a reaction back on the system producing the magnetic field.
- The magnetic field is created by moving charge or current. Therefore deflection of a moving charge in a magnetic field is really moving charges interacting with each other. There is symmetry just as for static charge to static charge interactions.

6.1 Magnetic field of a moving charge





$$|B| \propto |q|, \frac{1}{r^2}, \sin\phi, v$$

Introduce a constant of proportionality:

$$B = \left(\frac{\mu_o}{4\pi}\right)\frac{|q| v \sin\phi}{r^2}$$

Making this into a vector equation:

$$\underline{B} = \left(\frac{\mu_o}{4\pi}\right) q \frac{\underline{v} x \underline{\hat{r}}}{r^2}$$

This is the magnetic field of a charge q moving with velocity $\underline{v}$. $\underline{\hat{r}}$ is the position of the point where $\underline{B}$ is determined with respect to the position of the charge.
Of course, the charge also produces an electric field. This is neglected here.
Note the symmetry: the $\underline{B}$ field forms circles about the direction of motion.

Constant of proportionality:

$$\frac{\mu_o}{4\pi}$$

$$1T = 1NsC^{-1}m^{-1}$$
$$= 1NA^{-1}m^{-1}$$

Units of μ_o are $Ns^2C^{-2} = NA^{-2} = WbA^{-1}m^{-1} = TmA^{-1} = Hm^{-1}$

H is the Henry unit of induction.

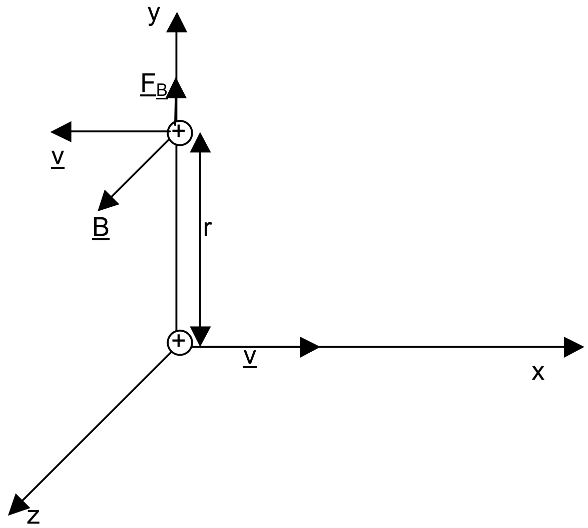
μ_o is defined to be $4\pi x 10^{-7} Hm^{-1}$

Remember $\varepsilon_o = 8.85 x 10^{-12} Fm^{-1}$ (definition)

$$\frac{1}{\sqrt{\varepsilon_o \mu_o}} = 3x10^8 ms^{-1} = c$$ the velocity of light in a vacuum.

Example 6-1

Force between two photons a distance r apart, moving in opposite directions with velocity v.



$\underline{F_E}$ is in the +y direction on the upper charge/.

$\underline{F_B}$ is also upwards.

$F_E = \frac{1}{4\pi\varepsilon_o}\frac{q^2}{r^2}$ repulsive.

Lower charge: $\underline{v} = v\hat{\underline{i}}$ and $\hat{\underline{r}} = \hat{\underline{j}}$

At upper charge: $\underline{B} = \left(\frac{\mu_o}{4\pi}\right)q\frac{v\hat{\underline{i}}x\hat{\underline{j}}}{r^2} = \frac{\mu_o}{4\pi}\frac{qv}{r^2}\hat{\underline{k}}$.

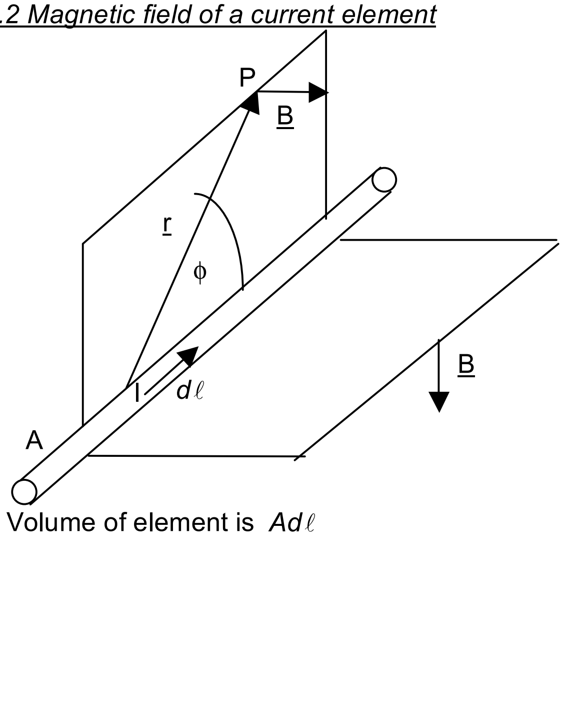
Force on the upper charge due to B created by the motion of the lower charge:

$\underline{F_B} = q\left(-v\hat{\underline{i}}\right)x\underline{B} = \frac{\mu_o}{4\pi}\frac{q^2v^2}{r^2}\hat{\underline{j}}$

This is also repulsive and is the inverse square.

Note $\frac{F_B}{F_E} = \varepsilon_o\mu_o v^2 = \frac{v^2}{c^2}$

6.2 Magnetic field of a current element



Volume of element is $Ad\ell$

Therefore the moving charge is:
$dQ = nAqd\ell$ which moves with a drift velocity v_d.
n is the charge carrier density
q is the charge of the carrier.

At point P:

$$dB = \left(\frac{\mu_o}{4\pi}\right)|dQ|\frac{v_d \sin\phi}{r^2}$$

$$= \left(\frac{\mu_o}{4\pi}\right)n|q|\frac{v_d A d\ell \sin\phi}{r^2}$$

But $n|q|v_d A = I$ the current in the element.

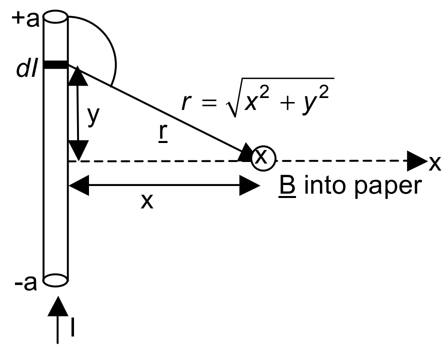
$$\therefore dB = \left(\frac{\mu_o}{4\pi}\right)I\frac{d\ell \sin\phi}{r^2}$$

$$\underline{dB} = \left(\frac{\mu_o}{4\pi}\right)I\frac{\underline{d\ell} x \hat{\underline{r}}}{r^2}$$

This is called the Biot-Savart Law.

For the total field, $\underline{B} = \left(\frac{\mu_o}{4\pi}\right)\int I\frac{\underline{d\ell} x \hat{\underline{r}}}{r^2}$.

