1. Introduction

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There are many examples of vibrating systems in everyday life, e.g. musical instruments, the clock pendulum, structural vibrations in bridges, earthquakes, vibrating molecules, etc. The beauty of this course is that we can often describe these diverse systems by relatively simple mathematics that we can solve.

There are also many lecture courses that you will take that depend on a thorough understanding of the physics of oscillating systems, for example quantum mechanics and also electromagnetism.

The aim of this course is to understand the physics of these systems, and to describe mathematically and quantify vibrations and waves.

1.1 Recommended Books

Vibrations and Waves, A.P. French (Course textbook)
Physics by Young & Freedman
Physics of Vibrations and Waves, H. J. Pain *
The Elements of Physics, Grant & Phillips

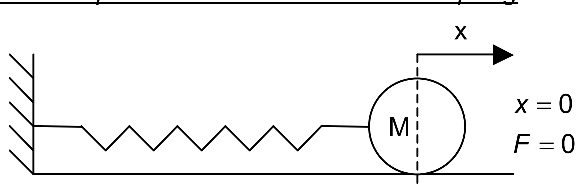
* Some tutorials and exam questions are taken from this book.

1.2 The Physical Characteristics of Oscillatory Motion

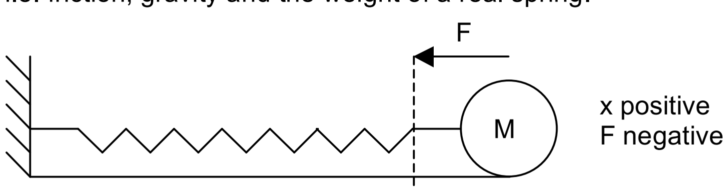
- Motion repeats periodically
- Equilibrium position
- Always a restoring force
- Overshoots the equilibrium
- Has inertia
- Maximum displacement but zero velocity at turning points
- Exchange of Potential Energy (E_p) and Kinetic Energy (E_k)
- Constant amplitude (Ideal system)

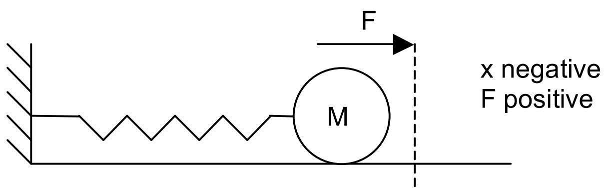
2. The Simple Harmonic Oscillator

2.1 Example of a mass on a horizontal spring



Assume a weightless spring and a frictionless surface.
NB: in physics we usually start with an idealized system and add the complications in later, i.e. friction, gravity and the weight of a real spring.





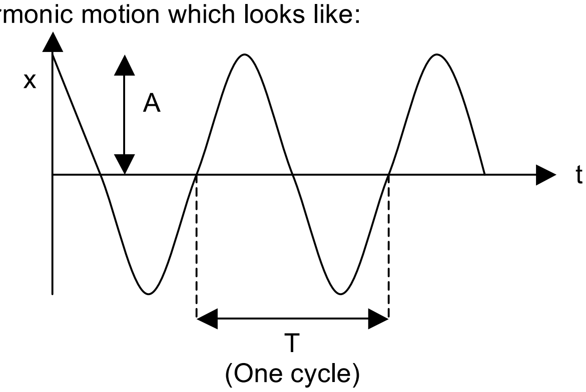
The force produced by the spring acts to push or pull the mass back to its equilibrium position. The restoring force, F, is given by $F = -kx$ (Hooke's Law), where x is the displacement and k the spring constant (i.e. the restoring force per unit displacement). The negative sign shows that the restoring force is in the opposite displacement to the displacement x.

NB: in many complicated situations e.g. diatomic molecules, Hooke's law is obeyed for small displacements, which is why our present discussion is so durable and far-reaching.

Aside: the correct form for larger distances is $F = -\left(k_1 x + k_2 x^2 + k_3 x^3 + \ldots\right)$. However, k_2 and k_3 are small compared to k_1, hence for small distances $F = -kx$ is true.

We also have Newton's law Force = Mass x Acceleration, or $a = \frac{d^2x}{dt^2} = \frac{F}{M} = -\frac{kx}{M}$.

This gives simple harmonic motion which looks like:



A: the amplitude of the oscillation
T: the period of the oscillation (i.e. time for one complete cycle)
f: Frequency of the oscillation $= \frac{1}{t}$ (Units of cycles.s^{-1} or Hertz Hz)

NB: Boundary condition $x = A$ at $t = 0$

SHM occurs when there is a restoring force (i.e. acting towards the equilibrium position) that is directly proportional to the displacement from equilibrium.

2.2 Solutions for x(t), v(y) and a(t)

We want expressions for displacement, velocity and acceleration as functions of time. Observing periodic motion of the mass on a spring we look for a function for x(t) that is also periodic – a cosine solution. So try $x = A\cos\left(\frac{2\pi t}{T}\right)$, where A is the amplitude.

NB: $\left(\frac{2\pi t}{T}\right)$ is the angle in radians.

As t goes from 0 → T, the angle goes from 0 to 2π.

Let $\omega = \frac{2\pi}{T}$ where ω is the angular frequency of the oscillator and is in radians/second.

We obtain $x = A\cos\omega t$.

Does this fit our observation? Yes. In particular $x = A$ at $t = 0$, i.e. boundary conditions.

$$\frac{dx}{dt} = \dot{x} = v(t) = -\omega A\sin\omega t$$

$$\frac{d^2x}{dt^2} = \ddot{x} = a(t) = -\omega^2 A\cos\omega t = -\omega^2 x$$

or $\ddot{x} = -\omega^2 x$

i.e. x(t) is a solution of the DE where $\omega^2 = \frac{k}{M}$

Some physics:

$$T = \frac{2\pi}{\omega}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\left(\frac{k}{M}\right)^{1/2}$$

The frequency is determined by the properties of the oscillator, k and M, and does not depend on the amplitude A of the oscillation. A depends on the boundary conditions.

$f \propto \frac{1}{m^{1/2}}$ ∴ heavier things vibrate at low frequency

$f \propto k^{1/2}$ ∴ the stronger the spring, the higher the frequency

Examples of frequencies and periods:

- Clock pendulum: 1Hz
- Middle C on a piano: 256Hz
- Crystal in watch / computer: MHz
- Molecular vibration: 10^{14}Hz
- Earthquake: several hours
- Oscillating universe: tens of billions of years

2.3 General solution for SHM

In general $x(t) \neq 0$ at $t = 0$ and motion looks like $A\cos(\omega t - \phi)$. Cosine curve has been displaced horizontally by angle ϕ, called the phase angle.

x(t) is described by $x = A\cos(\omega t - \phi)$ and doesn't reach its' maximum value until $\omega t = \phi$.

This is the general solution to our differential equation $\ddot{x} = -\omega^2 x$. Solutions to second order differential equations always contain two arbitrary constants.

NB: $\omega t \Rightarrow (\omega t + \phi)$ would shift the curve to the left.

Since $\cos(a - b) = \cos a\cos b + \sin a\sin b$:

$$A\cos(\omega t-\phi)=A\cos\omega t\cos\phi+A\sin\omega t\sin\phi$$
$$=a\cos\omega t+b\sin\omega t$$

where $a=A\cos\phi$ and $b=A\sin\phi$.

So the general solution of our differential equation can also be written as $x=a\cos\omega t+b\sin\omega t$ where a and b are again determined by the boundary conditions.

2.4 Energy considerations in SHM

Consideration of energy in a system (E_K and E_P) is a powerful tool in Vibrations and Waves for we are dealing with scalar rather than vector quantities.

For a mass on a spring $E_k=\frac{1}{2}mv^2$

$E_p=$ Work done in extending or compressing the spring $=F.d$

In an extension the force exerted by the spring is $F=-kx$, but the force exerted on the spring is $+kx$. Therefore $E_p=\int_0^x kxdx=\frac{1}{2}kx^2$.

Total energy must be a constant, say E.

$$E=E_p+E_k=\frac{1}{2}mv^2+\frac{1}{2}kx^2$$

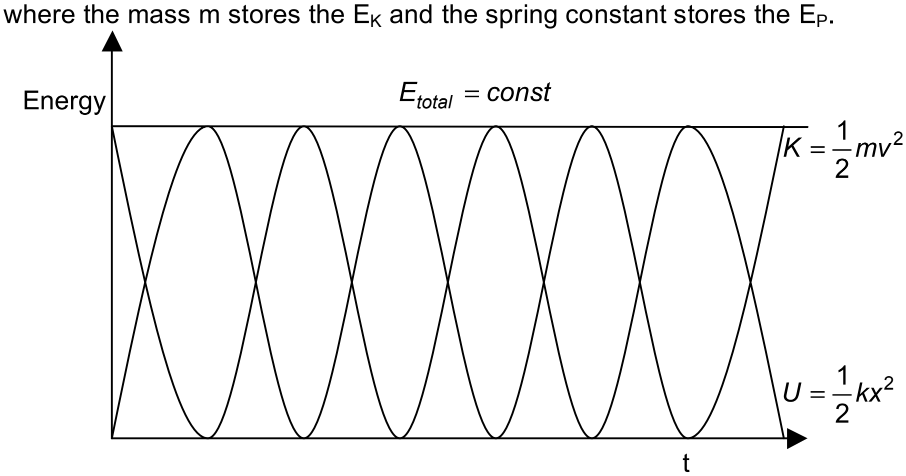
When mass is at maximum displacement i.e. x=A, E_k=0 since the mass is at rest. Therefore $E_p=E=\frac{1}{2}kA^2$ and $\frac{1}{2}mx^2+\frac{1}{2}kx^2=\frac{1}{2}kA^2$, which is true for any time t.

Taking $x=A\cos\omega t$ and so $v=-A\omega\sin\omega t$ gives $U(t)=\frac{1}{2}kx^2=\frac{1}{2}kA^2\cos^2\omega t$,

and $K(t)=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2A^2\sin\omega t$.

$$\therefore E=\frac{1}{2}kA^2\cos^2\omega t+\frac{1}{2}kA^2\sin\omega t.$$

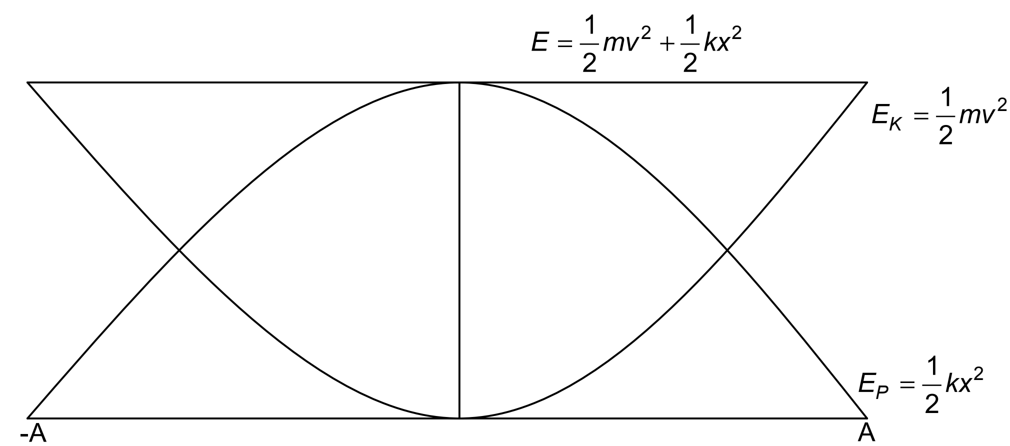
The energy flows between E_K and E_P. We can write total energy as $E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$, where the mass m stores the E_K and the spring constant stores the E_P.



NB: in general for a simple harmonic oscillator we have $E = \alpha(\dot{x})^2 + \beta(x)^2$ where α and β are constants. This is another signature of SHM.

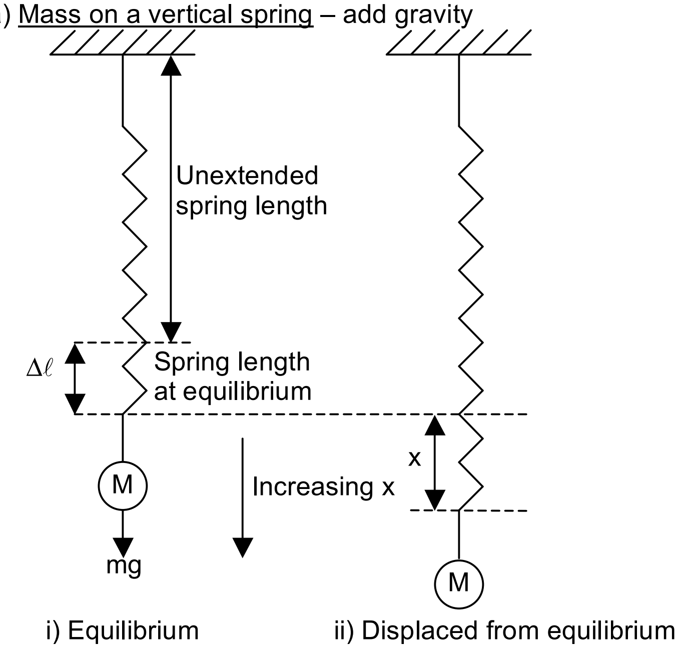
Aside: in thermal physics $\frac{1}{2}mv^2$ and $\frac{1}{2}kx^2$ are called 'degrees of freedom'. In thermal equilibrium they each have a mean value equal to $\frac{1}{2}k_BT$ where K_B is Boltzman's Constant. $k_BT \sim \frac{1}{40}eV$ at room temperature. This gives us "thermal jiggling".

Just for fun we can also plot E_p and E_k with respect to displacement.



2.5 Other examples of SHM

(a) Mass on a vertical spring – add gravity



When mass is attached to the spring, its' length is extended by $\Delta\ell$. Therefore the force on the mass at equilibrium is $F = mg - k\Delta\ell = 0$ at equilibrium.

When the mass is displaced downwards by distance x then:

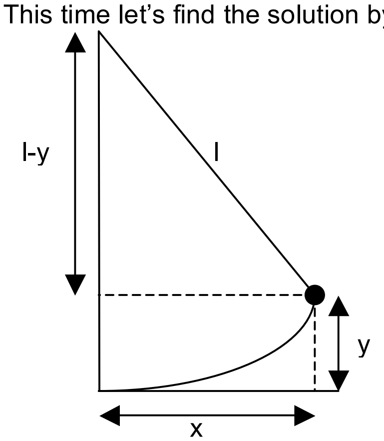
$F = ma = mg - k(\Delta\ell + x) = mg - k\Delta\ell - kx$

i.e. $m\ddot{x} = -kx$

Therefore SHM with $\omega = \sqrt{\frac{k}{m}}$ as before.

(b) The simple pendulum

This time let's find the solution by considering the energy of the system.



We have $l^2 = (l-y)^2 + x^2 \rightarrow l^2 = l^2 + y^2 - 2ly + x^2$ and so $2ly = y^2 + x^2$.

For small θ, x>>y so that y^2 is negligible giving us $y = \frac{x^2}{2l}$.

The total energy $E = \frac{1}{2}mv^2 + \frac{1}{2}mg\left(\frac{x^2}{l}\right)$.

Note similarity. $E = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}\frac{mg}{l}x^2$ with $E = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2$ - mass on a spring.

This is our signature of SHM.

When $x = A$ (Amplitude), $v = 0 \rightarrow E = \frac{1}{2}mg\frac{A^2}{l}$.

Therefore $\frac{1}{2}mg\frac{A^2}{l} = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}mg\frac{x^2}{l}$ true for all times.

$\therefore A^2 = \frac{l}{g}v^2 + x^2$

We need to solve this for x.

Rearranging $A^2 = \frac{l}{g}\left(\frac{dx}{dt}\right)^2 + x^2$

Giving us $\frac{dx}{dt} = \sqrt{\frac{g}{l}}\left(A^2 - x^2\right)^{1/2}$

Therefore $\int \frac{dx}{\left(A^2 - x^2\right)^{1/2}} = \sqrt{\frac{g}{l}}\int dt$

The first integral is a standard integral, equal to $\arcsin\left(\frac{x}{A}\right)$.

Therefore $\arcsin\left(\frac{x}{A}\right) = \sqrt{\frac{g}{l}}t + \phi$.

ϕ is the constant of integration which we recognize as the phase angle.

Therefore $x = A\sin(\omega t + \phi)$ i.e. SHM with $\omega = \sqrt{\frac{t}{l}}$ and $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$.

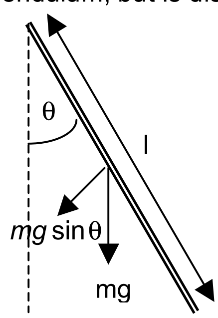
Measuring the period is a good way of determining g.

Note that for l=1m, $T = 2\pi = \sqrt{\frac{1}{9.81}} = 2.006s$.

NB: the original definition of a second was one half period of a 1m simple pendulum.

(c) <u>The physical pendulum</u>

In a physical pendulum the mass is not concentrated at one point, as in the simple pendulum, but is distributed. Take a uniform rod of length l pivoting around one end.