6.3 Magnetic field due to current I in a straight conductor



dB is perpendicular to dl and r. Therefore is into the paper.
$dl = dy$

$$\sin(\pi - \phi) = \sin\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$B = \frac{\mu_o I}{4\pi}\int_{-a}^{a}\frac{dl \sin\phi}{r^2} = \frac{\mu_o I}{4\pi}\int_{-a}^{a}\frac{xdy}{\left(x^2 + y^2\right)^{3/2}} = \frac{\mu_o I}{4\pi}\frac{2a}{x\left(x^2 + a^2\right)^{1/2}}$$

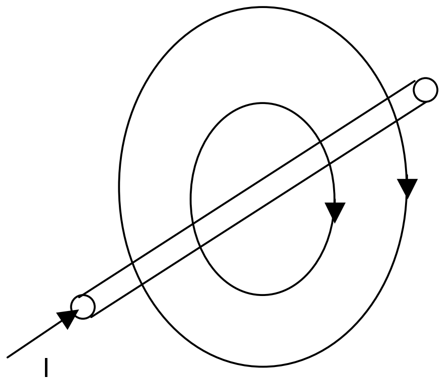
When $2a >> x$ (infinite wire approximation)

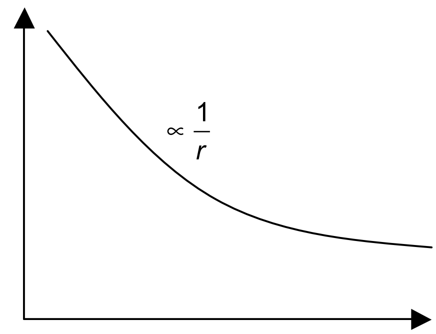
$$\left(x^2 + y^2\right)^{1/2} \approx a$$

$$\therefore B \approx \frac{\mu_o I}{2\pi x}$$

But because of axial symmetry:

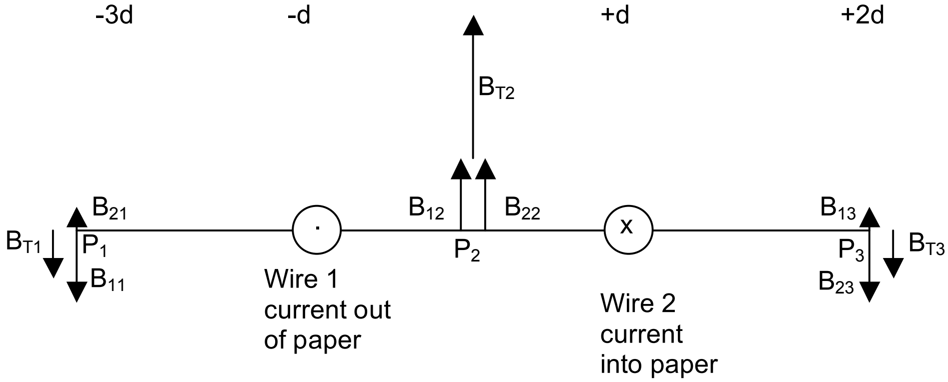
$B = \frac{\mu_o I}{2\pi r}$ where r is the distance of the point off the axis.





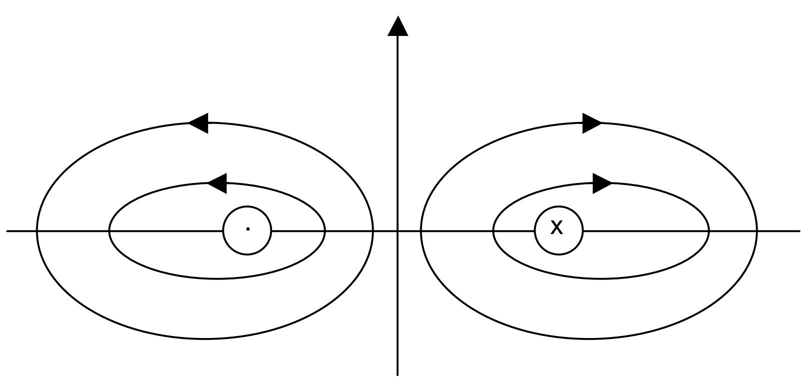
Example 6-2

2 long parallel wire, currents in opposite directions.

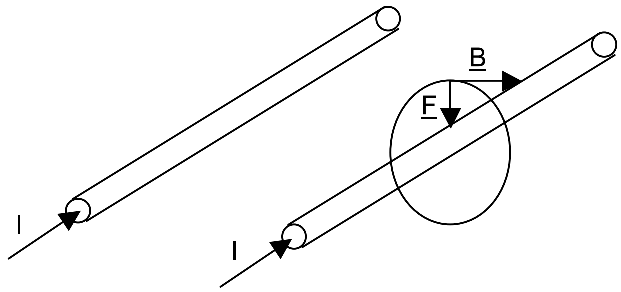


B_{ij} − i = wire, j = point (T = total)

At point P_1,	At point P_2,	At point P_3,
$B_{11} = -\frac{\mu_o I}{2\pi(2d)}$	$B_{12} = \frac{\mu_o I}{2\pi d}$	$B_{13} = \frac{\mu_o I}{2\pi(3d)}$
$B_{21} = \frac{\mu_o I}{2\pi(4d)}$	$B_{22} = \frac{\mu_o I}{2\pi d}$	$B_{23} = -\frac{\mu_o I}{2\pi d}$
$B_{T1} = -\frac{\mu_o I}{8\pi d}$	$B_{T2} = \frac{\mu_o I}{\pi d}$	$B_{T3} = -\frac{\mu_o I}{3\pi d}$



6.4 Force between parallel conductors carrying current (long wires)



B is the field at the upper wire due to the current I in the lower wire.

$$|\underline{B}| = \frac{\mu_o I}{2\pi r}$$

Force on length L of the upper wire is:

$$F = I'LB = \frac{\mu_o}{2\pi r} II'L$$

Therefore the force per unit length is:

$$\frac{F}{L} = \frac{\mu_o II'}{2\pi r}$$

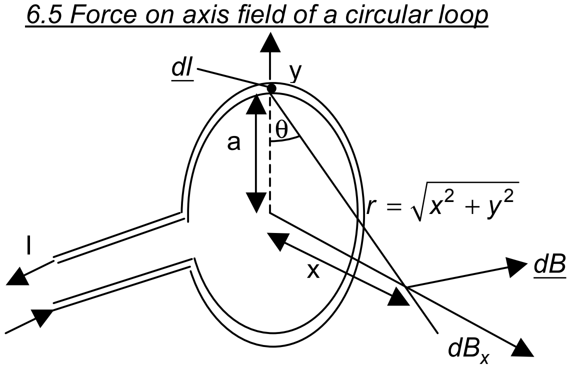
Equal and opposite forces act on the lower wire.
Currents in same sense attract – “pinch effect”.
Currents in opposite senses repel.
An example:

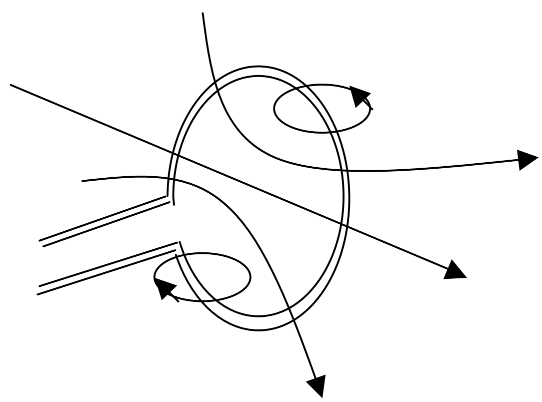
$$I = I'$$

$$r = 1m$$

$$\frac{F}{L} = 2x10^{-7}\,Nm^{-1}$$

6.5 Force on axis field of a circular loop





Neglect the connecting wires.

$$dB = \frac{\mu_o I}{4\pi}\frac{dl}{x^2 + a^2}$$

$$dB_x = dB\cos\theta$$

$$dB_y = dB\sin\theta$$

$$\left(\begin{array}{l} \cos\theta = \dfrac{a}{\sqrt{x^2 + a^2}} \\ \sin\theta = \dfrac{x}{\sqrt{x^2 + a^2}} \end{array}\right)$$

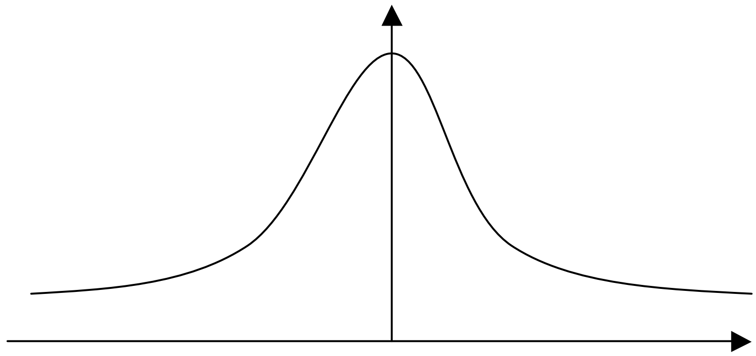
By symmetry, for the whole loop there is no net B_y.