Considering forces this time, and the physical pendulum as a rotating system. Then Newton's law $F = m\ddot{x}$ becomes $\tau = I\ddot{\theta}$ where I is the moment of inertia and τ is the torque.

$I = \frac{1}{3}ml^2$ for the rod, and $\tau = -\frac{l}{2}mg\sin\theta$

Therefore $\frac{1}{3}ml^2\ddot{\theta} = -\frac{l}{2}mg\sin\theta$

And using the small angle approximation $\sin\theta \approx \theta$ for small θ:

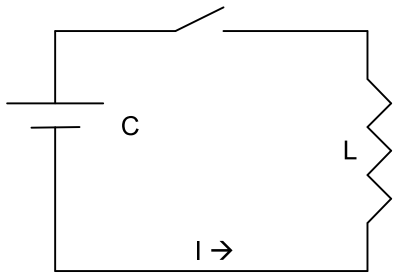
$$\ddot{\theta} = -\left(\frac{3g}{2l}\right)\theta$$

i.e. SHM with $\omega = \left(\frac{3g}{2l}\right)^{1/2}$ and $T = \frac{2\pi}{\omega} = 2\pi\left(\frac{2l}{3g}\right)^{1/2}$

For $l = 1m$, $T = 1.64s$.

Example: footsteps. Each period corresponds to two steps. 10 periods takes 15 seconds, giving T as 1.5 seconds.

(d) <u>The LC Circuit</u>



The capacitor is initially charged to some voltage V and then the switch is closed. This is an ideal system with no resistance in the circuit.

Kirchoff's law: "the sum of the voltages around any loop is zero".

Therefore $V_C + V_L = 0$ with $V_c = \frac{q}{C}$ and $V_L = L\frac{dI}{dt}$ where I is the current.

$$\frac{q}{C} + L\frac{dI}{dt} = 0$$

Since $I = \frac{dq}{dt}$:

$$\frac{q}{c} + Ld\ddot{q} = 0$$

$$\ddot{q} = -\frac{q}{LC}$$

i.e. SHM with $\omega = \frac{1}{\sqrt{LC}}$.

q is replacing x in the harmonic oscillator. From energy considerations:-

$$\frac{1}{2}CV_c^{\,2} + \frac{1}{2}LI^2 = const. = E$$

(Energy stored in capacitor C + energy stored in inductor L)
i.e. continuous exchange between electrostatic and magnetic energy.
Let's note the important similarity between the differential equations

$$L\ddot{q} + \frac{q}{c} = 0$$

$$m\ddot{x} + kx = 0$$

where $q \equiv x, L \equiv m, \frac{1}{C} \equiv k$

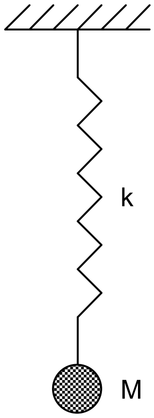
Also $\frac{1}{2}CV_c + \frac{1}{2}LI^2 = E$ becomes $\frac{1}{2C}q^2 + \frac{1}{2}L\dot{q}^2 = E$ and $\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = E$.

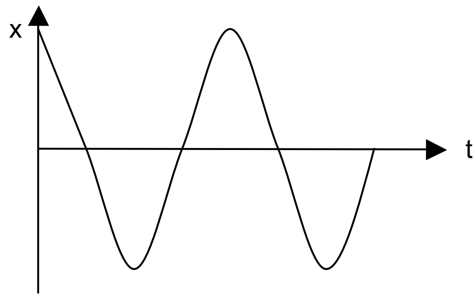
NB: Similarities in physics – the equations are the same. If you understand one system then you understand lots of others with the same differential equations. You can also make analogue computers.

3. The Damped Harmonic Oscillator

In reality oscillators are not ideal. There are various damping mechanisms present and the total energy reduces with time.

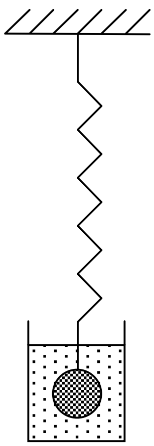
An example is a tuning fork. The sound intensity ($\propto A^2$) steadily decreases. Actually damping is very light – the fork vibrates for about 5 seconds at a frequency of 440Hz i.e. after several thousand complete oscillations.
Recall our mass on a spring, with no damping.

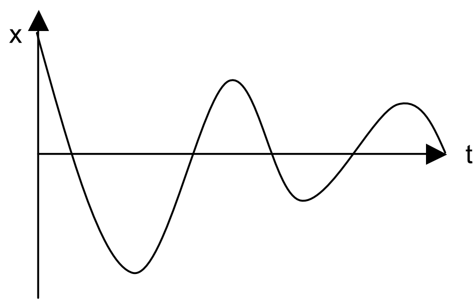




Undamped SHO

$x = A\cos\omega_o t \quad \omega_o = \sqrt{\dfrac{k}{m}}$





Damped SHO

Observe:
- decreasing amplitude
- Constant frequency

Let's imagine what x(t) might look like.

Expect x~(Amplitude which varies with time) times $\cos\omega t$ where ω is about, but not necessarily the same, as for undamped case. One clue is that the amplitude reduces by equal fractions in equal times i.e. exponentially
i.e. $x(t) = \left(Ae^{-\beta t}\right)\cos\omega t$ and $\omega \sim \omega_0$

Equation of Motion

The good news is that in practice the damping is often due to frictional forces that are proportional to the velocity of the mass v. Example: Stoke's Law, $F = 6\pi\eta av$ where η is the viscosity of the medium and a is the radius of the sphere, e.g. motor cars, raindrops. This damping force is in the opposite direction to that of velocity. So for our example of a mass on a spring, then the force on the mass can be represented by $F = -kx - bv$ where $-bv$ is the damping force.
Note units:

k: force/unit length (Nm^{-1})
b: force/unit velocity ($kg.s^{-1}$)

Applying Newton's law, we obtain $m\ddot{x} = -kx - b\dot{x}$, and making the useful substitutions, $\frac{b}{m} = \gamma$, $\frac{k}{m} = \omega_o{}^2$ where ω_o is the angular frequency for the undamped case, we obtain $\ddot{x} + \gamma\dot{x} + \omega_o{}^2 x = 0$.

3.1 Solutions for the damped SHO

We distinguish between light and heavy damping.

i) Light damping

We guessed the solution $x = Ae^{-\beta t}\cos\omega t$ where β is a constant and ω is approximately equal to ω_o.

Then find $\dot{x}$ and $\ddot{x}$ and substitute into our DE using the product rule

$$\frac{d}{dt}(f.g) = f\frac{dg}{dt} + g\frac{df}{dt}$$

where f and g are functions of t. We obtain:

$$\dot{x} = Ae^{-\beta t}(-\omega\sin\omega t) + \cos\omega t\left(-\beta Ae^{-\beta t}\right)$$
$$= Ae^{-\beta t}(-\omega\sin\omega t - \beta\cos\omega t)$$
$$\ddot{x} = Ae^{-\beta t}\left(-\omega^2\cos\omega t + \beta\omega\sin\omega t\right) + Ae^{-\beta t}(-\beta)(-\omega\sin\omega t - \beta\cos\omega t)$$
$$= Ae^{-\beta t}\left(2\beta\omega\sin\omega t + \left(\beta^2 - \omega^2\right)\cos\omega t\right)$$

Substitute into DE and collecting terms in $\sin\omega t + \cos\omega t$:

$$Ae^{-\beta t}\left((2\beta\omega - \gamma\omega)\sin\omega t + \left(\beta^2 - \omega^2 - \gamma\beta + \omega_o{}^2\right)\cos\omega t\right) = 0$$

This can only be true for all t if sin and cos terms are both equal to 0.

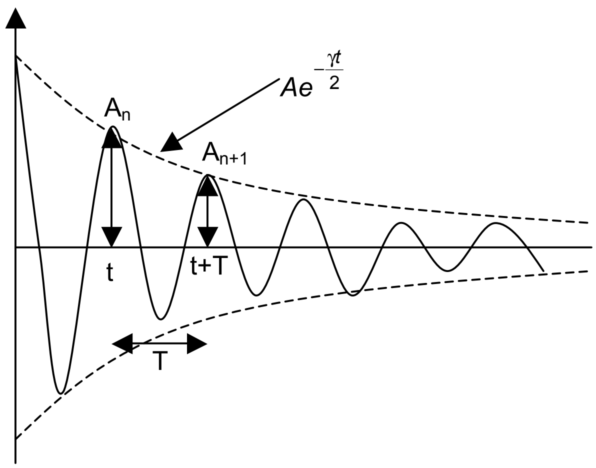
$$2\beta\omega - \gamma\omega = 0 \Rightarrow \beta = \frac{\gamma}{2}$$

and

$$\beta^2 - \omega^2 - \gamma\beta + \omega_o{}^2 = 0$$
$$\frac{\gamma^2}{4} - \omega^2 - \frac{\gamma^2}{2} + \omega_o{}^2 = 0$$
$$\omega^2 = \omega_o{}^2 - \frac{\gamma^2}{4}$$

The frequency of damped oscillation.

$$x(t) = Ae^{-\frac{\gamma t}{2}}\cos\left(\omega_o{}^2 - \frac{\gamma^2}{4}\right)^{1/2} t \text{ is a solution of } \ddot{x} + \gamma\dot{x} + \omega_o{}^2 x = 0$$



Consider successive maxima A_n and A_{n+1} occurring at times t and (t+T) where T is the period.
Then

$$A_n = x(t) = A^{-\frac{\gamma t}{2}}\cos\omega t$$

$$A_{n+1} = x(t+T) = Ae^{-\frac{\gamma(t+T)}{2}}\cos\omega(t+T)$$

$$\therefore \frac{A_n}{A_{n+1}} = \frac{e^{-\frac{\gamma t}{2}}}{e^{-\frac{\gamma(t+T)}{2}}} = e^{\frac{\gamma T}{2}}$$

since $\cos\omega t = \cos\omega(t+T)$.

i.e the amplitude decays by an equal fraction in an equal amount of time.

$\ln\frac{A_n}{A_{n+1}} = \frac{\gamma T}{2}$ is called the logarithmic decrement and measures the rate of which the amplitude dies away.

ii) Heavy damping

What would you expect for heavy damping? Few oscillations, if at all, and the mass returns slowly to rest. Then a cosine function is not appropriate. Let's try a general solution $f(t)$

i.e. $x(t) = Ae^{-\frac{\gamma t}{2}}f(t)$.

Substituting x(t) and its' derivatives into our DE, we obtain:

$$f'' + f\left(\omega_o{}^2 - \frac{\gamma^2}{4}\right) = 0.$$

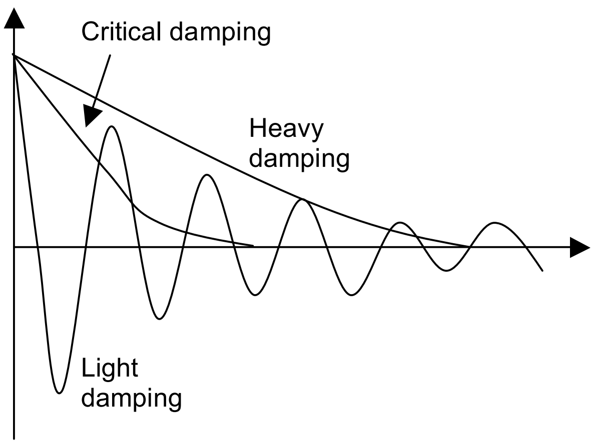
For heavy damping $\frac{\gamma^2}{4} >> \omega_o{}^2$.

Obtain $f'' = \alpha^2 f$ where $\alpha^2 = \left(\frac{\gamma^2}{4} - \omega_o{}^2\right)$ which is a positive quantity.

This has a general solution $f = ae^{\alpha t} + be^{-\alpha t}$ giving the displacement

$$x(t) = Ae^{-\frac{\gamma t}{2}}\left[ae^{-\alpha t} + be^{-\alpha t}\right]$$

As expected the mass moves slowly back to its equilibrium position without oscillation.



Critical Damping

This is a special case where $\frac{\gamma^2}{4} = \omega_o{}^2$. We obtain $f'' = 0$. This has a general solution, $f = a + bt$ (NB: two constants for 2nd order ODE) and hence

$$x(t) = Ae^{-\frac{\gamma t}{2}}(a + bt)$$

and for the special case of critical damping the mass returns to its equilibrium position most quickly with no oscillations.
Note that in many cases damping is a good thing. Critical damping in particular is an important consideration in things like electrical meters, car shock absorbers, and beds versus trampolines.

3.2 Rate of energy loss in damped oscillator

We know that the mechanical energy $(E_K + E_P)$ is not conserved: it is dissipated as heat. For lightly damped case we have $x(t) = Ae^{-\frac{\gamma t}{2}} \cos\omega t$ .We can write the amplitude dependence on time as $A(t) = A_o e^{-\frac{\gamma t}{2}}$ where A_o is the amplitude at $t = 0$.
We also have

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.$$

Note that the energy is proportional to the amplitude squared, i.e.

$$E(t) = \frac{1}{2}kA(t)^2 = \frac{1}{2}k\left(A_o e^{-\frac{\gamma t}{2}}\right) = \frac{1}{2}kA_o{}^2 e^{-\gamma t}$$

giving us that $E(t) = E_o e^{-\gamma t}$ where E_o is the energy at time $t = 0$.
The important result is that energy decays exponentially with time.

3.3 The quality factor of a damped oscillator

We want to quantify how "good" an oscillator is – a "figure of merit".

When $t = \frac{1}{\gamma}$, $E = E_o e^{-1} = \frac{E_o}{e}$. So γ is the reciprocal of the time taken for E to reduce by a factor of $\frac{1}{e}$. Note that ω_o and γ have the same dimensions (time^{-1}). γ is a characteristic of the exponential decay of the amplitude and ω_o is characteristic of the oscillatory part of the motion.

We define a <u>quality factor</u> $Q = \frac{\omega_o}{\gamma}$ where Q is a pure number. The larger Q is, the better the oscillator is.

<u>Example:</u>

The sound intensity, which is proportional to the energy, from a tuning fork with a frequency of 440Hz decreases by a factor of 5 in 4 seconds.

$$5 = \frac{E_o e^0}{E_o e^{-4\gamma}} = e^{-4\gamma}$$

$$4\gamma = \ln 5$$

$$\gamma = 0.4s^{-1}$$

$$Q = \frac{\omega_o}{\gamma} = \frac{2\pi 440}{0.4} \approx 7000$$

which is very high.

<u>Another example</u>

An excited atom emitting radiation.

Here $\gamma = \frac{1}{\tau}$ where τ is the atomic lifetime (how long the particle spends in an excited state), around 10^{-8}s, and $\omega_o = 2\pi f = \frac{2c}{\lambda}$ where λ is 500nmx10^{-9}m → $\frac{2\pi 3x10^8}{500x10^{-9}} = 4x10^{15} rad.s^{-1}$ →

$Q = \frac{\omega_o}{\gamma} = 4x10^7$ →very high indeed.

From above;

$$\gamma^2 = \frac{{\omega_o}^2}{Q^2}$$

$$\omega^2 = {\omega_o}^2 - \frac{\gamma^2}{4}$$

$$\omega = \omega_o \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

When damping is small, Q is large and so $\omega \approx \omega_o$.

E.g. of a modest value of Q of 5 then;

$$\omega = \omega_o \left(1 - \frac{1}{4x5^2}\right)^{1/2}$$

$$= \omega_o \left(1 - \frac{1}{100}\right)^{1/2}$$

$$\approx \omega_o \left(1 - \frac{1}{200}\right)$$

i.e. ω is different from ω_o by just 0.5%.

Recall again the variation of energy with time.

$$E(t) = E_o e^{-\gamma t}$$

Let

$$E_1 = E_o e^{-\gamma t}$$
$$E_2 = E_o e^{-\gamma(t+T)}$$

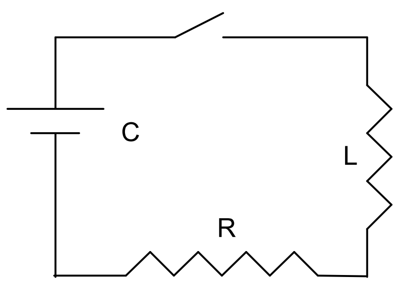
Therefore $\frac{E_2}{E_1} = \frac{e^{-\gamma(t+T)}}{e^{-\gamma t}} = e^{-\gamma T} \approx (1-\gamma T)$ for $\gamma T \ll 1$ i.e. for light damping.

$$\therefore E_2 \approx E_1 - E_1 \gamma t$$
$$\frac{(E_1 - E_2)}{E_1} \approx \gamma T \approx \frac{2\pi\gamma}{\omega_o} = \frac{2\pi}{Q}$$

assuming ω is roughly equal to ω_o.

So the fractional change in energy per cycle $= \frac{2\pi}{Q}$ and so the fractional change in energy per radian $= \frac{1}{Q}$.

3.4 Electrical example of damped oscillation



Here we have an inductor L, capacitor C and resistance R (damping element). These are all connected in series. We charge the capacitor to voltage V and then close the switch.

Kirchoff's law gives $L\frac{dI}{dt} + RI + \frac{q}{c} = 0$, i.e. $L\ddot{q} + R\dot{q} + \frac{q}{c} = 0$ or $\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{cL} = 0$

Compare this to $m\ddot{x} + b\dot{x} + kx = 0$ or $\ddot{x} + \gamma\dot{x} + \omega_o{}^2 = 0$ where $\omega_o{}^2 = \frac{k}{m}$.

Comparing the mechanical and electrical systems;

$x \equiv q$

$m \equiv L$

$k \equiv \frac{1}{C}$

as before. But also:

$b \equiv R$

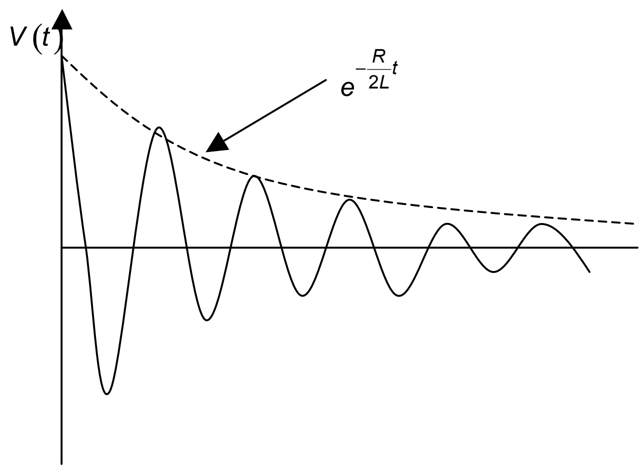
$\gamma = \frac{R}{L}$

and so we can say straight away that for the electrical system, $\omega_o^2 = \frac{1}{LC}$. Recalling that

$x(t) = Ae^{-\frac{\gamma t}{2}}\cos\left(\omega_o^2 - \frac{\gamma^2}{4}\right)^{1/2} t$ gives the solution for $q(t) = Ae^{-\frac{Rt}{2L}}\cos\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2} t$. This works for $\frac{R^2}{4L^2} << \frac{1}{LC}$.

Since $C = \frac{q}{V}$, $V(t) = \frac{q(t)}{C}$.

So voltage across the capacitor with time looks like $V(t) = V_o e^{-\frac{RT}{2L}}\cos\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2} t$



Quality Factor

$Q = \frac{\omega_o}{\gamma}$ becomes $Q = \frac{1}{\sqrt{LC}}\frac{1}{R/L} = \frac{1}{R}\sqrt{\frac{L}{C}}$.

Energy in the circuit will decrease as $e^{-\frac{RT}{L}}$.

NB: for $\frac{R^2}{2L^2} = \frac{1}{LC}$ we obtain critical damping and so the oscillations stop. For $\frac{R^2}{4L^2} >> \frac{1}{LC}$ we get heavy damping.
So by understanding the mechanical system we can also understand the equivalent electrical system, see the Circuits course PC 1382.

4. Use of the complex representation of SHM

For all the situations we encounter in this course, we could describe x(t), v(t) and a(t) in terms of sines and cosines. However there is a more elegant and powerful representation using the complex formulations. Moreover in some of your courses its use is essential e.g. Quantum Mechanics and Electromagnetism. The complex formulation may sound mysterious but for us it will be an important mathematical tool.

4.1 Complex Numbers

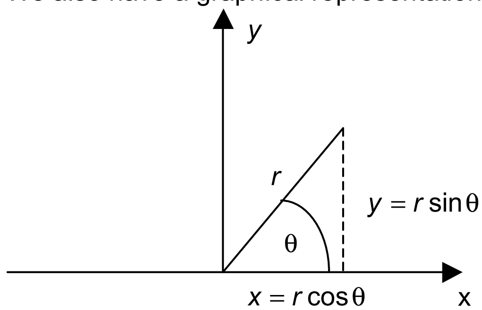
Let's recall the nature of a complex number z.

$z = x + jy$ where $j = \sqrt{-1}$. It consists of a real part (x) and an imaginary part (y).

We also have Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, or $re^{j\theta} = r\cos\theta + jr\sin\theta$.
Feynman says "this amazing jewel, the most remarkable formula in mathematics."
Clearly

$$x = r\cos\theta$$
$$y = r\sin\theta \; .$$

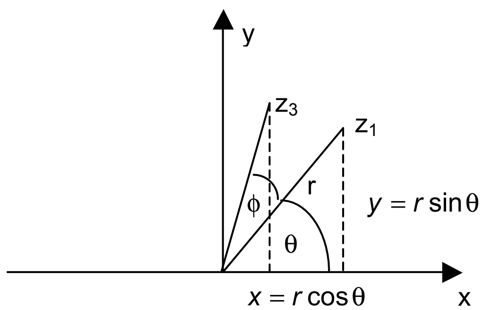
We also have a graphical representation of a complex number in the complex plane.



Here we recognize r as the length of the line and θ as the phase, and again $r = \sqrt{x^2 + y^2}$.
If

$$z_1 = re^{j\theta}$$
$$z_2 = e^{j\phi}$$

then $z_1 z_2 = re^{j(\theta+\phi)}$.



Thus If we multiply a complex number by $e^{j\phi}$ then we rotate the line by angle ϕ.

Example: if $\phi = \frac{\pi}{2}$ we rotate the line through 90°.

But $e^{j\theta}e^{j\frac{\pi}{2}} = e^{j\theta}\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = je^{j\theta}$.

i.e. multiplying a complex number by j is the same as multiplying by $e^{j\frac{\pi}{2}}$ i.e. rotation through 90°.
If $\phi = \pi$ we rotate through 180°. But $e^{j\theta}e^{j\pi} = e^{j\theta}(\cos\pi + j\sin\pi) = -1e^{j\theta}$ i.e. multiplying a complex number by -1 is the same as multiplying by $e^{j\pi}$ i.e. rotation through 180°.

4.2 The use of complex numbers to represent physical quantities
The idea is that the physical quantity is represented by the real part if complex number z. We have:

We have	Compare with
$x = A\cos\omega t$	$z = Ae^{j\omega t} = A\cos\omega t + jA\sin\omega t$

$\dot{x} = -A\omega \sin \omega t$	$\dot{z} = j\omega A e^{j\omega t} = j\omega z = -A\omega \sin \omega t + j\omega A \cos \omega t$
$\ddot{x} = -A\omega^2 \cos \omega t$	$\ddot{z} = j^2 A\omega^2 e^{j\omega t} = (j\omega)^2 z = -\omega^2 A \cos \omega t - j\omega^2 A \sin \omega t$

So we can represent our physical quantities by complex numbers, remembering that the real part of the complex number corresponds to the physical quantity. This is even the case when we operate on the complex number to give a new complex number to give a new complex number, e.g. $\dot{z}, \ddot{z}$. Notice also that we have replaced differentiation by multiplication because we are using exponentials – that makes life much easier – especially in complicated situations.

4.3 Graphical representation of physical quantities in the complex plane

Remember the moons of Jupiter. The projection of the moons across the field of view traced round Simple Harmonic Motion. In a similar way we represent the displacement x as the projection onto the real axis of vector z_1 that rotates in the complex plane i.e.

$z_1 = Ae^{j\omega t}$ where the length of the vector is amplitude A. The projection of the vector is $A \cos \omega t = x$.

Then $z_2 = \dot{z}_1 = j\omega A e^{j\omega t} = j\omega z_1$.Note the factor j; we have rotated z_1 through $\frac{\pi}{2}$. i.e.

$z_2 = A\omega e^{j\left(\omega t + \frac{\pi}{2}\right)}$.

The velocity v is the projection of z_2 onto the real axis $= A\omega \cos\left(\omega t + \frac{\pi}{2}\right) = A\omega \sin \omega t$ as required. The length of the line $= A\omega$.

Similarly for acceleration a

$z_3 = \dot{z}_2 = -\omega^2 A e^{j\omega t} = -\omega^2 z_1$

Note the factor (-1). This means we have rotated z_2 through $\pi/2$, or z_1 through π.

i.e. $z_3 = \omega^2 A e^{j(wt+\pi)}$ and the length of the line is $A\omega^2 ss$.

A is a projection of z_3 on the real axis $= \omega^2 A \cos(\omega t + \pi) = -A\omega^2 \cos \omega t$ as required.

Then these three vectors rotate in the complex plane maintaining constant relative phases.

5. Forced Oscillations

So far we have considered systems that oscillate at their natural frequency. We now want to consider the situation where we drive the oscillator at different frequencies i.e. the application of a periodic driving force e.g. pushing a swing, a house in an earthquake, the oceans. When we apply a periodic driving force, the resulting force acting on the mass = driving force + restoring force + damping force.

$m\ddot{x} = F_o \cos \omega t - kx - b\dot{x}$ where ω is the driving frequency of the periodic force.

Therefore $m\ddot{x} + b\dot{x} + kx = F_o \cos \omega t$. This is the general equation of a driven oscillator.

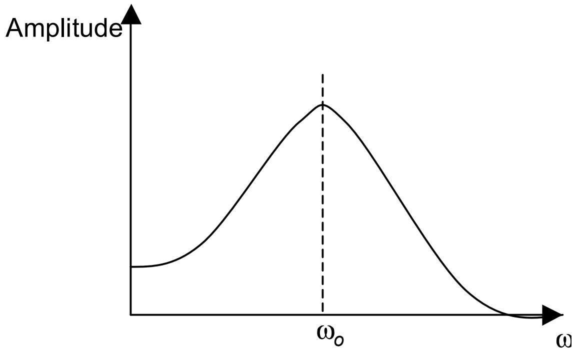
5.1 Physical characteristics of Forced Oscillation

Take the example of a simple pendulum, driven by moving the point of suspension horizontally. We are interested in how the amplitude of the oscillations changes with driving frequency, and how the phase difference between the driving force and the displacement changes with frequency.

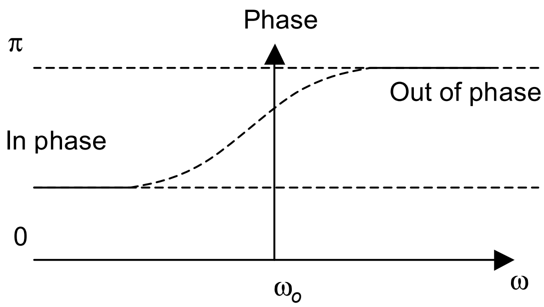
We observe, where ω_o s the natural frequency of oscillation:

1. The frequency of the oscillator is the same as that of the driving force.

2. For $\omega < \omega_o$ the mass closely follows the driving force; there is no phase difference between the two
3. For $\omega \sim \omega_o$ there is a very large amplitude of oscillation – <u>resonance</u>.
4. For $\omega > \omega_o$ the amplitude decreases but moves in the opposite direction i.e. out of phase.
5. For $\omega >> \omega_o$ the displacement tends to 0. The mass can't follow the driving force because of its' inertia.



The driving displacement is finite and equal to $\frac{\pi}{2}$.



<u>5.2 Mass on a spring driven by periodic force (damping neglected)</u>

The force on the mass is equal to k times the extension, $F = k(x - s)$

therefore $m\ddot{x} = -k(x - s) = -kx + ks$.

If $s = a\cos\omega t$ then we obtain $m\ddot{x} + kx = ka\cos\omega t$ where we recognize $ka\cos\omega t$ as the driving force $(F_o \cos\omega t)$ and $F_o = ka$.

Since $\frac{k}{m} = \omega_o^{\ 2}$, we have $\ddot{x} + \omega_o^{\ 2} x = a\omega_o^{\ 2}\cos\omega t$.

Let's try the solution $x = A\cos\omega t$ where we assume that the system will oscillate at driving frequency ω. Strictly speaking $A = A(\omega)$. Therefore;

$\dot{x} = A\omega \sin ít$

$\ddot{x} = -A\omega^2 \cos\omega t$

$$-A\omega^2 \cos\omega t + \omega_o{}^2 A\cos\omega t = \omega_2{}^2 a\cos\omega t$$

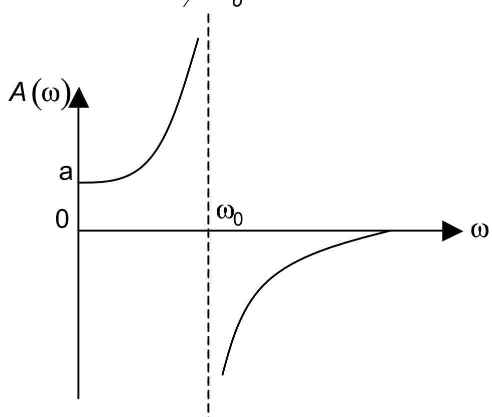
$$A\left(\omega_o{}^2 - \omega^2\right) = \omega_o{}^2 a$$

$$A = \frac{a\omega_o{}^2}{\omega_o{}^2 - \omega^2} = \frac{a}{1 - \omega^2 / \omega_o{}^2}$$

We see that the amplitude depends on the driving frequency.

1. When $\omega << \omega_o$ $A \approx a$
2. When $\omega >> \omega_o$, $A \to -0$ i.e. it approaches 0 from the negative direction. Now the motion is controlled by mass i.e. inertia. i.e. $A \approx \frac{a}{-\omega^2 / \omega_o{}^2} = -\frac{ak}{m\omega^2}$
3. When $\omega = \omega_o$, $A = \infty$.

A plot of $A = \frac{a}{1 - \omega^2 / \omega_o{}^2}$ looks like:



Of course, the situation where $A \to \infty$ at $\omega = \omega_o$ is unrealistic because we have not included damping. Thus $x = A\cos\omega t$ is a solution of the differential equation but A changes sign as ω passes through ω_o. This change in sign is saying something about the phase of the displacement with respect to the driving force.

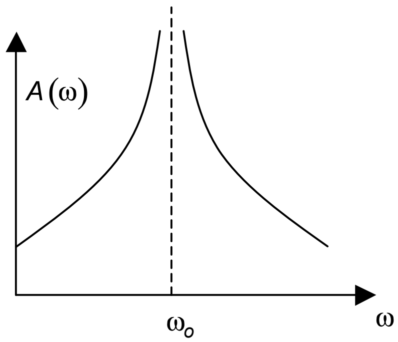
Let's try $x = A\cos\left(\omega t + \phi\right)$ where A is always positive but ϕ takes on different values.

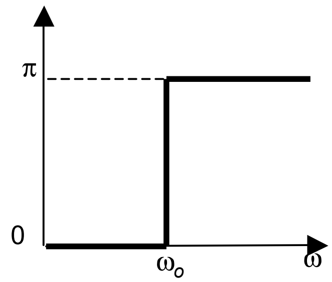
Recall the identity $\cos\left(\omega t + \phi\right) = \cos\omega t\cos\phi - \sin\omega t\sin\phi$

For $\phi = 0$, $\cos\left(\omega t + \phi\right) = \cos\omega t$.

For $\phi = \pi$, $\cos\left(\omega t + \phi\right) = -\cos\omega t$ since $\cos\pi = -1$.

Therefore $x = A\cos\left(\omega t + \phi\right)$, where $\phi = 0$ for $\omega < \omega_o$ and $\phi = \pi$ for $\omega > \omega_o$ i.e. we have reproduced the sign of A using a phase angle. For $\omega < \omega_o$ the system oscillates exactly in phase with the driving force and for $\omega > \omega_o$ the system oscillates exactly out of phase.





Now let's try complex representation of the system.
We have

$$\ddot{x} + \omega_o{}^2 x = a\omega_o{}^2 \cos\omega t \ (1)$$

$a\omega_o{}^2 \cos\omega t$ is the real part of $a\omega_o{}^2 e^{j\omega t}$.
Thus the corresponding complex differential equation is

$$\ddot{z} + \omega_o{}^2 z = \omega_o{}^2 a e^{j\omega t} \ (2)$$

where z is the solution of this differential equation and x is $\mathrm{Re}(z)$.

Note that equation 1 is the real part of equation 2. We can see this as follows.

$$\frac{d^2}{dt^2}\underbrace{\left(A\cos\omega t + kA\sin t\right)}_{z} + \omega_o{}^2\underbrace{\left(A\cos\omega t + kA\sin t\right)}_{z} = \omega_o{}^2\left(a\cos\omega t + ja\sin\omega t\right)$$

Equating the real parts:

$$\frac{d^2}{dt^2}\left(A\cos\omega t\right) + \omega_o{}^2\left(A\cos\omega t\right) = \omega_o{}^2\left(a\cos\omega t\right).$$

$$\ddot{x} = \omega_o{}^2 = a\omega_o{}^2\cos\omega t$$

Assuming solution for equation (2) is $z = Ae^{j\omega t}$, we have

$$(j\omega)^2 Ae^{j\omega t} + A\omega_o{}^2 e^{j\omega t} = \omega_o{}^2 a e^{j\omega t}$$

NB we have replace differentiation by multiplication. Hence:

$$-\omega^2 A + A\omega_o{}^2 = \omega_o{}^2 a$$

and $A = \dfrac{a}{\left(1 - \omega^2/\omega_o{}^2\right)}$ as before.