Therefore

$$B = B_x = \int \frac{\mu_o I}{4\pi} \frac{a dl}{\left(a^2 + x^2\right)^{3/2}} = \frac{\mu_o I}{4\pi} \frac{a}{\left(a^2 + x^2\right)^{3/2}} \int dl = \frac{\mu_o I a^2}{2\left(a^2 + x^2\right)^{3/2}} = \frac{\mu_o \mu}{2\pi\left(a^2 + x^2\right)^{3/2}}$$

where $\mu = I\pi a^2$ the magnetic dipole moment of the loop. NB: $\int dl = 3\pi a$

On axis field:



At the centre, $x = 0$. $B = \frac{\mu_o \mu}{2\pi a^3} = \frac{\mu_o I}{2a}$

For $x >> a$, $B = \frac{\mu_o \mu}{2\pi x^3} = \frac{\mu_o I a^2}{2x^3}$ i.e. $B \propto \frac{1}{r^3}$ (on axis).

Compare with the electric dipole.

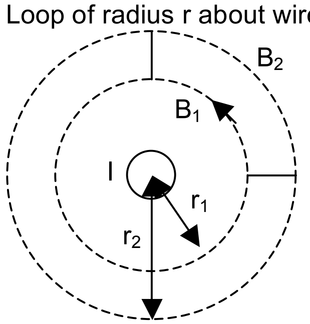
6.6 Ampere's Law

Using the Biot-Savart law, we found B for an infinite straight wire carrying current I:

$B = \frac{\mu_o I}{2\pi r}$ at radius r.

Take $\oint \underline{B} \cdot \underline{dl} = 2\pi r B = \mu_o I$

Loop of radius r about wire



$$B_1 = \frac{\mu_o I}{2\pi r_1}$$

$$B_2 = \frac{\mu_o I}{2\pi r_2}$$

$$\therefore \oint \underline{B} \cdot \underline{dl} = B_1 \frac{\pi r_1}{2} + B_2\left(-\frac{\pi r_2}{2}\right) = 0$$

First path enclosed the current → $\mu_o I$

Second path did not enclose the current → 0.

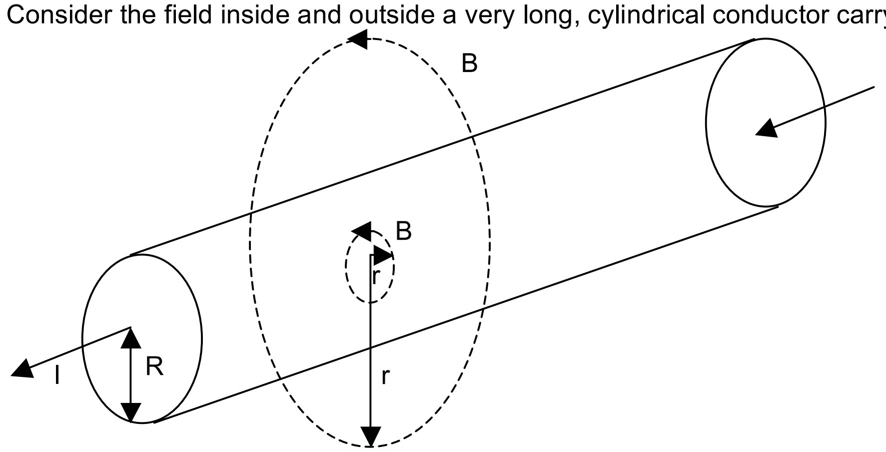
The actual path does not matter.

$$\oint_{loop} \underline{B} \cdot \underline{dl} = \mu_o I_{enclosed}$$ Ampere's Law

I is the current enclosed by the loop.

Example 6-3

Consider the field inside and outside a very long, cylindrical conductor carrying a charge.



Around coaxial circles $\oint \underline{B} \cdot \underline{dl} = B 2\pi r$

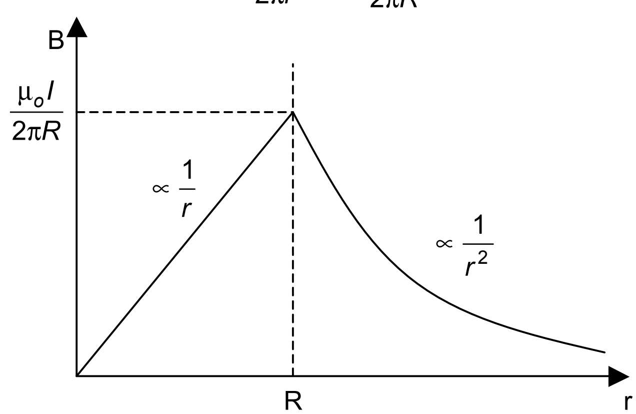
For large circles $(r \geq R)$ $I_{enclosed} = I$

Therefore $B_{outside}(r \geq R) = \dfrac{\mu_o I}{2\pi r}$

For the small path $(r < R)$

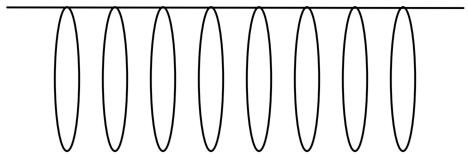
$I_{enclosed} = I \dfrac{r^2}{R^2}$ assuming uniform current density

Therefore $B_{inside} = \dfrac{\mu_o I_{enclosed}}{2\pi r} = \dfrac{\mu_o I}{2\pi R^2} r$

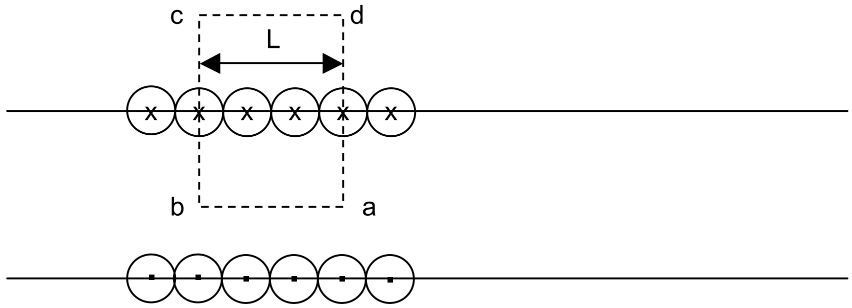


Example 6-4

Field of a Solenoid (long):
A Solenoid is a helical coil of cylindrical form, with n turns per unit length.



Regard as having n adjacent circular current loops per unit length.
Consider central part of the very long solenoid.