5.3 Forced Oscillation With Damping. Mass on a spring.

Mass is damped by a viscous fluid.

We now have the equation $m\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+kx=F_o\cos\omega t$

NB: $b\dfrac{dx}{dt}$ is the damping force, kx is the restoring force, $F_o\cos\omega t$ is the driving force.

As before, the extension $=(x-s)$. Therefore $F=-k(x-s)$. Therefore

$m\dfrac{d^2x}{dt^2}=-k(x-s)-b\dfrac{dx}{dt}$ and $s=a\cos\omega t$

Therefore $m\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+kx=ks=ka\cos\omega t=F_o\cos\omega t$.

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_o{}^2x=\omega_o{}^2a\cos\omega t$$

where $\gamma=\dfrac{b}{m},\omega_o{}^2=\dfrac{k}{m}$.

Assume $x=A\cos(\omega t-\phi)$.

$\dfrac{dx}{dt}=-A\cos(\omega t-\phi)$ and $\dfrac{d^2x}{dt^2}=-A\omega^2\cos(\omega t-\phi)$

Substitute into equation of motion to obtain:

$-A\omega^2(\cos\omega t\cos\phi+\sin\omega t\sin\phi)-A\omega\gamma(\sin\omega t\cos\phi-\cos\omega t\sin\phi)$

$+\omega_o{}^2A(\cos\omega t\cos\phi+\sin\omega t\sin\phi)=\omega_o{}^2a\cos\omega t$

Eqn coefficients of $\cos\omega t$:

$-A\omega^2\cos\phi+A\omega\gamma\sin\phi+\omega_o{}^2A\cos\phi=\omega_o{}^2a$

$\Rightarrow A\left((\omega_o{}^2-\omega^2)\cos\phi+\omega\gamma\sin\phi\right)=\omega_o{}^2a$

Eqn coefficients of $\sin\omega t$:

$-A\omega^2\sin\phi-A\omega\gamma\cos\phi+\omega_o{}^2A\sin\phi=0$

$\Rightarrow(\omega_o{}^2-\omega^2)\sin\phi=\omega\gamma\cos\phi$

and:

$$\tan\phi=\frac{\omega\gamma}{(\omega_o{}^2-\omega^2)}$$

$$\therefore\sin\phi=\frac{\omega\gamma}{\left((\omega_o{}^2-\omega^2)^2+\omega^2\gamma^2\right)^{1/2}}$$

and $\cos\phi=\dfrac{(\omega_o{}^2-\omega^2)}{\left((\omega_o{}^2-\omega^2)^2+\omega^2\gamma^2\right)^{1/2}}$

Substitute for $\sin\phi$ and $\cos\phi$ into the previous equation:

$$A\left[\frac{\left(\omega_o^{\ 2}-\omega^2\right)\left(\omega_o^{\ 2}-\omega^2\right)}{\left(\left(\omega_o^{\ 2}-\omega^2\right)^2+\omega_o^{\ 2}\gamma^2\right)^{1/2}}+\frac{\omega^2\gamma^2}{\left(\left(\omega_o^{\ 2}-\omega^2\right)^2+\omega^2\gamma^2\right)^{1/2}}\right]=\omega_o^{\ 2}a$$

$$A\frac{\left(\left(\omega_o^{\ 2}-\omega^2\right)^2+\omega^2\gamma^2\right)}{\left(\left(\omega_o^{\ 2}-\omega^2\right)^2+\omega^2\gamma^2\right)^{1/2}}=\omega_o^{\ 2}a$$

$$A=\frac{a\omega_o^{\ 2}}{\left(\left(\omega_o^{\ 2}-\omega^2\right)+\omega^2\gamma^2\right)^{1/2}}$$

When $\gamma=0$ i.e. undamped oscillator then $A=\dfrac{a}{\left(1-\omega^2/\omega_o^{\ 2}\right)}$ as before.

For $\omega\to 0$, $A\to\dfrac{\omega_o^{\ 2}a}{\omega_o^{\ 2}}\to a$

For $\omega\to\infty$, $A\to 0$.

For $\omega=\omega_o$, $A=\dfrac{\omega_o^{\ 2}a}{\omega_o\gamma}=\dfrac{\omega_o}{\gamma}a=Qa$ which is not infinitely large.

The maximum amplitude no longer occurs at ω_o. For $A(\omega)$ to be a maximum the denominator must be a minimum. This means:

$$\frac{d}{d\omega}\left(\left(\omega_o^{\ 2}-\omega^2\right)^2+\omega^2\gamma^2\right)=0$$

$$\frac{d}{d\omega}\left(\omega_o^{\ 4}+\omega^4-2\omega^2\omega_o^{\ 2}+\omega^2\gamma^2\right)=0$$

$$4\omega^3-\omega\left(4\omega_o^{\ 2}-2\gamma^2\right)=0$$

$$\omega=\omega_o\left(1-\frac{\gamma^2}{2\omega_o^{\ 2}}\right)^{1/2}=\omega_{max}$$

Therefore A_{max} occurs at <u>lower</u> frequency than ω_o.

We can find A_{max} by substituting ω_{max} into the equation for $A(\omega)$. We obtain (exercise for student)

$$A_{max}=\frac{a\,\omega_o/\gamma}{\left(1-\gamma^2/4\omega_o^{\ 2}\right)^{1/2}}.$$

Consider $\tan\phi$ where ϕ is the phase angle between the driving force and the resultant displacement of the mass.

For $\omega\to 0$, $\tan\phi\to 0$ i.e. $\phi\to 0$

For $\omega\to\infty$, $\tan\phi=\dfrac{1}{-\infty}$, i.e. small and negative i.e. $\phi\to\pi$.

For $\omega = \omega_o$, $\tan\phi = \infty$, i.e. $\phi = \frac{\pi}{2}$.

As ω varies from 0 to ∞, the function $\tan\phi = \frac{\omega\gamma}{\left(\omega_o^2 - \omega^2\right)}$ varies continuously from 0 to π. (See graphs from Forced Oscillations)

The overall response of the system is similar to the undamped case but:

1. The amplitude A remains finite.
2. The maximum value of A shifts to frequency below ω_o
3. ϕ changes continuously from 0 to π.

These changes occur because of damping.

Quality Factor

We recall the quality factor Q from our discussion of damped free oscillations. Q has important significance when we now drive the oscillator. Using $Q = \frac{\omega_o}{\gamma}$ we obtain

$$\omega_{\max} = \omega_o \left(1 - \frac{1}{2Q^2}\right)^{1/2}$$

And

$$A_{\max} = a \frac{Q}{\left(1 - 1/4Q^2\right)^{1/2}}.$$

When $Q >> 1$:

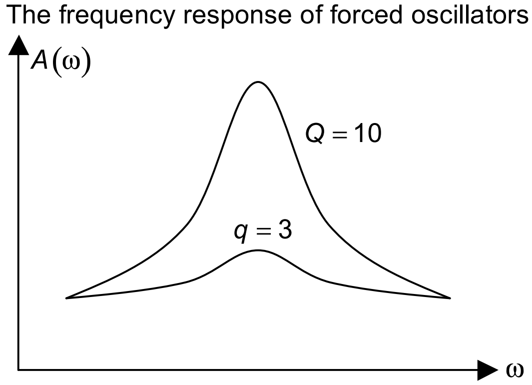
$$\omega_{\max} \simeq \omega_o$$

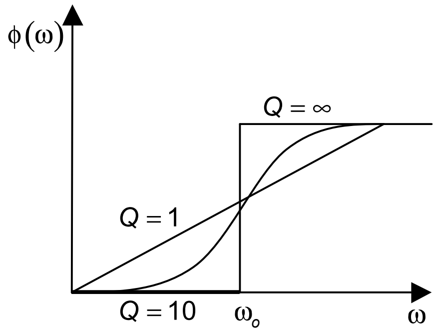
$$A_{\max} \simeq aQ$$

where Q is like an amplification factor. The important point is that small forces can produce large oscillations when applied at the resonance frequency.

For example, a simple pendulum with $Q = 500$ and $a = 1mm$, $A = 0.5m$.

The frequency response of forced oscillators for various Q values look like;





See also French 4.9.

5.4 The Complex representation of forced oscillation with damping

Our general expression was:

$$\ddot{x}+\gamma\dot{x}+\omega_o^{\,2}x=\frac{F_o}{m}\cos\omega t\ .$$

This becomes:

$$\ddot{z}+\gamma\dot{z}+\omega_o^{\,2}z=\frac{F_o}{m}e^{j\omega t}\ .$$

We assume the general solution $z=Ae^{j\omega t-\phi}$ with $x=\mathrm{Re}\{z\}$. Substitute for z in the differential equation to obtain:

$$\left(-\omega^2A+j\gamma\omega A+\omega_o^{\,2}A\right)e^{j(\omega t-\phi)}=\frac{F_o}{m}e^{j\omega t}$$

$$\left(\omega_o^{\,2}-\omega^2\right)A+j\gamma\omega A=\frac{F_o}{m}e^{j\omega\phi}$$

Equate the real parts:

$$\left(\omega_o^{\,2}-\omega^2\right)A=\frac{F_o}{m}\cos\phi$$

Equate the imaginary parts:

$$\gamma\omega A=\frac{F_o}{m}\sin\phi$$

Using $\sin^2\phi+\cos^2\phi=1$:

$$A(\omega)=\frac{F_o/m}{\left(\left(\omega_o^{\,2}-\omega^2\right)+\left(\gamma\omega^2\right)^2\right)^{1/2}}$$

And

$$\tan\phi(\omega)=\frac{\gamma\omega}{\left(\omega_o^{\,2}-\omega^2\right)}$$

5.5 Power absorption during forced oscillations

The power absorbed by the driven oscillator can be deduced from $P=Fv$.

$$v=\dot{x}=-A\omega\sin(\omega t-\phi)$$

$$P = -F_o \cos\omega t A\omega \sin(\omega t - \phi)$$
$$= -F_o A\omega \cos\omega t(\sin\omega t \cos\phi - \cos\omega t \sin\phi)$$
$$= -(F_o A\omega \cos\phi)\sin\omega t \cos\omega t + F_o A\omega \sin\phi \cos^2 \omega t$$

If we integrate power input over a complete cycle, the first term gives 0, i.e.

$$\int_0^T \sin\omega t \cos\omega t dt = 0$$

However, the average of:

$$\int_0^T \cos^2 \omega t dt = \frac{1}{2}$$

(over one cycle). So the average power is given by:

$$P_{average} = \frac{1}{2}\omega A F_o \sin\phi .$$

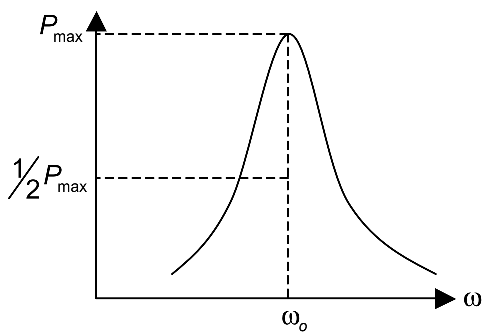
Using expressions for A and $\sin\phi$, we obtain:

$$P_{average}(\omega) = \frac{\omega^2 F_o^{\,2}\gamma}{2m\left(\left(\omega_o^{\,2} - \omega^2\right)^2 + (\gamma\omega)^2\right)}$$

(i) for $\omega \to 0$, $P_{average}(\omega) \to 0$

(ii) For $\omega \to \infty$, $P_{average}(\omega) \to 0$

(iii) $P_{average}(\omega)$ passes through a maximum at exactly $\omega = \omega_o$ i.e. maximum power absorbed at $\omega = \omega_o$. This leads to the power resonance curve.



NB the power curve is symmetrical about the maximum, except for large γ or low Q values.

$$P_{max} = \frac{\omega_o^{\,2} F_o^{\,2}\gamma}{2m(\gamma\omega_o)^2} = \frac{F_o^{\,2}}{2m\gamma}$$

An important parameter of the resonance curve is its <u>width</u> at half maximum. This width characterizes the sharpness of the response of the oscillator. We can find the point at which the power falls to $P_{max}/2$ in the following way by considering equation 1.

$$P_{average}(\omega) = \frac{\omega^2 F_o^{\,2}\gamma}{2m\left(\left(\omega_o^{\,2} - \omega^2\right)^2 + (\gamma\omega)^2\right)} \text{ -- equation 1.}$$

$$\left(\omega_o^{\,2} - \omega^2\right) = (\omega_o - \omega)(\omega_o + \omega) \approx \Delta\omega 2\omega_o$$

where $\Delta\omega = (\omega_o - \omega)$ and $(\omega_o + \omega) \approx 2\omega_o$ where we are assuming that over the narrow region of resonance $\omega \approx \omega_o$.

Therefore equation (1) becomes:

$$P_{average}(\omega) = \frac{\omega_o^2 F_o^2 \gamma}{2m\left\{(2\omega_o\Delta\omega)^2 + \gamma^2\omega_o^2\right\}} = \omega_o^2 \frac{F_o^2\gamma}{2m\gamma^2} \frac{1}{4\omega_o^2 (\Delta\omega)^2 / \gamma^2 + \omega_o^2} = \frac{F_o^2}{2m\gamma} \frac{1}{4\dfrac{(\Delta\omega)^2}{\gamma^2} + 1}$$

$P_{\max} = \dfrac{F_o^2}{2m\gamma}$ and $\dfrac{P_{\max}}{2}$ will occur when $\dfrac{4(\Delta\omega)^2}{\gamma^2} = 1$.

So the width of the power resonance curve at half-height is:

$\Delta\omega = (\omega_o - \omega) = \pm\dfrac{\gamma}{2}$.

$\omega = \omega_o \pm \dfrac{\gamma}{2}$ at half height.

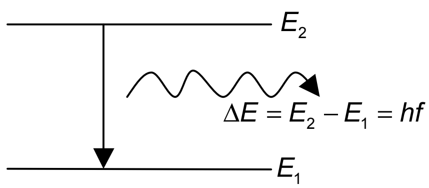
The width $= \dfrac{2\gamma}{2} = \gamma$

Width $= \gamma = \dfrac{\omega_o}{Q}$ i.e. to the reciprocal of time needed for free oscillations to decay to $\dfrac{1}{e}$ of their initial energy (see section 3.4).

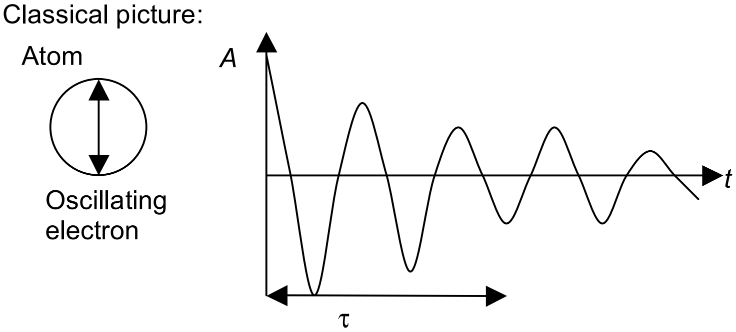
This power resonance curve is common in physics, for example optical resonance. Light is absorbed or emitted by atoms at well-defined frequencies.

<u>Emission</u>

Quantum picture:



Classical picture:



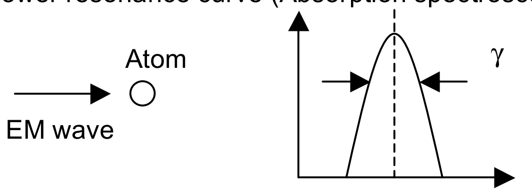
The oscillating electron produces electromagnetic radiation. This is analogous with the tuning fork. The intensity falls to $\dfrac{1}{e}$ of the initial value in the lifetime τ. $\tau \sim 10^{-8}\,s$. This gives $\gamma = 10^8\,s^{-1}$.

$\lambda = 500nm, \omega_o = 4x10^{15}\,rads^{-1}$

$$Q = \frac{\omega_o}{\gamma} = \frac{4x10^{15}}{10^8} = 4x10^7.$$

Absorption

The atom interacts with the oscillating field of electromagnetic radiation. It absorbs over a narrow range of frequencies to give the absorption line. This is essentially the same as the power resonance curve (Absorption spectroscopy).



Absorption spectrum

The width of the resonance curve $= \gamma = \frac{1}{\tau}$.

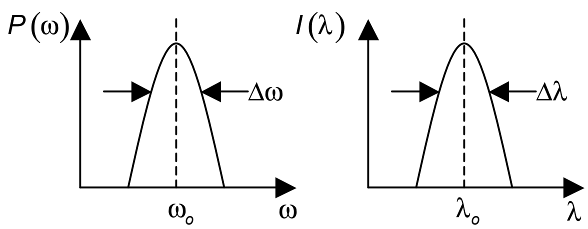
So by measuring the width of the absorption line we can deduce the lifetime of the excited state. Usually we measure the absorption spectrum by varying the wavelength of the light. We have:

$$c = f\lambda$$

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$$

$$d\lambda = -\frac{2\pi c}{\omega^2}d\omega = -\frac{\lambda\omega}{\omega^2}d\omega = -\frac{\lambda}{\omega}d\omega$$

$$\frac{d\lambda}{\lambda} = -\frac{d\omega}{\omega}$$



We are interested in the width of the peaks, so we have:

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\omega}{\omega}$$

Order of magnitude calculation:

$$\lambda = 500nm$$

$$\Delta\omega = \gamma = 10^8 s^{-1}$$

$$\omega = 4x10^{15} rads^{-1}$$

$$\therefore \Delta\lambda = \frac{10^8}{4x10^{15}}500nm = 10^{-5}nm$$

This is the width of the spectral lines.