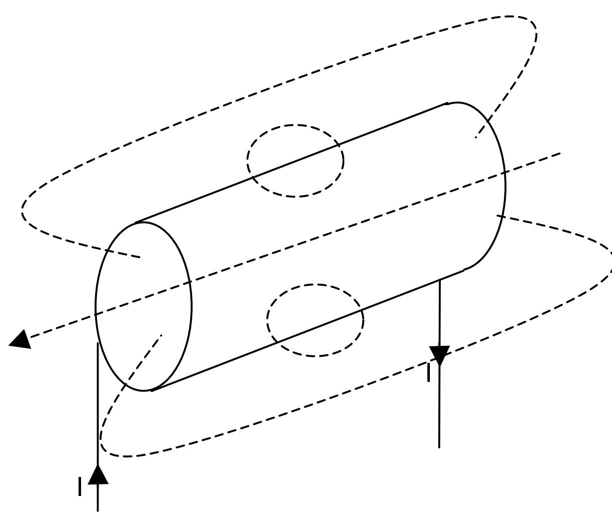


Appealing to symmetry;
B_{inside} is in the ← direction.
$B_{outside}$ might be in the → direction.
If loop abcd is outside the solenoid, there is no current enclosed therefore $B_{outside} = 0$.
For the loop straddling the solenoid;

$$\oint \underline{B} \cdot \underline{dl} = B_{inside} L = \mu_o I n L$$

$$\therefore B_{inside} = \mu_o n I$$

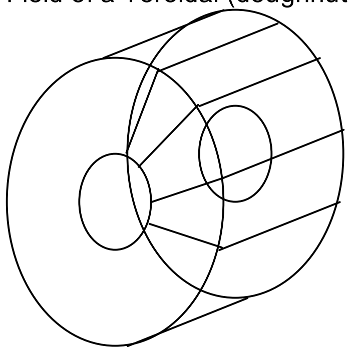
and it is the same everywhere inside.
For a solenoid of finite length:

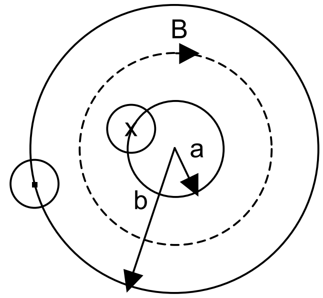


Could use Biot-Savart law. Solution on axis. Numerical solution off axis.

Example 6-5

Field of a Toroidal (doughnut) solenoid





Total of N turns.

$r > b \quad \oint \underline{B}\cdot\underline{dl} = B2\pi r = 0$

$r < a \quad \oint \underline{B}\cdot\underline{dl} = 0$

This is because there is no current enclosed.

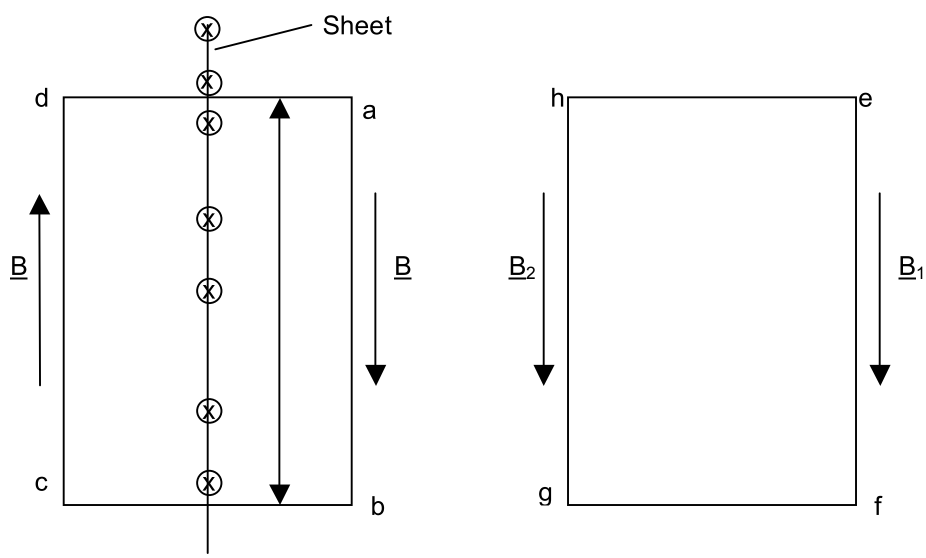
$a < r < b \quad \oint \underline{B}\cdot\underline{dl} = B2\pi r = \mu_o nI$

Therefore $B = \dfrac{\mu_o nI}{2\pi r}$ everywhere inside the toroid.

We can use Biot-Savart generally, but you have to integrate which can be complicated.
You can derive Ampere's law from Biot-Savart. Ampere's law can be used in problems where there is some sort of symmetry – this eases the calculations.
Have to know that B is uniform round the loop (or perpendicular to $\underline{dl}$).
Ampere's Law is important because Maxwell found an error – extra "displacement" current which needs to be added..
Corrected Ampere's law analogous to Faraday's law → EM waves.

<u>Example 6-6</u>

Infinite current sheet $k\ Am^{-1}$



Loop 1:

$$\oint \underline{B} \cdot \underline{dl} = Bl + 0 + (-B)(-L) + 0 = 2BL = \mu_o kL$$

$$B = \frac{\mu_o k}{2}$$

Loop 2:

$$\oint \underline{B} \cdot \underline{dl} = B_1 L + 0 + B_2(-L) + 0 = (B_1 - B_2)L = \mu_o 0 = 0$$

$$\therefore B_1 = B_2$$

Electric	Magnetic
Coulomb's Law $\underline{dE} = \frac{1}{4\pi\varepsilon_o}\frac{dq\hat{r}}{r^2}$	Biot-Savart $\underline{dB} = \left(\frac{\mu_o}{4\pi}\right) I \frac{dl x \hat{r}}{r^2}$
Gauss's Law $\Phi_E = \int \underline{E} \cdot \underline{dA} = \frac{Q_{enclosed}}{\varepsilon_o}$	Ampere's Law $\oint \underline{B} \cdot \underline{dl} = \mu_o I_{enclosed}$
Infinite line charge with $\lambda\ Cm^{-1}$ $E = \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{2\lambda}{r}\right)$	Infinite current $I\ Cs^{-1}$ $B = \frac{\mu_o}{4\pi}\left(\frac{2I}{r}\right)$
Infinite charged sheet $\sigma\ Cm^{-2}$ $E = \left(\frac{1}{4\pi\varepsilon_o}\right)(2\pi\sigma)$	Infinite current sheet $K\ Am^{-1}$ $B = \left(\frac{\mu_o}{4\pi}\right)(2\pi K)$

For fields diverging in 2 "dimensions" → $\frac{1}{r^2}$

Fields diverging in 1 "dimensions" → $\frac{1}{r}$

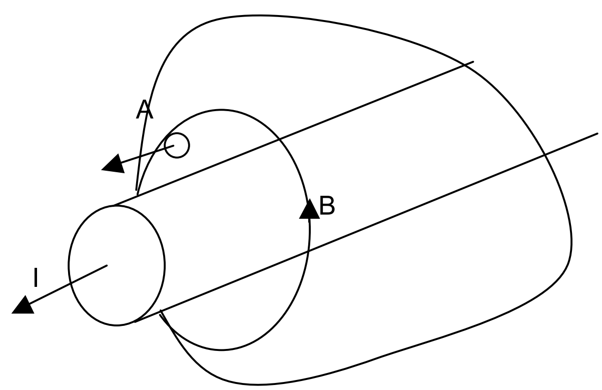
Fields not diverging at all → no dependence on r.

29.10 Displacement Current

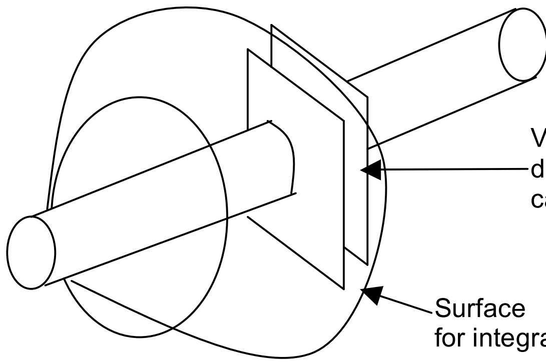
If the current is spread over an area as a current density J $\left(Am^{-2}\right)$:

$$\oint \underline{B} \cdot \underline{dl} = \mu_o I_{enclosed} = \mu_o \int \underline{J} \cdot \underline{dA}$$

The last integral is any surface which has the loop as a perimeter.