5.6 Transient phenomena

So far we have considered steady state solutions that occur when the system has had time to settle down to the driving force. When we first apply the force the system will respond by also oscillating at its natural frequency. This transient decays away because it is a damped motion just like free oscillations. Mathematically we have:

$$\ddot{x} + \gamma\dot{x} + \omega_o{}^2 x = \frac{F_o}{m}\cos\omega t \text{ -- (1)}$$

If $x_1(t)$ is a solution of

$$\ddot{x}_1 + \gamma\dot{x}_1 + \omega_o{}^2 x_1 = \frac{F_o}{m}\cos\omega t$$

and $x_2(t)$ is a solution of

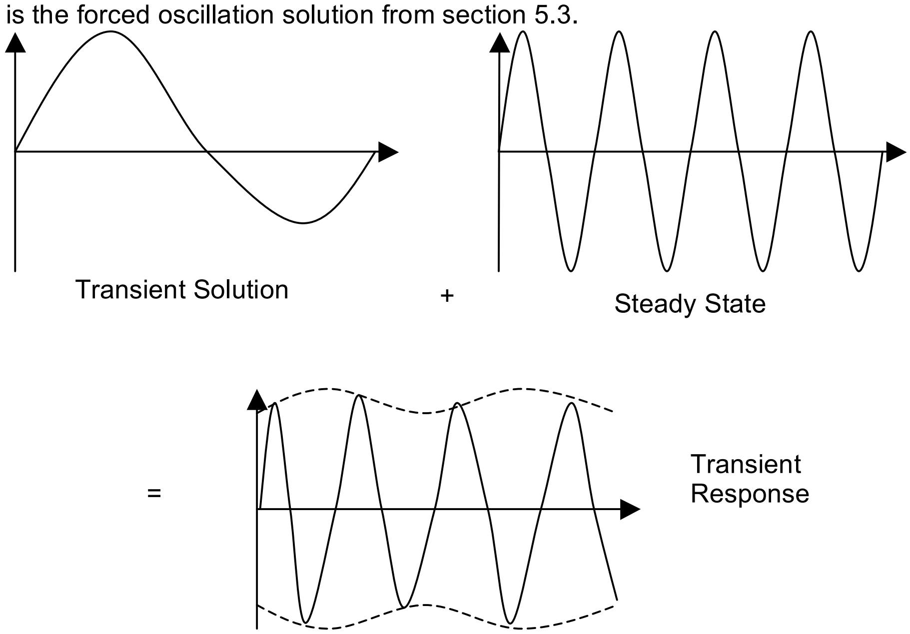
$$\ddot{x}_2 + \gamma\dot{x}_2 + \omega_o{}^2 x_2 = 0$$

Then $\frac{d^2}{dt^2}\left[x_1 + x_2\right] + \gamma\frac{d}{dt}\left[x_1 + x_2\right] + \omega_o{}^2\left[x_1 + x_2\right] = \frac{F_o}{m}\cos\omega t$

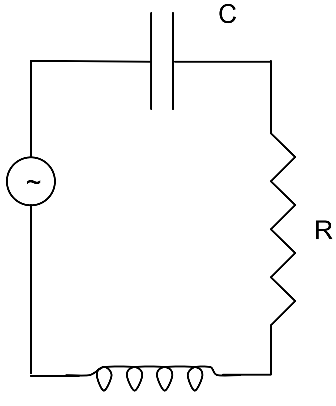
i.e. $\left[x_1 + x_2\right]$ is also a solution of equation (1). It is in fact the complete solution. For example, in the case of light damping the complete solution is:

$$x(t) = Be^{-\frac{\gamma t}{2}}\cos\left(\omega_o{}^2 - \frac{\gamma^2}{4}\right)^{1/2} + A\cos(\omega t - \phi)$$

$Be^{-\frac{\gamma t}{2}}\cos\left(\omega_o{}^2 - \frac{\gamma^2}{4}\right)^{1/2}$ is the damped oscillation solution from section 3.1, and $A\cos(\omega t - \phi)$ is the forced oscillation solution from section 5.3.



5.7 Electrical Resonance Circuits



Voltage generator $V_o \cos\omega t$.

Our LCR circuit now contains an AC voltage source.
Applying Kirchoff's law:

$$L\ddot{q} + R\dot{q} + \frac{1}{c}q = V_o \cos\omega t$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = \frac{V_o}{L}\cos\omega t$$

Compare with the general equation for the mechanical system:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_o}{m}\cos\omega t$$

$$x \equiv q$$

$$m \equiv L$$

$$b \equiv R$$

$$k \equiv \frac{1}{C}$$

$$\gamma = \frac{R}{L}$$

And driving force $F =$ driving voltage V.

All that we learnt for the mechanical oscillator, we can transfer to electrical oscillations.
For example, we can immediately write down a solution for LCR circuit. Compare:

$x(t) = A\cos(\omega t - \phi)$ where the amplitude $A(\omega) = \dfrac{F_o/m}{\left(\left(\omega_o{}^2 - \omega^2\right)^2 + (\gamma\omega)^2\right)^{1/2}}$ and $\omega_o = \sqrt{\dfrac{k}{m}}$

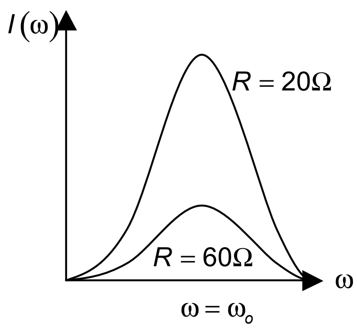
becomes $q = q_o \cos(\omega t - \phi)$ with $q_o = \dfrac{V_o/L}{\left(\left(\omega_o{}^2 - \omega^2\right)^2 + \left(\dfrac{R\omega}{L}\right)^2\right)^{1/2}}$ and $\omega_o = \sqrt{\dfrac{1}{LC}}$.

$$I = \frac{dq}{dt} = -q_o\omega\sin(\omega t - \phi) = -\frac{\omega_o \frac{V_o}{L}\sin(\omega t - \phi)}{\left(\left(\omega_o{}^2 - \omega^2\right)^2 + \left(\frac{R\omega}{L}\right)^2\right)^{1/2}} = -\frac{V_o \sin(\omega t - \phi)}{\left(L^2\left(\omega_o{}^2/\omega^2 - 1\right)^2 + R^2\right)^{1/2}} = I_o(\omega)\sin(\omega t - \phi)$$

The maximum current will occur when $\omega = \omega_o$ i.e. when $\omega = \sqrt{\frac{1}{LC}}$ the resonance frequency and $I_{o,\max} = \frac{V_o}{R}$.

e.g. $C = 10^{-8}$ farad, $L = 10^{-4}$ henrys, $R = 20$ ohms, $V_o = 100$ volts.

$\omega_o = \frac{1}{\sqrt{LC}} = 10^6 \, rads^{-1}$ and $I_o \max = \frac{100}{20} = 5$ amps.



Finally $Q = \frac{\omega_o}{\gamma} = \frac{\omega_o L}{R} = \frac{\sqrt{L/C}}{R}$

6. Coupled Oscillators

So far we have discussed systems which have only one natural frequency of oscillation, e.g. the simple pendulum. Many systems are capable of oscillating at two or more frequencies. These systems consist of two or more oscillators coupled together.

6.1 Physical Characteristics of coupled oscillators

An example is coupled pendulums.

We observe;

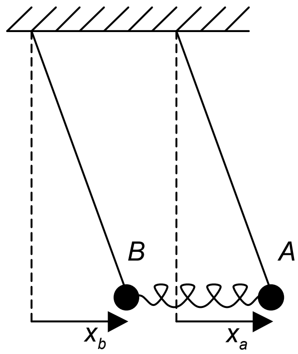
- The system can oscillate in two well-defined ways, or the normal modes of vibration.
- Once started in a particular normal mode, the system stays in that mode, i.e. it doesn't change to the other. No exchange of energy between them. The two normal modes are independent of each other. Each normal mode has a well-defined frequency.
- Can have more complicated (general) motions that are suggested to be a combination of the two independent modes.

Definition

A normal mode of vibration is one where all parts of the system vibrate at the same frequency, with constant amplitude and with a constant phase relationship.

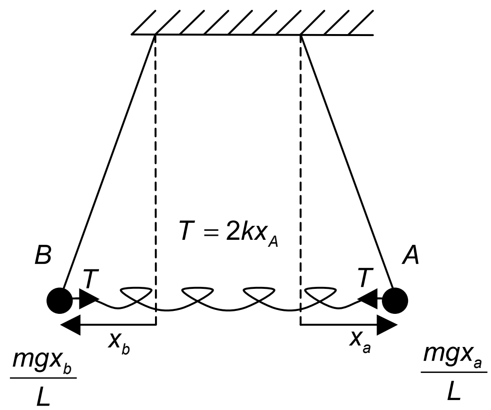
6.2 Normal modes of two pendulums with spring coupling

i) First normal mode. Here the distance between the two masses remains constant and equal to the unstretched length of the spring.



I.e. $x_A = x_B$. This is an easy solution. It means that the pendulums are essentially uncoupled so that $x_A = A\cos\omega_o t$ and $x_B = A\cos\omega_o t$ with $\omega_o = \sqrt{\frac{g}{l}}$ where for simplicity we emit phase angle ϕ i.e. $\phi = 0$. Also, A is determined by the initial boundary conditions.

ii) Second normal mode. Now we displace the masses by equal and opposite distances. This gives us alternate compression and extension of the spring, i.e. $x_A = -x_B$.



The equation of motion for mass A,

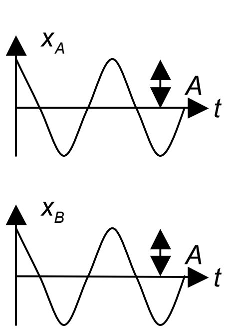
$m\ddot{x}_A = -mg\theta - 2kx_A = -\frac{mgx_A}{L} - 2kx_A$ where $\frac{mgx_A}{L}$ is the simple pendulum result, and $2kx_A$ is the extension of $2x_A$. Therefore:

$\ddot{x}_A + \left(\omega_o^{\ 2} + 2\omega_x^{\ 2}\right)x_A = 0$ where $\omega_o^{\ 2} = \frac{g}{l}$ and $\omega_C^{\ 2} = \frac{k}{m}$

Let $\omega' = \left(\omega_o^{\ 2} + 2\omega_c^{\ 2}\right)^{1/2}$ to give the SHO equation $\ddot{x}_A + \omega'^2 x_A = 0$.

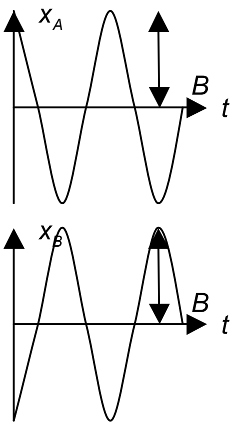
NB: The spring constant increases the restoring force and therefore increases the frequency compared to the first mode.

Then $x_A = B\cos\omega' t$ and so $x_B = -x_A = -B\cos\omega' t$ and B is determined by the initial boundary conditions.



First Normal Mode

$$\omega_o = \sqrt{\frac{g}{L}}$$



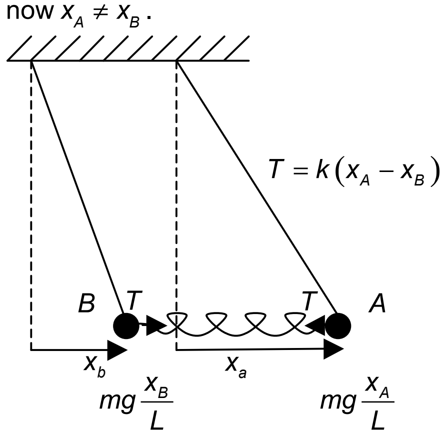
Second Normal Mode

$$\omega' = \left(\omega_o^{\,2} + 2\omega_c^{\,2}\right)^{1/2} > \omega_o$$

NB: there is a higher frequency for the second normal mode.
For each of these normal modes, the two masses move at the same frequency as each other, have constant amplitude of oscillation and there is a well-defined phase relationship between them. i.e. 0 or π.
Once started in a particular mode the motion remains in that mode: <u>there is no interaction between them</u>.

6.3 General motion of coupled pendulums (the super-position of normal modes)

Now let's consider the more general case where mass A is at x_A and mass B is at x_B and now $x_A \neq x_B$.



Spring is stretched by $\left(x_A - x_B\right)$. This produces a tension T in the spring that pulls on the masses A and B with a force $k\left(x_A - x_B\right)$. Therefore the restoring force on A is $\frac{mgx_A}{L} + k\left(x_A - x_B\right)$ and restoring force on B is $\frac{mgx_B}{L} - k\left(x_A - x_B\right)$. (Minus sign because T is in the opposite direction to restoring force).
Therefore the equations of motion:

$$m\ddot{x}_A + \frac{mg}{L}x_A + k\left(x_A - x_B\right) = 0 \; (1)$$

$$m\ddot{x}_B + \frac{mg}{L}x_B - k\left(x_A - x_B\right) = 0 \; (2)$$

These two differential equations cannot be solved separately but must be solved simultaneously. We do this as follows:
Adding (1) and (2) we obtain:

$$m\frac{d^2}{dt^2}\left(x_A + x_B\right) + \frac{mg}{L}\left(x_A + x_B\right) = 0$$

This is Simple Harmonic Motion in a new variable $\left(x_A + x_B\right)$.

If we let $\left(x_A + x_B\right) = q_1$ then $\ddot{q}_1 = -\frac{g}{L}q_1$ i.e. simple harmonic motion with frequency $\sqrt{\frac{g}{l}} = \omega_o$.
As we shall see, this change of variable is not just algebra, it is an important bit of physics.
Subtracting (2) and (1), we obtain

$$m\frac{d^2}{dt^2}\left(x_A - x_B\right) + \frac{mg}{L}\left(x_A - x_B\right) + 2k\left(x_A - x_B\right) = 0$$

or

$$\frac{d^2}{dt^2}\left(x_A - x_B\right) + \left(\omega_o^{\ 2} + 2\omega_c^{\ 2}\right)\left(x_A - x_B\right) = 0$$

where again $\omega_o^{\ 2} = \frac{g}{l}$ and $\omega_c^{\ 2} = \frac{k}{m}$.

This is simple harmonic motion in another new variable $\left(x_A - x_B\right)$. If we let $q_2 = \left(x_A - x_B\right)$ then:

$$\ddot{q}_2 = -\left(\omega_o^{\ 2} + 2\omega_c^{\ 2}\right)q_2.$$

i.e. simple harmonic motion with frequency $\omega' = \left(\omega_o^{\ 2} + 2\omega_c^{\ 2}\right)^{1/2}$

These two frequencies ω_o and ω' correspond exactly to those of the normal modes we identified before.
Thus we obtain the possible solutions of $q_1 = D\cos\omega_o t$ and $q_2 = E\cos\omega' t$ where D and E depend on the initial boundary conditions.
We identify these two with the normal modes of vibration we saw earlier. The variables q_1 and q_2 are called the <u>normal coordinates</u> while ω_o and ω' are called the <u>normal frequencies</u>.

NB:

$$x_A = \frac{1}{2}\left(\left(x_A + x_B\right) + \left(x_A + x_B\right)\right) = \frac{1}{2}\left(q_1 + q_2\right)$$
$$= \frac{1}{2}\left(D\cos\omega_o t + E\cos\omega' t\right)$$
$$x_B = \frac{1}{2}\left(\left(x_A + x_B\right) - \left(x_A - x_B\right)\right) = \frac{1}{2}\left(q_1 - q_2\right)$$
$$= \frac{1}{2}\left(D\cos\omega_o t - E\cos\omega' t\right)$$

Thus the oscillations of both masses can be described by a linear combination of <u>the two normal modes</u>.

Recap:
From our discussion of the coupled pendulums we have:
1. Spotted the two normal modes
2. Deduced their associated frequencies $\omega_o + \omega'$ in terms of m, L and k.
3. Deduced the normal coordinates $q_1 + q_2$.

4. Found general expressions for $x_A + x_B$ in terms of a linear combination of the two normal modes.

6.4 Examples of the linear combination of normal modes

Consider examples of different initial conditions i.e. the boundary conditions at $t = 0$.

Example 1:

Boundary conditions: $x_A = A_o$, $x_B = A_o$, at $t = 0$.