Maxwell pointed out a problem with Ampere's law.



No current through the surface but there is a time varying electric field.
E.g. parallel plate capacitor.

$q = CV = \left(\frac{\varepsilon_o A}{d}\right) Ed = \varepsilon_o AE = \varepsilon_o \Phi_E$

Charging current $I_c = \frac{dq}{dt} = \varepsilon_o \frac{d\Phi_E}{dt}$

Maxwell defined this to be a "displacement current" $I_D = \varepsilon_o \frac{d\Phi_E}{dt}$ and modified Ampere's Law to be $\oint \underline{B} \cdot \underline{dl} = \mu_o (I_c + I_D)$

The first current is the conduction current (the current in the wire which can be measured), while the second is the displacement current (which is fictitious).

Note also that the displacement current density $J_D = \frac{I_D}{A} = \varepsilon_o \frac{dE}{dt}$

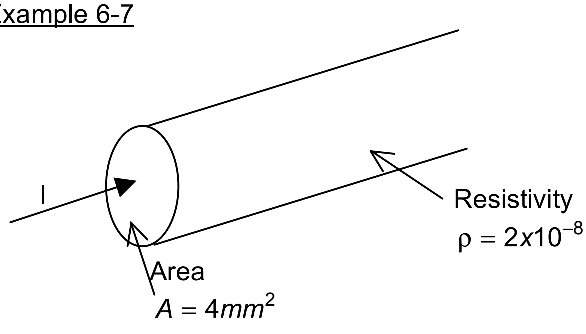
With a dielectric in the capacitor, part of the displacement current is due to the charge movement as atoms polarise. <u>But</u> I_D is in a vacuum.

Faraday's law $\frac{dB}{dt} \rightarrow E$

Maxwell $\frac{dE}{dt} \rightarrow B$

This symmetry leads to EM waves.

<u>Example 6-7</u>



a) What is <u>E</u>?

$R = \frac{\rho L}{A}$, $V = IR = \frac{I\rho L}{A}$

Therefore $E = \frac{V}{L} = \frac{I\rho}{A} = 0.15Vm^{-1}$

b) If $\frac{dI}{dt} = 6000 As^{-1}$ what is $\frac{dE}{dt}$?

$$\frac{dE}{dt} = \frac{\rho}{A}\frac{dI}{dt} = 30 Vm^{-1}s^{-1}$$

c) What is I_D?

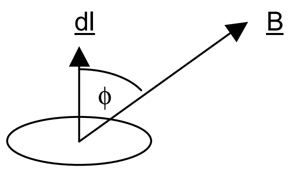
$$I_D = \varepsilon_o \frac{d\Phi_E}{dt} = \varepsilon_o A \frac{dE}{dt} = \varepsilon_o \rho \frac{dI}{dt} \sim 10^{-15} A$$

7. Magnetic Induction

(Chapter 30, Young and Freedman)

We know that if we move a magnet near a coil, we get an induced EMF. If you change a current on one coil, you induce an EMF in a nearby coil (mutual inductance) and an EMF in the first coil (self inductance)

7.1 Faraday's Law



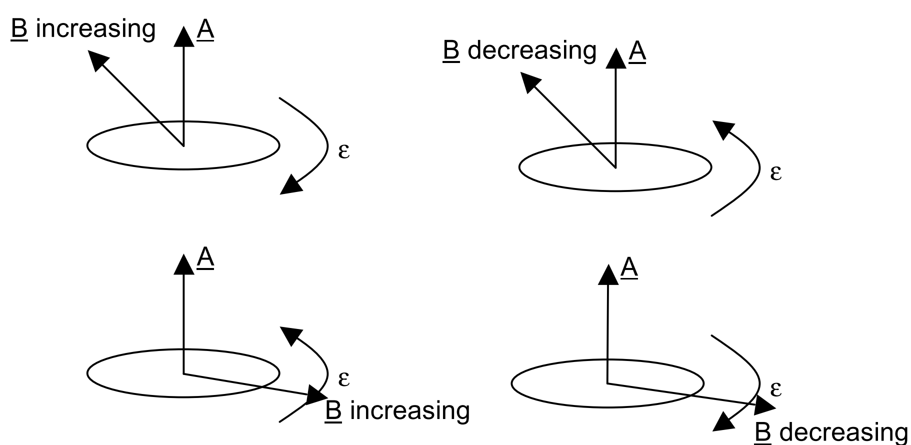
Magnetic flux:

$$d\Phi_B = \underline{B} \cdot \underline{dA}$$
$$B_\perp dA$$
$$= BdA\cos\phi$$

$$\Phi_B = \int d\Phi_B = \int \underline{B} \cdot \underline{dA}$$

Faraday's law states that the EMF induced in a closed loop is equal to the minus of the rate of change of magnetic flux through the loop i.e. $\varepsilon = -\frac{d\Phi_B}{dt}$ where ε is the EMF. Alternatively, $\varepsilon = -N\frac{d\Phi_B}{dt}$ if the coil has N terms.

Direction of induced EMF:

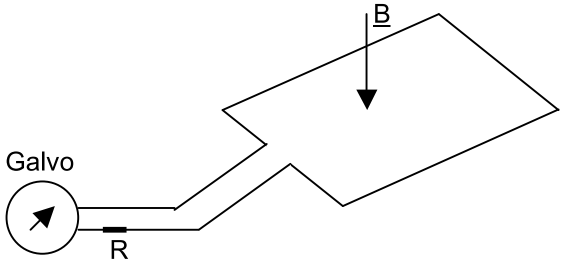


7.2 Examples

Example 7-1

Search coil to measure B.

A search coil is a coil of wire with N turns, area *A* in B. Resistance of the whole system is R.



Step 1: align B with A (Maximise Φ_B through the loop) $\Phi = BA$

Step 2: Quickly rotate the coil through 90 degrees. $\Phi \to 0$

EMF $\varepsilon = -NA\dfrac{dB}{dt}$

Current $I = \dfrac{\varepsilon}{R}$

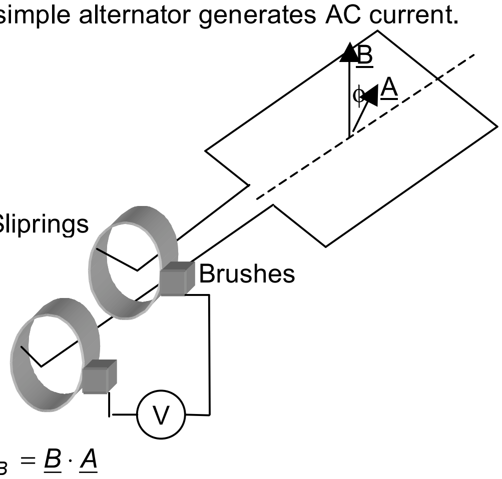
Therefore a charge Q flows through the galvanometer.

$$Q = \int Idt = -\frac{NA}{R}\int \frac{dB}{dt}dt = \frac{NAB}{R}$$

The special galvanometers calibrated in Q.

Example 7-2

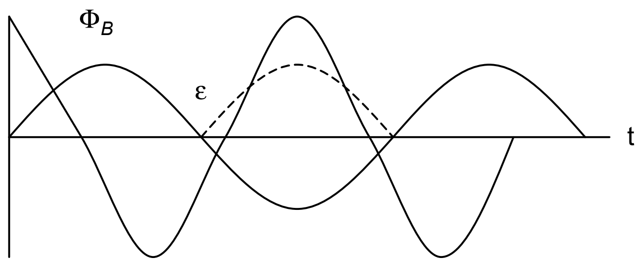
A simple alternator generates AC current.



$$\begin{aligned}\Phi_B &= \underline{B}\cdot\underline{A}\\ &= BA\cos\phi\end{aligned}$$

$$\begin{aligned}e &= -\frac{d\Phi_B}{dA}\\ &= \omega BA\sin(\omega t)\end{aligned}$$

$$\theta = \omega t$$



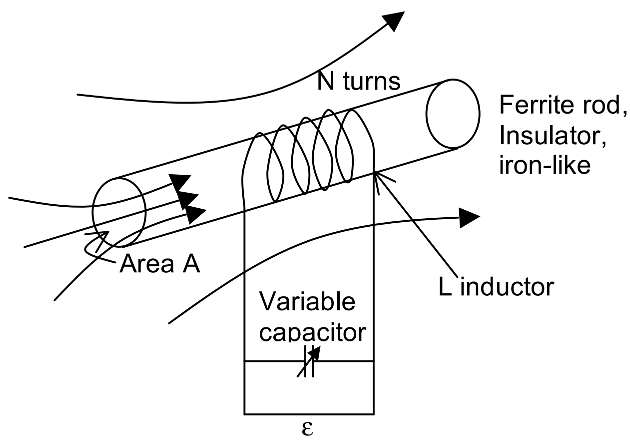
If the slip rings are replaced by a commulator, the path in dotted lines is followed.

Example 7-3

AM Radio

e.g. BBC 5 live, 909kHz.

$$\lambda = \frac{c}{f} = 330m$$



An LC resonant circuit is formed with $Q \approx 100$

$$|\varepsilon| = \left| N \frac{d\Phi_B}{dt} \right| = n \frac{d}{dt}(ABF)$$

F is the value by which the ferrite concentrates B in the rod.

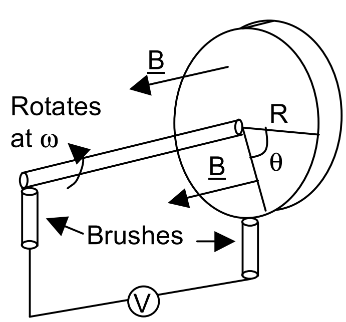
If $B = B_o \cos(2\pi ft)$ (at fixed position) and

$$B_o \approx 3.3x10^{-12} T,\ \ F \approx 10,\ \ N = 30,\ \ A = 2x10^{-4} m^2$$

$$|\varepsilon| \approx Q.1.1x10^{-6} V = 1.1x10^{-4} V$$

Example 7-4

Faraday's Disc Dynamo



$$Area\ A = \frac{R^2\theta}{2}$$

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

$$\Phi_B = BA$$

$$\frac{d\Phi_B}{dt} = B\frac{dA}{dt}$$

$$\frac{dA}{dt} = \frac{R^2}{2}\frac{d\theta}{dt} = \frac{R^2\omega}{2}$$

$$\therefore |\varepsilon| = |B|\frac{R^2\omega}{2}$$

e.g. $B = 1T,\ R = 0.1m,\ \omega = 300rs^{-1}\left(f = \frac{\omega}{2\pi} \sim 50Hz\right)$

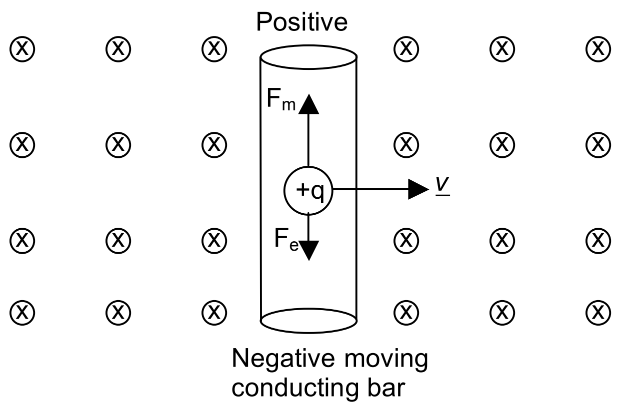
$\rightarrow |\varepsilon| = 1.5V$

7.3 Lenz's Law

"The direction of any magnetic induction effect is such as to oppose the cause of the effect"
E.g. braking torque on Faraday generator and examples on the sense of ε.

7.4 Motional EMF

All charges in any material moving in a magnetic field $\underline{B}$ experience magnetic forces. Only the free charge carriers in a conductor are free to move.



B uniform.
Charge builds up on the ends of the conductor to produce an internal electric field E. In equilibrium, $qvB = qE$, therefore the motional EMG between the ends of the bar $\omega = EL$ where L is the length of the bar. Therefore $\varepsilon = vBL$

Example 7-5

An airplane.

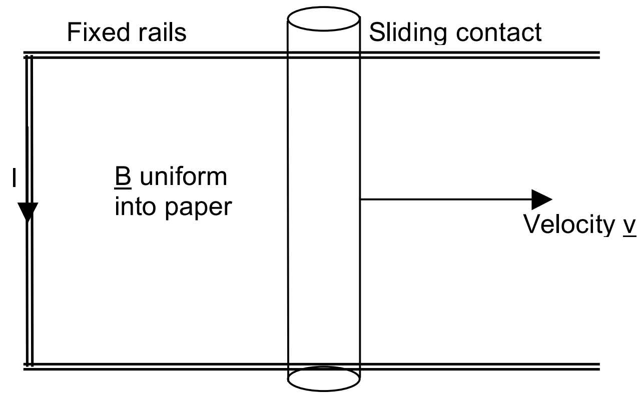
$v = 300ms^{-1}$

$B = 10^{-4}T$ (Earth's B field)

$L = 33m$ (using span).

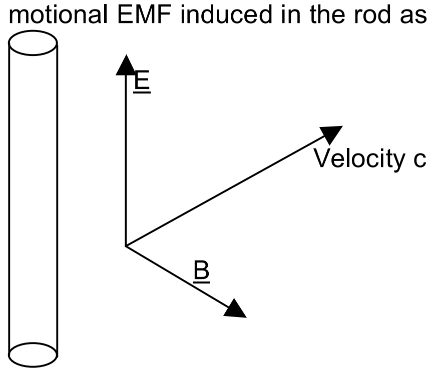
The motional EMF between the tips of the wings $\omega = vBL = 1volt$.

Even though ε appears across the ends of the bar it cannot be used for anything since moving connections to it would have the same motional EMF induced in it, meaning that there is no reason for charge to flow around the loop.
Now consider:



Example 7-6

Consider a rod antenna (length $L << \lambda$ wavelength) in the E field of an EM wave, parallel to the rod. The EMF induced in the rod $\varepsilon_E = EL$. We could equally well determine the motional EMF induced in the rod as the B component of the wave as it passes at velocity c.



Motional EMF $\varepsilon_B = cBL$

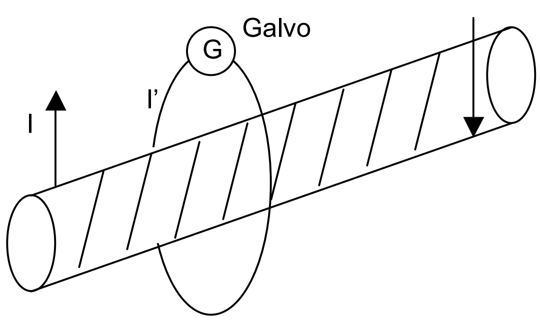
We want $\varepsilon_E = \varepsilon_B$

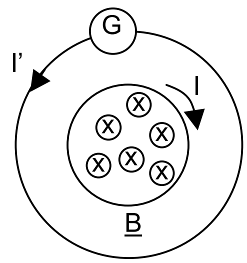
Therefore $\frac{E}{B} = C = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 3x10^8\, ms^{-1}$

This is true in a vacuum.