Since $x_A = \frac{1}{2}\left(D\cos\omega_o t + E\cos\omega' t\right)$ then $\frac{1}{2}\left(D+E\right) = A_o$ since $t = 0$.

Similarly $x_B = \frac{1}{2}\left(D-E\right) = A_o$

Adding, obtain $D = 2A_o$ and so $E = 0$ i.e.

$x_A = \frac{1}{2}D\cos\omega_o t = A_o\cos\omega_o t$ and $x_B = \frac{1}{2}D\cos\omega_o t = A_o\cos\omega_o t$.

We recognize this as the first normal mode i.e. for $E = 0$ all the motion is in a single normal mode.

Example 2:

Boundary conditions $x_A = A_o$, $x_B = -A_o$ at $t = 0$.

As before, $x_A = \frac{1}{2}\left(D\cos\omega_o t + E\cos\omega' t\right) = A_o$ i.e. $\frac{1}{2}\left(D+E\right) = A_o$ since $t = 0$.

$x_B = \frac{1}{2}\left(D-E\right) = -A_o$

Subtracting: obtain $E = 2A_o$ and so $D = 0$ i.e.

$x_A = \frac{1}{2}E\cos\omega' t = A_o\cos\omega' t$

$x_B = -\frac{1}{2}E\cos\omega' t = -A_o\cos\omega' t$

We recognise this as the second normal mode i.e. for $D = 0$ all motion is in a single mode.

Example 3:

Boundary conditions $x_A = A_o$, $x_B = 0$ (at rest) at $t = 0$.

Then $x_A = \frac{1}{2}\left(D\cos\omega_o t + E\cos\omega' t\right) = A_o$ → $x_A = \frac{1}{2}\left(D+E\right) = A_o$

$x_B = \frac{1}{2}\left(D-E\right) = 0$

Therefore $D = A_o$ and $E = A_o$ i.e. $D = E = A_o$.

Again, a linear combination of normal modes with equal amounts of each mode.

$x_A = \frac{1}{2}A_o\left(\cos\omega_o t + \cos\omega' t\right)$

$x_B = \frac{1}{2}A_o\left(\cos\omega_o t - \cos\omega' t\right)$

Recast these solutions in a different way. Recall the trig identities

$\cos\left(A \pm B\right) = \cos A\cos B \mp \sin A\sin B$

Giving:

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$
$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$

Let $(A+B)=\omega'$ and $(A-B)=\omega_o$.

Therefore by addition $2A=\omega'+\omega_o \Rightarrow A=\dfrac{(\omega'+\omega_o)}{2}$

And by subtraction $2B=\omega'-\omega_o \Rightarrow B=\dfrac{(\omega'-\omega_o)}{2}$

Therefore:

$$\cos\underbrace{(\omega')}_{(A+B)}+\cos\underbrace{(\omega_o)}_{(A-B)}=2\cos\underbrace{\left(\frac{\omega'+\omega_o}{2}\right)}_{A}\cos\underbrace{\left(\frac{\omega'-\omega_o}{2}\right)}_{B}$$

$$\Rightarrow x_A=A_o\underbrace{\cos\left(\frac{\omega'+\omega_o}{2}t\right)}_{\textit{high frequency}}\quad\underbrace{\cos\left(\frac{\omega'-\omega_o}{2}t\right)}_{\textit{low frequency MODULATION}}$$

Compare with “beating” (aka interference patterns)
Similarly,

$$\cos\omega'-\cos\omega_o=-2\sin\left(\frac{\omega'+\omega_o}{2}\right)\sin\left(\frac{\omega'-\omega_o}{2}\right)$$

$$\cos\omega_o-\cos\omega'=2\sin\left(\frac{\omega'+\omega_o}{2}\right)\sin\left(\frac{\omega'-\omega_o}{2}\right)$$

$$x_B=A_o\sin\left(\frac{\omega'+\omega_o}{2}t\right)\sin\left(\frac{\omega'-\omega_o}{2}t\right)$$

$$=A_o\cos\left(\frac{\omega'+\omega_o}{2}t+\frac{\pi}{2}\right)\cos\left(\frac{\omega'-\omega_o}{2}t+\frac{\pi}{2}\right)$$

i.e. each term is $\dfrac{\pi}{2}$ out of phase with respect to the x_A term.

(French fig. 5.3)

Note: there is an exchange of energy between the two masses but there is no exchange of energy between the two normal modes. We can see this as follows:

$$x_A=\frac{1}{2}A_o(\cos\omega_o t+\cos\omega' t)$$

$$x_B=\frac{1}{2}A_o(\cos\omega_o t-\cos\omega' t)$$

Therefore:

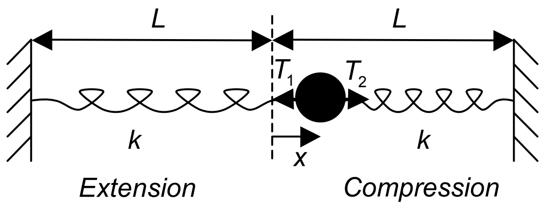
$$q_1=x_A+x_B=A_o\cos\omega_o t$$

$$q_2=x_A-x_B=A_o\cos\omega' t$$

Constant amplitude means constant energy.

6.5 Oscillating masses coupled by springs

This is an example of coupled motion that is closely related to vibrational modes of molecules. Let’s start with the simpler system of a single mass connected by two springs.



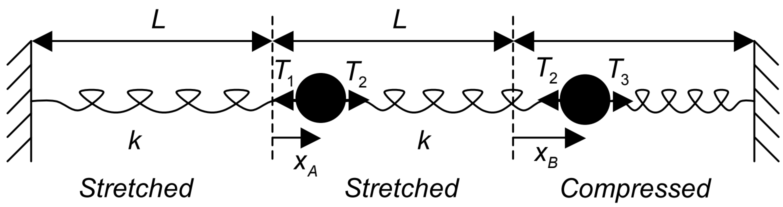
$|T_1| = |T_2| = kx$

$m\ddot{x} = -kx - kx = -2kx$

This is twice as strong as a single spring.

$\omega_o^2 = \dfrac{2k}{m}$

Now consider two masses connected by springs.



L is the equilibrium length of the springs.

Assume $x_B > x_A$ with both x_A and x_B positive.

$|T_1| = kx_A,\ |T_2| = k\left(x_B - x_A\right),\ |T_3| = kx_B,$

Then $m\ddot{x}_A = -kx_A + k\left(x_B - x_A\right)$ and $m\ddot{x} \;\; = -k\left(x_B - x_A\right) - kx_B$

Hence:

$$\ddot{x}_A + \frac{2k}{m}x_A = \frac{k}{m}x_B$$

$$\ddot{x}_B + \frac{2k}{m}x_B = \frac{k}{m}x_A$$

Letting $\omega_1^2 = \dfrac{2k}{m}$ and $\omega_2^2 = \dfrac{k}{m}$ we obtain;

$$\ddot{x}_A + \omega_1^2 x_A = \omega_2^2 x_B$$

$$\ddot{x}_B + \omega_1^2 x_B = \omega_2^2 x_A$$

These are our coupled equations. We could proceed as before, i.e. add and subtract the equations to get $\left(x_A + x_B\right)$ and $\left(x_A - x_B\right)$ etc. But let's try a more general approach. Here we don't "spot" the normal modes, but assume that for each normal mode the masses will move at the same frequency, and that their motion will be periodic. Also let's try the complex representation.

Let $x_A = Ae^{j\omega t},\ x_B = Be^{j\omega t}$.

Substitute these into the coupled equations.

$$-\omega^2 Ae^{j\omega t} + \omega_1^2 Ae^{j\omega t} = \omega_2^2 Be^{j\omega t}$$

$$-\omega^2 Be^{j\omega t} + \omega_1^2 Be^{j\omega t} = \omega_2^2 Ae^{j\omega t}$$

Thus:

$$A\left(\omega_1^2 - \omega^2\right) = B\omega_2^2$$

$$B\left(\omega_1^2 - \omega^2\right) = A\omega_2^2$$

Multiplying

$$AB\left(\omega_1^2-\omega^2\right)\left(\omega_1^2-\omega^2\right)=AB\omega_2^4$$

$$\omega^4-2\omega^2\omega_1^2+\omega_1^4-\omega_2^4=0$$

$$\omega^4-2\omega^2\omega_1^2+\left(\omega_1^4-\omega_2^4\right)=0$$

This is a quadratic equation in ω^2

Therefore:

$$\omega^2=\frac{2\omega_1^2\pm\sqrt{4\omega_1^4-4\left(\omega_1^4-\omega_2^4\right)}}{2}=\omega_1^2\pm\omega_2^2$$

$$\omega=\left(\omega_1^2+\omega_2^2\right)^{1/2}=\left[\frac{3k}{m}\right]^{1/2}$$

$$\omega=\left(\omega_1^2-\omega_2^2\right)^{1/2}=\left[\frac{k}{m}\right]^{1/2}$$

We have obtained the two normal frequencies of the system.

We also have from above;

$$A=\frac{B\omega_2^2}{\left(\omega_1^2-\omega^2\right)}$$

For $\omega^2=\omega_1^2+\omega_2^2$, $B=-A$, i.e. the masses move in opposite directions,

For $\omega^2=\omega_1^2-\omega_2^2$, $B=A$ i.e. the masses move in the same direction.

NB: this is giving us the <u>phase information</u>.

$$\boxed{\begin{aligned}x_A&=A\cos\left(\omega_1^2-\omega_2^2\right)^{1/2}t\\x_B&=A\cos\left(\omega_1^2-\omega_2^2\right)^{1/2}t\end{aligned}}$$

or

$$\boxed{\begin{aligned}x_A&=A\cos\left(\omega_1^2+\omega_2^2\right)^{1/2}t\\x_B&=-A\cos\left(\omega_1^2+\omega_2^2\right)^{1/2}t\end{aligned}}$$

Each pair corresponds to a normal mode of vibration. As usual, the general solution will be a linear combination of these i.e.

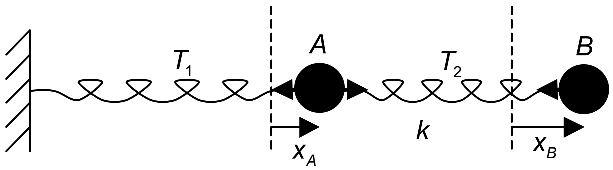
$$x_A=D\cos\left(\omega_1^2-\omega_2^2\right)^{1/2}t+E\cos\left(\omega_1^2+\omega_2^2\right)^{1/2}t$$

$$x_A=D\cos\left(\omega_1^2-\omega_2^2\right)^{1/2}t-E\cos\left(\omega_1^2+\omega_2^2\right)^{1/2}t$$

This is our familiar pattern where D and E are determined by the boundary conditions.

<u>A final example</u>

Two masses attached by springs of constant k as shown. What are the normal frequencies of the system?



$|T_1| = kx_A \quad |T_2| = k(x_B - x_A)$

Equation of motion:

$$m\ddot{x} = k(x_B - x_A) - kx_A = kx_B - 2kx_A$$

$$m\ddot{x} = -k(x_B - x_A) = -kx_B + kx_A$$

$\Rightarrow$

$$\ddot{x}_A = \omega_o^2 x_B - 2\omega_o^2 x_A \quad \left(\omega_o^2 = \frac{k}{m}\right)$$

$$\ddot{x}_B = -\omega_o^2 x_B + \omega_o^2 x_A$$

Assume $x_A = Ae^{j\omega t}$ and $x_B = Be^{j\omega t}$

Then

$$-\omega^2 Ae^{j\omega t} = \omega_o^2 Be^{j\omega t} - 2\omega_o^2 Ae^{j\omega t}$$

$$-\omega_o^2 Be^{j\omega t} = -\omega_o^2 Be^{j\omega t} + \omega_o^2 Ae^{j\omega t}$$

$$2\omega_o^2 A - \omega_o^2 B = \omega^2 A$$

$$-\omega_o^2 A + \omega_o^2 B = \omega^2 B$$

We can write this in matrix form.

$$\begin{pmatrix} 2\omega_o^2 & -\omega_o^2 \\ -\omega_o^2 & \omega_o^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \omega^2 \begin{pmatrix} A \\ B \end{pmatrix}$$

This is called an <u>eigenvalue equation</u> and the solutions of this equation for ω^2 are called the <u>eigenvalues</u>. The column vector with components A and B is an <u>eigenvector</u>. We also have:

$$\left(2\omega_o^2 - \omega^2\right)A - \omega_o^2 B = 0$$

$$-\omega_o^2 A + \left(\omega_o^2 - \omega^2\right)B = 0$$

giving

$$\begin{bmatrix} \left(2\omega_o^2 - \omega^2\right) & -\omega_o^2 \\ -\omega_o^2 & \left(\omega_o^2 - \omega^2\right) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

This has a non-zero solution if the determinant of the matrix vanishes i.e. if

$$\left(2\omega_o^2 - \omega^2\right)\left(\omega_o^2 - \omega^2\right) = \omega_o^4$$

$$2\omega_o^4 + \omega^4 - \omega^2\omega_o^2 = \omega_o^4$$

$$\omega^4 - 2\omega_o^2\omega^2 + \omega_o^4 = 0$$

This is a quadratic equation in ω^2.

Therefore

$$\omega^2 = \frac{\left(2\omega_o^{\,2} \pm \sqrt{9\omega_o^{\,4} - 4\omega_o^{\,4}}\right)}{2}$$
$$= \omega_o^{\,2}\frac{\left(3 \pm \sqrt{5}\right)}{2}$$
$$= \frac{k}{2m}\left(3 \pm \sqrt{5}\right)$$

These are the normal frequencies of the two normal modes.

NB: we could find the corresponding values of A and B by substituting solutions for ω^2 into our eigenvalue equation.

7. Waves

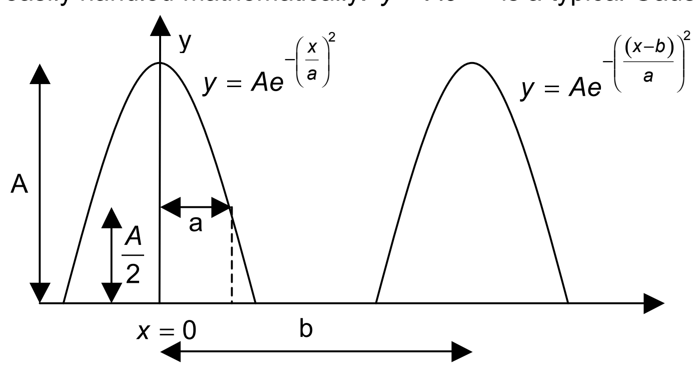
We are all familiar with waves in everyday life e.g. sound waves, ripples spreading on a lake. Here some sort of displacement travels through a medium but the medium itself is not transported. The energy is transported. The waves can be transverse so the displacement is perpendicular to the propagation axis e.g. pulses travelling down a string, or they can be longitudinal where the displacement is along the propagation axis e.g. sound waves. There is a strong link between and coupled oscillators as the number of masses increases from 2→N corresponding eventually to a continuous distribution of mass along a string.

7.1 Travelling Waves

(a) Pulses of constant shape

We see that the wave pulse travels along the string at a certain velocity, say v, and in our demonstration (roughly) holds its shape.

Let's model our pulse by a Gaussian function, this has roughly the correct shape and can be easily handled mathematically. $y = Ae^{-\frac{x^2}{a^2}}$ is a typical Gaussian function.



When $x = 0$, $y = A$

When $x = \pm a$, $y = \frac{A}{e}$

I.e. the "width" of the Gaussian is 2a.

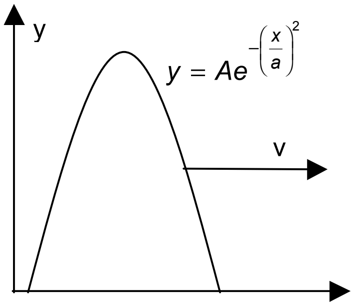
Let's now consider $y = Ae^{-\frac{(x-b)^2}{a^2}}$

Now $y = A(peak\max)$ when $x = b$, i.e. changing the variable from x to $(x-b)$ has moved the Gaussian to the right.

Now let $b = vt$ where v is the velocity and t is time.

$$y = Ae^{-\left(\frac{x-vt}{a}\right)^2}$$

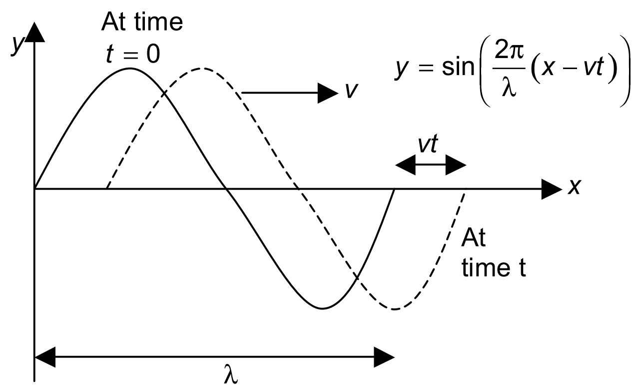
This function corresponds to the Gaussian moving at constant velocity v to the right, just like our pulse on the string. So changing the variable from x to $x - vt$ makes the Gaussian move.



(b) Travelling sine waves

Consider first $y = A\sin\left(2\pi\frac{x}{\lambda}\right)$. x and λ both have dimensions of length. The sine function goes through a complete cycle as x goes from 0 to λ, where λ is the wavelength of the wave.

Now change the variable from x to $x - vt$ i.e. $y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$



This represents a continuous sine wave that is travelling from left to right.
NB: y is a function of two variables x and t.
(i) if t is fixed, it is like a snapshot of waves on a lake.
(ii) if x is fixed, bobbing up and down on the lake.

7.2 The Wave Equation

A particular example (solution), $y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$.

Then, $\frac{\partial y}{\partial x} = \frac{2\pi}{\lambda}A\cos\frac{2\pi}{\lambda}(x - vt)$ and $\frac{\partial^2 y}{\partial x^2} = \left(\frac{2\pi}{\lambda}\right)^2 A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$.

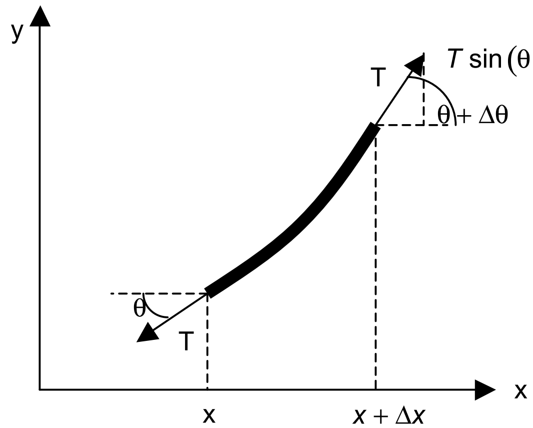
Similarly, $\frac{\partial y}{\partial t} = -v\left(\frac{2\pi}{\lambda}\right)A\cos\left(\frac{2\pi}{\lambda}(x - vt)\right)$ and $\frac{\partial^2 y}{\partial t^2} = -(-v)^2\left(\frac{2\pi}{\lambda}\right)^2 A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$.

Thus $\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$. This is the one-dimensional wave equation (1DWE).

Any function of $x - vt$, i.e. $f(x - vt)$ is a solution of the 1DWE.

7.3 The velocity of waves on a stretched string

Magnified portion of string:



String is stretched to uniform tension T mass/unit length $= \mu$

Consider a small element of the string between x and $x+dx$ and resolve forces acting on element into x and y components.

Assume transverse displacement is small so that θ and $(\theta+\Delta\theta)$ are small, i.e. $\sin\theta=\theta$.

Therefore $F_y \approx T(\theta+\Delta\theta)-T\theta = T\Delta\theta$

Similarly $F_x \approx T\cos(\theta+\Delta\theta)-T\cos\theta = T-T=0$ since $\cos\theta\approx 1$ for small θ.

Now using Newton's law $F=ma$ we obtain $T\Delta\theta = ma_y$ (1).

$m=\mu\Delta x$ and $a_y = \dfrac{\partial^2 y}{\partial t^2}$

Therefore $ma_y = \mu\Delta x \dfrac{\partial^2 y}{\partial t^2}$ (2).

We can find $\Delta\theta$ through the following way:

We have $\tan\theta = \dfrac{\partial y}{\partial x}$. Therefore $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial}{\partial x}\tan\theta = \dfrac{d}{d\theta}\tan\theta\dfrac{\partial\theta}{\partial x} = \sec^2\theta\dfrac{\partial\theta}{\partial x}$

$\sec^2\theta = \dfrac{1}{\cos^2\theta} \approx 1$ for small θ.

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} \approx \frac{\partial\theta}{\partial x} \approx \frac{\Delta\theta}{\Delta x}$$

$$\Rightarrow \Delta\theta \approx \frac{\partial^2 y}{\partial x^2}\Delta x$$

Therefore $T\Delta\theta = T\dfrac{\partial^2 y}{\partial x^2}\Delta x$ (3)

Combining (1), (2) and (3) we obtain:

$$\mu\Delta x\frac{\partial^2 y}{\partial t^2} = T\Delta x\frac{\partial^2 y}{\partial x^2}$$

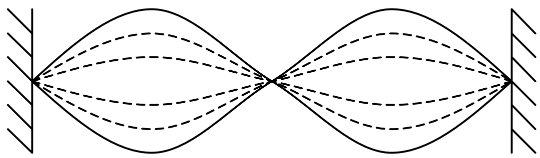
or $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{T}{\mu}\dfrac{\partial^2 y}{\partial x^2}$

This is the 1DSE for a stretched string.

Compare with $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. By comparison $v^2 = \frac{T}{\mu}$ → $v = \left(\frac{T}{\mu}\right)^{1/2}$ where v is the speed of travelling waves along the string.

7.4 Standing Waves on a stretched string

Now consider a stretched string fixed between points at $x = 0$ and $x = L$, e.g. a plucked guitar string.



The displacement of each element of the string can be described by $y(x,t) = f(x)\cos\omega t$ where $f(x)$ is the amplitude variation of the oscillation of each element of the string. $\cos\omega t$ is because all the elements of the string vibrate at the same angular frequency, c.f. coupled oscillators.

Therefore

$$\frac{\partial y}{\partial t} = -\omega f(x)\sin\omega t$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos\omega t$$

Similarly $\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f(x)}{dx^2}\cos\omega t$.

Substitute these into the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$ to obtain:

$$-\omega^2 f(x)\cos\omega t = v^2 \frac{d^2 f(x)}{dx^2}\cos\omega t$$

i.e. $\frac{d^2 f(x)}{dx^2} = -\frac{\omega^2}{v^2} f(x)$.

This is a familiar differential equation – compare this with SHM.

Since $f(x) = 0$ at $x = 0$ then the appropriate solution is $f(x) = A\sin\left(\frac{\omega}{v}\right)x$.

We also have the boundary condition that $f(x) = 0$ at $x = L$.

Therefore $A\sin\left(\frac{\omega L}{v}\right) = 0$ which is satisfied if $\frac{\omega L}{v} = n\pi$ where n is an integer.

$\Rightarrow \frac{\omega}{v} = \frac{n\pi}{L}$ and $f(x) = A\sin\left(\frac{n\pi x}{L}\right)$.

Thus ω depends on n and so we write it as ω_n.

Therefore the permitted angular frequency for each <u>mode</u>, n is:

$\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\left(\frac{T}{\mu}\right)^{1/2}$ and $y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$.

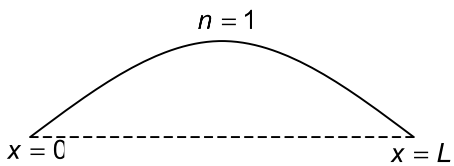
NB: since $\omega = 2\pi f$ and $v = f\lambda$ then:

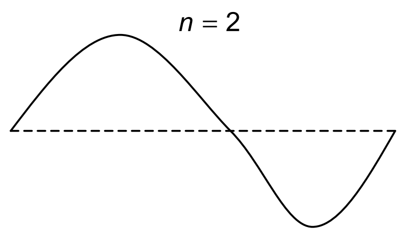
$$\frac{\omega_n L}{v} = n\pi$$

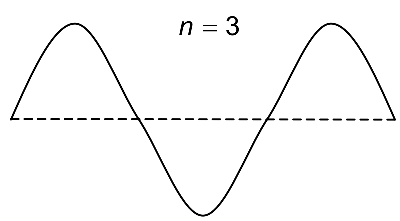
$$\frac{2\pi f L}{f\lambda_n} = n\pi$$

$$\lambda_n = \frac{2L}{n}$$

NB: we also use greek letter ν (nu) for the frequency.







We can look at this result in a different way. Let's use the identity

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

giving:

$$\sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t = \frac{1}{2}\left(\sin\left(\frac{n\pi x}{L} - \omega_n t\right) + \sin\left(\frac{n\pi x}{L} + \omega_n t\right)\right)$$

If we now substitute for $\omega_n = \frac{n\pi v}{L}$ we obtain:

$$y_n(x,t) = \frac{A_n}{2}\left(\sin\frac{n\pi}{L}(x - vt) + \sin\frac{n\pi}{L}(x + vt)\right).$$

Consider the first part and for simplicity take $n = 1$ i.e. $\frac{A_1}{2}\sin\frac{\pi}{L}(x - vt)$.

Recall $y = \sin\frac{2\pi}{\lambda}(x - vt)$ - a continuous sine wave moving at velocity v. By comparison, we recognise $\frac{A_1}{2}\sin\frac{\pi}{L}(x - vt)$ as a wave travelling in the positive x-direction with $\frac{\pi}{L} = \frac{2\pi}{\lambda}$ i.e. $\lambda = 2L$.

Similarly $\frac{A_1}{2}\sin\frac{n\pi}{L}(x + vt)$ is a wave travelling in the negative x-direction.

Conclusion: our standing wave is the sum of two travelling waves in opposite directions.

7.5 The wave number, k

We have for a travelling wave:

$$y(x,t) = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi vt}{\lambda}\right).$$

The wave number, k is defined as $\frac{2\pi}{\lambda}$. We also have $v = \lambda f$ where $f = \frac{\omega}{2\pi}$.

$$\Rightarrow \frac{2\pi v}{\lambda} = \frac{2\pi\lambda f}{\lambda} = \omega$$

giving $y(x,t) = A\sin(kx - \omega t)$.

NB: In some text books (including French) k is defined as $\frac{1}{\lambda}$. Of course none of the physics changes and so if you keep consistently to one of these definitions then there will be no problem.

7.6 The Dispersion of Waves

In general the velocity of a wave varies with wavelength. This is called dispersion e.g. dispersion occurs when white light (containing different frequencies) is passed through a glass prism. The light splits up into a rainbow of colours because the light of different colours (wavelengths) travels at different velocities in glass. $v_{red} < v_{blue}$ and so the angle of refraction also varies.

7.7 Phase & Group Velocities

For dispersion we need to think in terms of two distinctly different velocities, Consider two waves travelling in the same medium with the same amplitude but with slightly different frequencies.

$$y_1 = A\sin(k_1 x - \omega_1 t)$$

$$y_2 = A\sin(k_2 x - \omega_2 t)$$

where $v_1 = \frac{\omega_1}{k_1}$ $v_1 = \frac{\omega_2}{k_2}$ $v_1 \neq v_2$

v is the velocity
k is the wave number
ω is the frequency

The superposition of these waves is the resultant displacement

i.e. $y = A\left(\sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t)\right)$

Using $\sin(A+B) + \sin(A-B) = 2\sin A \sin B$ s

and letting $A + B = (k_1 x - \omega_1 t)$ and $A - B = k_2 x - \omega_2 t$, we obtain:

$$y = 2A\sin\left(\left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\omega_1 + \omega_2}{2}\right)t\right)\cos\left(\left(\frac{k_1 - k_2}{2}\right)x - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right)$$

Now let $\frac{k_1 - k_2}{2} - \Delta k$, $\frac{\omega_1 - \omega_2}{2} = \Delta\omega$, $\frac{k_1 + k_2}{2} = k$ and $\frac{\omega_1 + \omega_2}{2} = \omega$ i.e. the average values of wave number and frequency, so that:

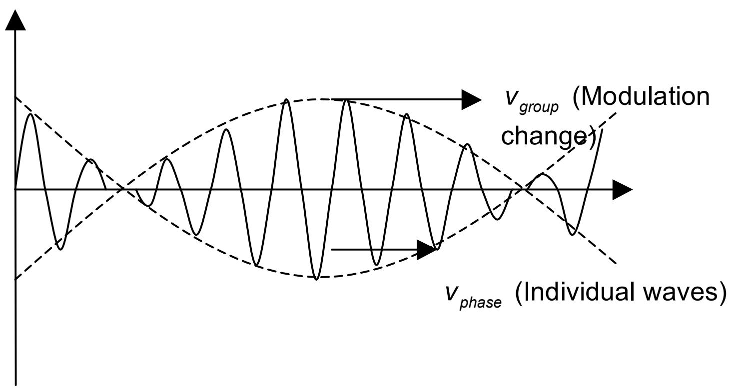
$$y = 2A\underbrace{\sin(kx - \omega t)}_{C}\underbrace{\cos(\Delta kx - \Delta\omega t)}_{D}$$

Part C represents a travelling wave with velocity $v_p = \frac{\omega}{k} = f\lambda$. This is the phase velocity (sometimes called the wave velocity).

Part D represents a travelling wave with velocity $v_g = \frac{\Delta\omega}{\Delta k} \rightarrow \frac{d\omega}{dk}$. This is the group velocity.

http://www.phys.virginia.edu/classes/109N/more_stuff/Applets/sines//groupVelocity.html

Note that the individual wave crests (the high frequency waves) move with the wave velocity, v_p, and the low frequency envelope (modulation) moves with the group velocity.
See the PC1203 Webpage on the teachweb.
It is important to note that energy and information are carried at the group velocity.



Example: Waves in deep water are strongly dispersive. Here the phase velocity $v_p = \left(\frac{g\lambda}{2\pi}\right)^{1/2}$

NB: note the dependence on λ and on g, the acceleration due to gravity.

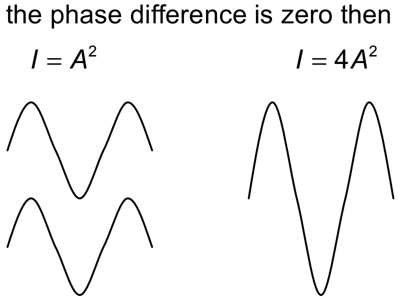
$k = \frac{2\pi}{\lambda}$ giving $v_p = \left(\frac{g}{k}\right)^{1/2} = \frac{\omega}{k}$. Rearranging gives $\omega = (gk)^{1/2}$. Therefore:

$$\frac{d\omega}{dk} = \frac{1}{2} g^{1/2} k^{-1/2} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \frac{1}{2} v_p .$$

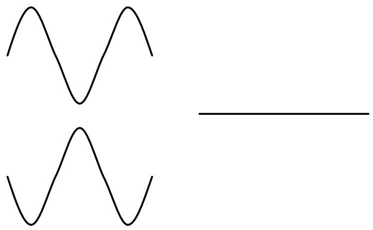
In this example the group velocity is half the phase velocity.

7.8 Interference of Waves

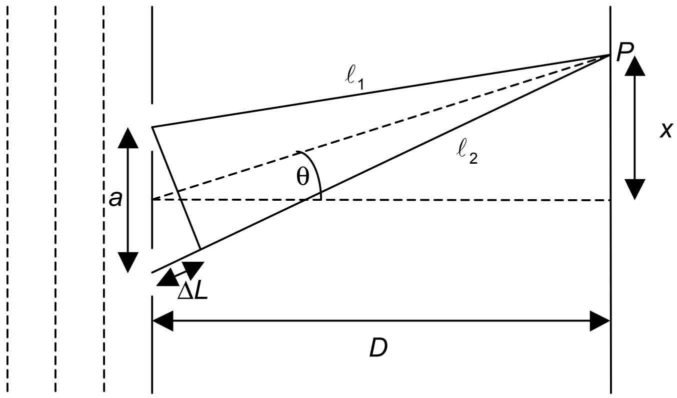
When waves with a well defined phase relationship come together, interference will occur. If the phase difference is zero then we have constructive interference.



If the phase difference is π then we have destructive interference.



A good example of this is Young's double-slit experiment.



Monochromatic light of wavelength λ is incident on the two slits. The waves come from the same source, the laser and so they have a well-defined phase relationship. They are said to be coherent.
The distance between slits is equal to a.
The distance to the screen is equal to D.
$D >> \lambda, D >> a$

Each slit acts as a source of secondary waves. The resultant displacement at P due to these two waves, each having amplitude A_1 is:

$Y = A_1 \cos\left(k\ell_1 - \omega t\right) + A_1 \cos\left(k\ell_2 - \omega t\right)$

Recall $\cos\left(\alpha - \beta\right) + \cos\left(\alpha + \beta\right) = 2\cos\alpha\cos\beta$.

As before letting $\left(\alpha - \beta\right) = \left(k\ell_1 - \omega t\right)$ and $\left(\alpha + \beta\right) = \left(k\ell_2 - \omega t\right)$ then:

$$\alpha = \frac{k\left(\ell_2 + \ell_1\right)}{2} - \omega t$$
$$\beta = \frac{k\left(\ell_2 - \ell_1\right)}{2}$$

Therefore:

$$Y = 2A_1 \cos\left[\frac{k\left(\ell_2 + \ell_1\right)}{2} - \omega t\right]\cos\left[\frac{k\left(\ell_2 - \ell_1\right)}{2}\right]$$

Since $a << D$, $\ell_1 \approx \ell_2 \approx \dfrac{D}{\cos\theta} \approx D$ since θ is small:

$$Y = 2A_1 \cos\left[kD - \omega t\right]\cos\left[\frac{k\Delta\ell}{2}\right]$$

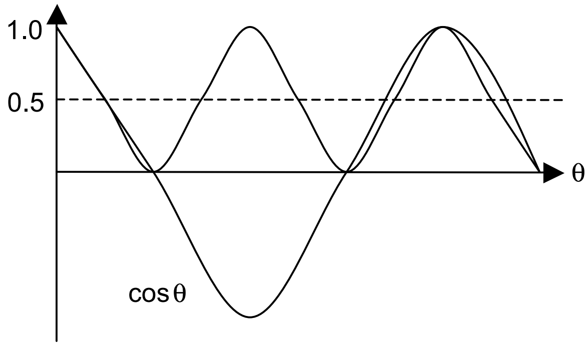
The intensity $= y^2 = 4A_1^2 \cos^2\left[kD - \omega t\right]\cos^2\left(\dfrac{k\Delta\ell}{2}\right)$

Note the intensity is 4 times greater than that of a single wave.