7.5 Induced Electric Fields

Long solenoid. $B_{inside} = \mu_o nI$ n turns per unit length.





To force a current I' through the loop there must be a tangential E field.

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\mu_o nA\frac{dI}{dt}$$

$$\oint \underline{E} \cdot \underline{dl} = \varepsilon$$

Therefore we can state Faraday's Law as:

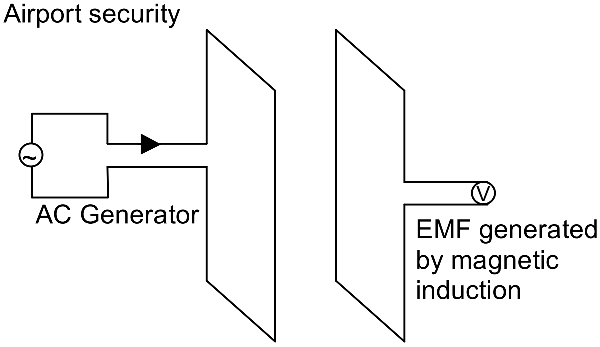
$$\oint \underline{E} \cdot \underline{dl} = -\frac{d\Phi_B}{dt}$$

7.6 Eddy Currents

Any conductor moving in B or experiencing changing B has EMFs induced in it. In the latter case, currents can flow. Where the induced currents are unwanted, they are called eddy currents, e.g. in electric motors and transformers. These have ferromagnetic cores (iron or steel) and are made of metal. Eddy currents cause heating, meaning that power is lost.

Example 7-7

Airport security



7.7 Maxwell's Equations

These are a collection of the fundamental equations of E&M.

Gauss's Law for E: $\oint \underline{E} \cdot \underline{dA} = \frac{Q_{enclosed}}{\varepsilon_o}$

Gauss's Law for B: $\oint \underline{B} \cdot \underline{dA} = 0$

Ampere's Law $\oint \underline{B} \cdot \underline{dl} = \mu_o\left[I_c + \varepsilon_o\frac{d\Phi_E}{dt}\right]$

Faraday's Law $\oint \underline{E} \cdot \underline{dl} = -\frac{d\Phi_B}{dt}$

In empty space, there are no charges or currents so:

Gauss's Law for E $\oint \underline{E} \cdot \underline{dA} = 0$

Gauss's Law for B $\oint \underline{B} \cdot \underline{dA} = 0$

Ampere's Law $\oint \underline{B} \cdot \underline{dl} = \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$

Faraday's Law $\oint \underline{E} \cdot \underline{dl} = -\frac{d\Phi_B}{dt}$

There is an obvious symmetry between the two Gauss's Laws and Ampere's and Faraday's laws, except for the sign and some constant.

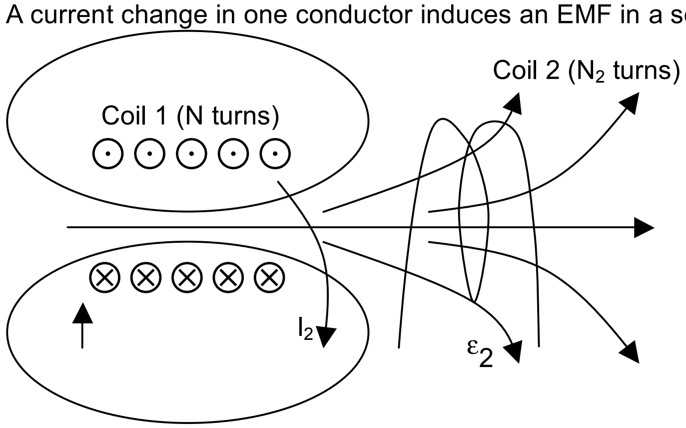
8. Inductance

(Chapter 31, Young and Freedman)

A current I in a conductor produces a magnetic field, and as a result there is a magnetic flux linking any nearby conductors. Changing I will result in EMFs being induced in all conductors nearby, including the original conductor.

8.1 Mutual Induction

A current change in one conductor induces an EMF in a second conductor.



The EMF induced in the second coil is $\varepsilon_2 = -N_2 \frac{d\Phi_{B2}}{dt}$ where Φ_{B2} is the magnetic flux through one turn of the second coil due to current I_1 in the first coil, as the coils are linked magnetically.

$N_2\Phi_{B2} \propto I_1$

$= M_{21} I_1$

where M_{21} is the mutual inductance. $N_2\Phi_{B2}$ is the total magnetic flux through the second coil.

$M_{21} = \frac{N_2\Phi_{B2}}{I_1} =$ total flux through the second coil due to I_1.

Therefore $\varepsilon_2 = -N_2 \frac{d\Phi_B}{dt} = -M_{21}\frac{dI_1}{dt}$

Similarly, a changing current I_2 n coil 2 induces an EMF ε_1 in coil 1.

$\varepsilon_1 = -N_1 \frac{d\Phi_{B1}}{dt} = -M_{12}\frac{dI_2}{dt}$

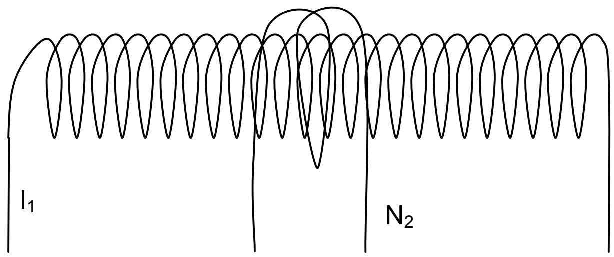
This is not self-evident but $M_{12} = M_{21} = M$.

Therefore $\varepsilon_2 = -M\frac{dI_1}{dt}$ and $\varepsilon_1 = -M\frac{dI_2}{dt}$ where $M = \frac{N_2\Phi_{B2}}{I_1} = \frac{N_1\Phi_{B1}}{I_2}$.

Unit of inductance is the Henry.

$1H = 1WbA^{-1} = 1VsA^{-1} = 1\Omega s$

Example 8-1



Treat this as an infinitely long solenoid.

$B_{outside} = 0$

$B_{inside} \approx \mu_o \left(\frac{N_1}{L}\right) I_1$

Magnetic flux through 1 turn of the second coil is $\Phi_{B2} = BA = \mu_o \left(\frac{N_1}{L}\right) A I_1$

Therefore $M = \frac{N_2 \Phi_{B2}}{I_1} = \mu_o \left(\frac{N_1}{L}\right) N_2 A$.

e.g.

$L = 0.5m$

$A = 10^{-3} m^2$

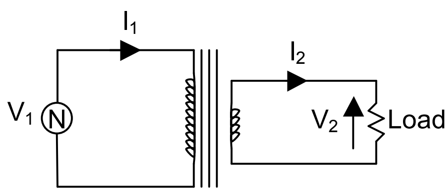
$N_1 = 1000$

$N_2 = 10$

$\rightarrow M = 25x10^{-6} H = 25\mu H$

Applications:

Transformer – maximizes flux linkage as far as possible.



$V_2 = \frac{N_2}{N_1} V_1$

$I_2 = \frac{N_1}{N_2} I_1$

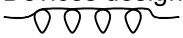
$V_2 I_2 = \frac{N_2}{N_1} V_1 \frac{N_1}{N_2} I_1 = V_1 I_1$.