$\cos^2\left(kD - \omega t\right)$ is the time varying part.

$\cos^2\left(\frac{k\Delta\ell}{2}\right)$ is the spatially varying part.

Time average of $\cos^2(kD - \omega t)$ is equal to ½.



Mathematically $\frac{1}{T}\int_0^T \cos^2\left[kD - \omega t\right]dt = \frac{1}{2}$

Therefore the time average of intensity $\bar{I} = 2A_1^2\cos^2\left(\frac{k\Delta\ell}{2}\right) = I_o\cos^2\left(\frac{k\Delta\ell}{2}\right)$

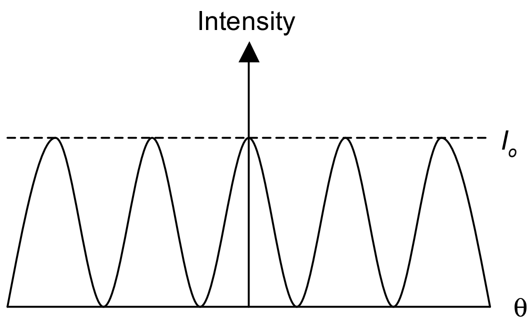
where I_o is the maximum intensity of the interference fringes.

$\Delta\ell \approx a\sin\theta$

$\bar{I} = I_o\cos^2\left(\frac{ka\sin\theta}{2}\right) = I_o\cos^2\left(\frac{\pi a\sin\theta}{\lambda}\right)$

The intensity maxima occur when $\left(\frac{\pi a\sin\theta}{\lambda}\right) = n\pi$ i.e. when $\sin\theta = \frac{n\lambda}{a}$ with n being an integer or zero.

For small θ, $\sin\theta \approx \theta \approx \frac{x}{D}$ and intensity maxima occur at distances x from point 0 given by $x = \frac{n\lambda D}{a}$.

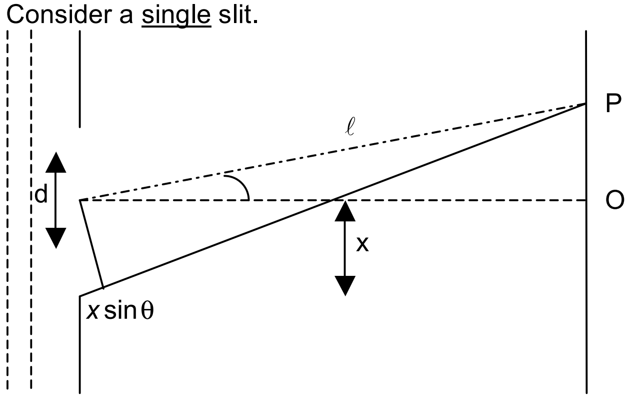


NB: in our analysis we neglected the finite length of the slits.

7.9 Diffraction of Waves

Waves bend around corners.

Consider a single slit.



Huygens Principle: each point on a wavefront acts as a source of secondary wavelets. We divide the slit up into infinitely small elements of width dx, and consider the wavelet from the small element at a distance x.

At point P the wavelet has an amplitude which is proportional to dx. $A = \beta dx$ where β is a constant of proportionality. The path difference between elements at x and $x = 0 = x\sin\theta$. Therefore the wavelet at P from element dx of slit at x can be represented by $dY = \beta dx\cos\left(k\left(\ell + x\sin\theta\right) - \omega t\right)$. Compare with $y = A\cos\left(kx - \omega t\right)$. Adding the contributions from all wavelets across the slit, the resultant disturbance at P is

$Y = \int_{-d/2}^{d/2} \beta dx \cos\left[k\left(\ell + x\sin\theta\right) - \omega t\right]$ assuming that θ has a constant value across the slit.

This integral can be readily evaluated to give:

$$Y = \beta d\cos\left(k\ell - \omega t\right)\frac{\sin\left[\frac{1}{2}kd\sin\theta\right]}{\frac{1}{2}kd\sin\theta}$$

$$\therefore I = \beta^2 d^2\cos^2\left(kl - \omega t\right)\frac{\sin^2\left(\frac{1}{2}kd\sin\theta\right)}{\left(\frac{1}{2}kd\sin\theta\right)^2}$$

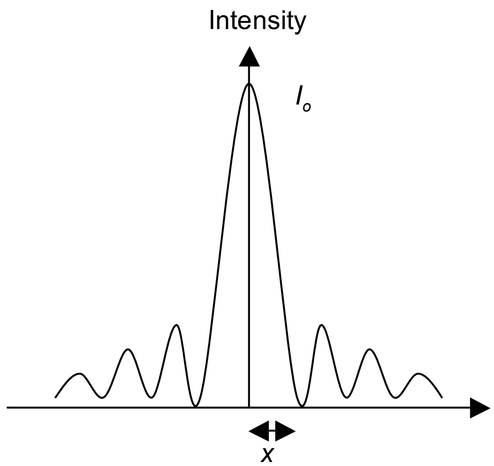
and the time averaged intensity:

$$\bar{I} = I_o\frac{\sin^2\left(\frac{1}{2}kd\sin\theta\right)}{\left(\frac{1}{2}kd\sin\theta\right)^2}$$

The function $\dfrac{\sin^2\left(\frac{1}{2}kd\sin\theta\right)}{\left(\frac{1}{2}kd\sin\theta\right)^2} = \dfrac{\sin^2\alpha}{\alpha^2}$ has its' maximum value, unity, when $\alpha = 0$.

i.e. $\dfrac{\sin^2\alpha}{\alpha^2} \to \dfrac{\alpha^2}{\alpha^2} \to 1$ when $\alpha \to 0$.

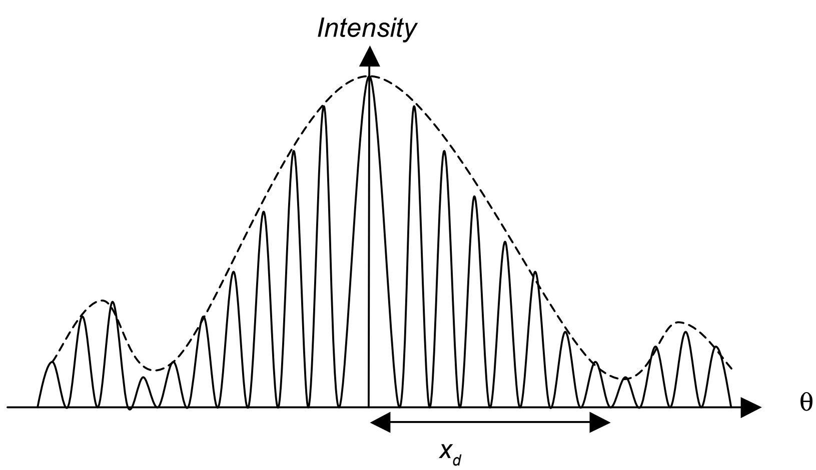
The maximum intensity thus occurs when $\theta = 0$. The function has zeros when $\sin\alpha = 0$ but $\alpha \neq 0$, e.g. the first zero occurs at $\alpha = \frac{1}{2}kd\sin\theta = \pi = \frac{1}{2}\frac{2\pi}{\lambda}d\sin\theta$ or $d\sin\theta = \lambda$. For light $\lambda << d$ and thus $\sin\theta \sim \theta$ giving the first zero at $\theta = \dfrac{\lambda}{d}$. Resultant diffraction pattern looks like:



Limit to angular resolution
Diffraction limits the angular resolution of optical instruments and hence their imaging capabilities, e.g. the ability of a telescope to distinguish two stars very close together. E.g. the Lovell telescope at Jodrell Bank $d = 76m$, looking at radiation of $1GHz\left(=10^9 Hz\right)$, $\theta \sim \dfrac{3x10^8}{10^9 x76} \sim 0.23^o$.

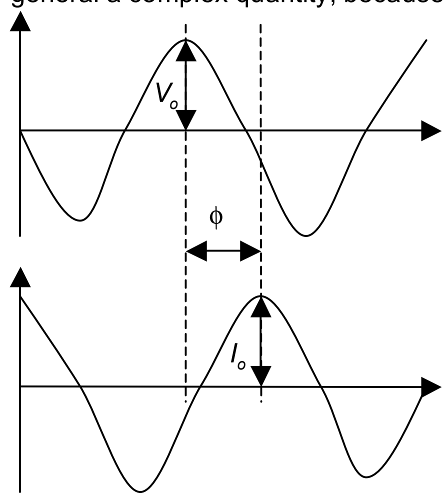
Double-slit with non-zero widths
Above we ignored the finite width of the two slits. In fact, the finite width of both slits produces a diffraction pattern that modulates the intensity of the interference fringes. The more exact distribution is given by:

$$\bar{I} = I_o \underbrace{\cos^2\left(\frac{\pi a \sin\theta}{\lambda}\right)}_{\text{int}\,erference\ fringes} \underbrace{\frac{\sin^2\left(\frac{1}{2} kd \sin\theta\right)}{\left(\frac{1}{2} kd \sin\theta\right)^2}}_{diffraction\ pattern}$$



8. AC Circuits

Recap. For DC circuits we have Ohm's law $I = \frac{V}{R}$ where R is the resistance or impedance of the circuit, and I, V and R are all real quantities e.g. $V = 15\,Volts$, $R = 50\,ohms \rightarrow I = 0.3\,amps$.
In an AC circuit, the voltage and current alternate (oscillate). Also the circuit impedance is in general a complex quantity, because it usually contains C's and L's as well as R's.

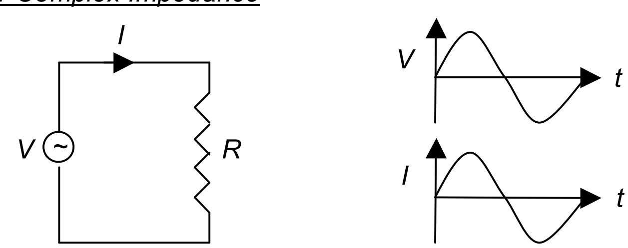


Thus $V(t) = V_o \cos\omega t$ where V_o is the amplitude and ω is the angular frequency. The current alternates at the same frequency as the voltage, but there is in general a phase shift between $I(t)$ and $V(t)$ i.e. $I(t) = I_o \cos(\omega t + \phi)$. Our job is to find I_o and ϕ for a given $V(t)$ and circuit components R, L and C.
It is very convenient to use the complex representation for $V(t)$ and $I(t)$ since it carries information about both the amplitude and the phase.
Thus $V(t)$ is represented by $V_o e^{j\omega t}$ where the real part of $V_o e^{j\omega t}$ is the actual voltage $(= V_o \cos\omega t)$. Similarly $I(t)$ is represented by $I_o e^{j(\omega t + \phi)}$.
Ohm's law now becomes:

$$I_o e^{j(\omega t + \phi)} = \frac{V_o e^{j\omega t}}{Z}$$

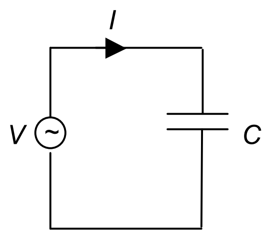
where z is the complex impendence of the circuit.

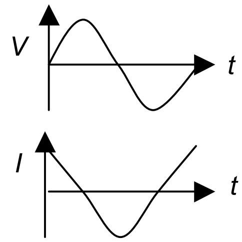
8.1 Complex Impedance



As noted above, the impedance of a resistor is real. Thus the phase difference between $I(t)$ and $V(t)$ is zero.

Complex Impedance of a Capacitor





We have $C = \frac{q}{V}$ and so $I = \frac{dq}{dt} = C\frac{dV}{dt}$

In our case $V = V_o e^{j\omega t}$ and $C\frac{d}{dt}\left(V_o e^{j\omega t}\right) = j\omega C V_o e^{j\omega t}$

Therefore $I = I_o e^{j(\omega t+\phi)} = j\omega C V_o e^{j\omega t}$ (1)

c.f. $I_o e^{j(\omega t+\phi)} = \frac{V_o e^{j\omega t}}{Z}$

The complex impedance of a capacitor is $\frac{1}{j\omega C}$.

The factor k in equation (1) tells us that the current leads the voltage by $\pi/2$.

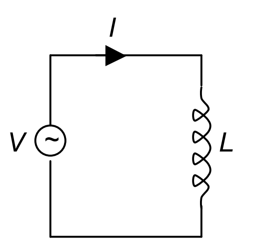
Note also that the ratio of the amplitudes:

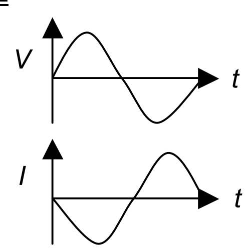
$$\frac{V_o}{I_o} = \frac{1}{\omega C} = \left|\frac{1}{j\omega C}\right|$$

The magnitude of the impedance.

In this way, we talk about the magnitude of the impedance of a capacitor at a particular frequency, e.g. the impedance of a $1x10^{-6}$ farad capacitor at $1kHz$ is $\frac{1}{2\pi x 1x10^{3} x 1x10^{-6}} = 159 ohms$.

Complex impedance of inductor L





$V = L\frac{dI}{dt} \Rightarrow I = \frac{1}{L}\int Vdt$

Again $V = V_o e^{j\omega t}$ and $\frac{1}{L}\int Vdt = \frac{1}{j\omega L} V_o e^{j\omega t}$

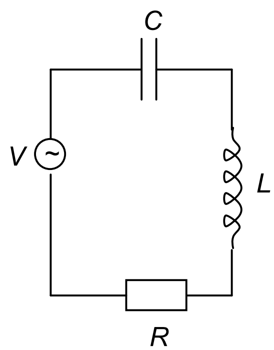
Therefore $I = I_o e^{j(\omega t+\phi)} = \frac{1}{j\omega L} V_o e^{j\omega t}$ (2)

c.f. $I_o e^{j(\omega t+\phi)} = \frac{V_o e^{j\omega t}}{Z}$

i.e. the complex impedance of an inductor is $j\omega L$. The magnitude is ωL.

The factor $\frac{1}{j} = -j$ in equation (2) tells us that the current <u>lags</u> the voltage by $\pi/2$.

8.2 The LCR Circuit



Since L, C and R are in series:

$$Z_{LCR} = Z_R + Z_C + Z_L$$
$$= R + \frac{1}{j\omega C} + j\omega L$$
$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Therefore $I_o e^{j(\omega t + \phi)} = \dfrac{V_o e^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$

Cancelling through by $e^{j\omega t}$;

$$I_o e^{j\phi} = \frac{V_o}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

To find I_o and ϕ, we multiply the top and bottom by $\left(R - j\left(\omega L - \frac{1}{\omega C}\right)\right)$ i.e. the complex conjugate.

$$I_o e^{j\phi} = \frac{V_o}{R + j\left(\omega L - \frac{1}{\omega C}\right)} x \frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{R - j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{V_o}{R^2 \left(\omega L - \frac{1}{\omega C}\right)^2}\left(R - j\left(\omega L - \frac{1}{\omega C}\right)\right)$$

Remembering that $I_o e^{j\phi} = I_o\left(\cos\phi + j\sin\phi\right)$ and equating real and imaginary parts:

$$I_o \cos\phi = \frac{V_o R}{R^2 \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$I_o \sin\phi = \frac{-V_o\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Squaring and adding:

$$I_o^{\,2} = \frac{V_o^{\,2}\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)}{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)^2} = \frac{V_o^{\,2}}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$I_o = \frac{V_o}{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)^{1/2}}$$

Dividing:

$$\tan\phi = \frac{{-V_o\left(\omega L - \frac{1}{\omega C}\right)}\Big/{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)}}{{V_o R}\Big/{\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)}}$$

$$= \frac{-\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

Note that $I_o = \frac{V_o}{\left|Z_{LCR}\right|}$ and $\tan\phi = \frac{\mathrm{Im}\left(Z_{LCR}\right)}{\mathrm{Re}\left(z_{LCR}\right)}$ which are the general results for any AC circuit.

Note also that the phase angle changes as ω is varied. At a particular frequency ω_o:

$$\omega_o L - \frac{1}{\omega_o C} = 0$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

At $\omega = \omega_o$, $I_o = \frac{V_o}{R}$ and it has its maximum value and $\tan\phi = 0$ so that the current and the voltage are in phase.

ω_o
is the resonant frequency of the circuit, which nicely takes us back to our previous discussion of the resonant LCR circuit.