8.2 Self Induction

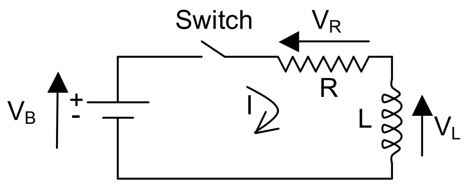
If you try to change the current I through a coil, you will change the magnetic flux giving an induced EMF.

The induced EMF $\varepsilon = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$. L is the self-inductance of the coil.

$L = \frac{total\ flux}{current} = \frac{N\Phi_B}{I}$.

Devices designed to exhibit inductance are called inductors, which are represented by  (a schematic of a coil). Real inductors have "winding" resistances i.e. the wire from which they are wound has finite resistance – equivalent to , with finite Q.
What is the sense of EMFs in L and M?
For L – it tends to oppose the change in I so it helps to keep I flowing.
For M – it tends to produce I in secondary to produce a magnetic field B which opposes the change in Φ_B .

Example 8-2



Close switch at $t = 0$. $I = 0$ for $t < 0$.

$$V_B = V_R + V_L$$

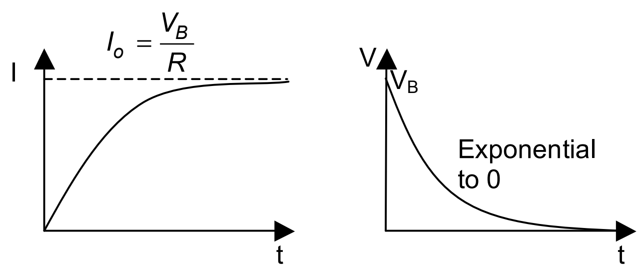
$$= IR + L\frac{dI}{dt}$$

Solution:

$$I = I_o\left(1 - e^{\left(-\frac{t}{2}\right)}\right)$$

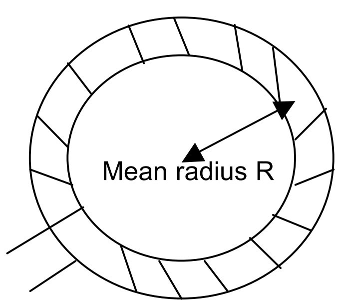
where

$$I_0 = \frac{V_B}{R} \quad \tau = \frac{L}{R}\text{g}$$



Example 8-3

Self-Inductance of a toroidal coil.



Apply Ampere's Law:

$$\oint \underline{B} \cdot \underline{dl} = \mu_o I_{enclosed}$$
$$= \mu_o NI$$

Assume B inside is uniform ($r >> \sqrt{A}$)
Therefore:

$$\oint \underline{B} \cdot \underline{dl} = B2\pi r$$

$$B = \frac{\mu_o NI}{2\pi r}$$

Φ_B through each turn is $\Phi_B \approx BA = \dfrac{\mu_o NIA}{2\pi r}$.

Therefore $L = \dfrac{N\Phi_B}{I} = \dfrac{\mu_o N^2 A}{2\pi r}$.

e.g.:
$N = 200$

$A = 5x10^{-4} m^2$

$r = 0.1m$

$\rightarrow L = 40\mu H = 40x10^{-6} H$

Example 8-4

Self inductance per unit length of a long solenoid, with length L, N turns and area A.

For an infinite solenoid, $B_{outside} = 0$ and $B_{inside} = \mu_o \left(\dfrac{N}{L}\right) I$.

Flux linking each turn $\Phi_B = B_{inside} A$.

Therefore for unit length (i.e. $\dfrac{N}{L}$ turns) the total flux linkage is $\dfrac{N}{L}\Phi_B = \mu_o \left(\dfrac{N}{L}\right)^2 AI$.

Therefore the inductance per unit length is $\dfrac{dL_{inductance}}{dL} = \mu_o \left(\dfrac{N}{L}\right)^2 A \;\left(Hm^{-1}\right)$.

8.3 Magnetic Field Energy

$P = VI$

$V = L\dfrac{dI}{dt}$ for an inductor.

During time dt

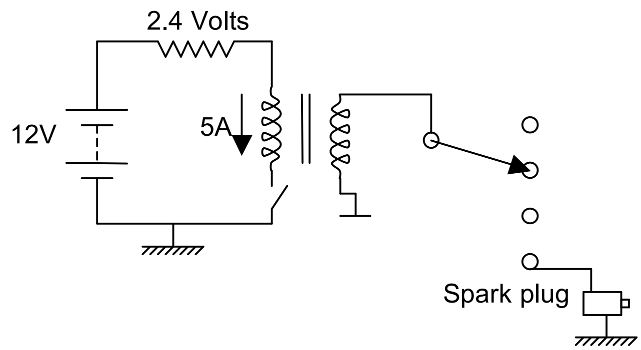
$$dU = Pdt$$
$$= LI\frac{dI}{dt}dt$$
$$= LIdI$$
$$U = \frac{1}{2}LI^2$$

(Energy stored in an inductor)

Compare this with $U = \dfrac{1}{2}CV^2$ for a capacitor.

Example 8-5

The ignition coil of a car. 200 sparks / second, 6000rpm, 4 cylinders.



Time constant $\tau = \frac{L}{R} < \frac{1}{200} \sec onds$

$L \leq \frac{2.4}{200} = 1.2x10^{-2} H$

Energy per spark $U = \frac{LI^2}{2} = 0.15J$ c.f. camera flash $U = 0.05J$.

8.4 Magnetic Energy Density

Consider the toroidal solenoid.
Volume occupied by $\underline{B} \approx A2\pi r$

Therefore the energy density $u = \frac{U}{2\pi rA} = \frac{1}{\pi rA}\frac{1}{2}LI^2 = \frac{1}{2}\frac{\mu_o N^2}{(2\pi r)^2}I^2$

But $B \approx \frac{\mu_o NI}{2\pi r}$

Therefore $u = \frac{1}{2}\frac{B^2}{\mu_o}$

This is a general result but it is modified in materials.

9. Course Summary

See PC 1342 Teaching Web.

Electrostatics

- Charges
  - Insulators and Conductors
  - Charges on materials
  - Conduction
- Superposition Principle
- Coulomb's Law
- Electric field
  - Electric dipoles, moments, torque, energy
- Electric flux Φ_E
- Gauss's Law
- Potential Energy
- Electric Potential
- Capacitance
  - Energy in a charged capacitor
  - Energy in the $\underline{E}$ field inside a capacitor

Magnetism

- Magnetic field B
- Force on a charge moving in a magnetic field
- Magnetic flux Φ_B
- Gauss's Law for B
- Force on a current element in a B field
  - Magnetic dipole moment of current I in loop of area A
  - Torque of magnetic dipole, energy.
- Magnetic field due to a moving charge
- Magnetic field due to a current element (Biot-Savart)
- Ampere's Law
  - Maxwell introduced displacement current → modified Ampere's Law
- Electromagnetic Induction
  - Faraday's Law
  - Lenz's Law
  - Motional EMF
- Maxwell's (Summery of) equations
  - Gauss's Law for E
  - Gauss's Law for B
  - Ampere's Law
  - Faraday's Law
- Self and Mutual Inductance
  - Energy in an inductor
  - Energy in a B field

10. Techniques

Look for symmetries, e.g. choose the surface to use while applying Gauss's Law - the integrations are trivial if you use the right surface.

Break the problem down to a simpler, more manageable problem e.g. a distributed charge.

Use a segment small enough to be regarded as a point charge.
$dq \rightarrow \underline{dE} \rightarrow \underline{E}$

You can attack problems from above or below.
$q \rightarrow \underline{E} \rightarrow V$
But given $\underline{E} \rightarrow \underline{V}$, or $V \rightarrow \underline{E}$. Sometimes you can also find q.