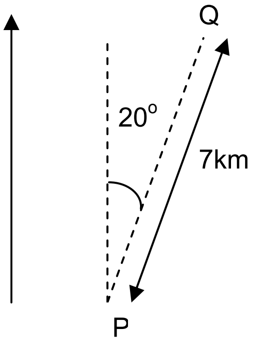
1. Vectors

1.1 Definition, Addition and Subtraction

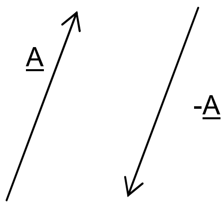
As physicists we are concerned with objects which represent physical quantities.

Scalar function / field: T(x). This could tell us the temperature at position X. It is specified by a number. No direction is mentioned. E.g. Mass.
Vectors: Objects characterised by both a *magnitude* and *direction*. e.g. the relative displacement of two points, P and Q. An example is velocity, which might be represented as $\underline{V}$. Other examples include Momentum and Angular Velocity.

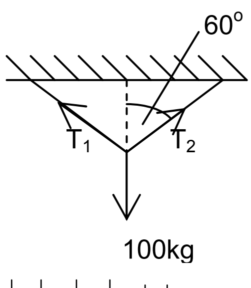


PQ is the displacement vector from P to Q. The vector tells us that Q lies 7km *(magnitude)* from P at a bearing of 20° *(Direction)*

Vector addition: if $\underline{A}$ denotes the displacement north by 3km and $\underline{B}$ the displacement east by 4km, then we can say $\underline{C} = \underline{A} + \underline{B}$. Clearly $\underline{A} + \underline{B} = \underline{B} + \underline{A}$, although addition is commutative. It is also clear that $(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C})$ for any $\underline{A}$, $\underline{B}$, and $\underline{C}$.
We can define the negativity of a vector as



So we can introduce subtraction of vectors, obvious $\underline{A}+(-\underline{A})=0$, or $\underline{A}-\underline{A}=\underline{0}$ ← $\underline{0}$ is a null vector.
Find the property that is $\lambda(\underline{A} + \underline{B}) = \lambda\underline{A} + \lambda\underline{B}$ (λ is a scalar).
i.e. addition is distributive.
It follows from rescaling the vectors.
We can solve numerous problems using this geometric effect:
e.g. a mass of 100kg is at the midpoint of the rope suspended as in the sketch. What is the tension in the rope?



$$|\underline{T_1}| = |\underline{T_2}| = |\underline{T}|$$

$$|\underline{w}| = 100kg \approx 10^3 N$$

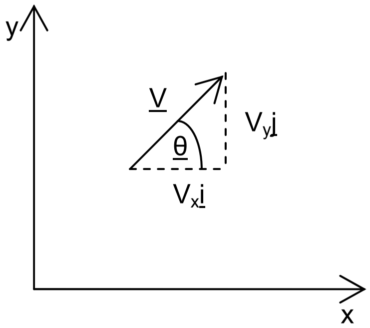
In equilibrium the sum of all the forces is zero.

$$\underline{T_1} + \underline{T_2} + \underline{w} = 0$$

Hence $\left|\underline{T_1}\right| = \left|\underline{T_2}\right| = \left|\underline{N}\right| \approx 10^3 N$

1.3 Components

So far we needed to draw a diagram in order to add or subtract vectors.
Seek an algebraic representation.
e.. any 2D vector can be expressed as a sum of two other vectors. Convenient to choose "basic vectors" to be parallel to the co-ordinate axis.



$\underline{V} = V_x \underline{i} + V_y \underline{j}$

$\left|\underline{V}\right| \cos\theta = V_x$

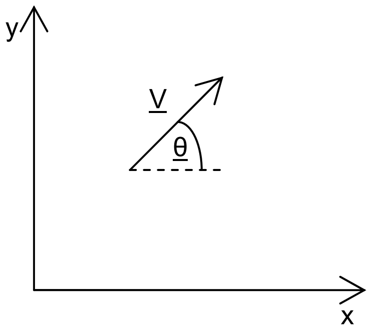
$\left|\underline{V}\right| \sin\theta = V_y$

then:

$\left|\underline{i}\right| = 1$

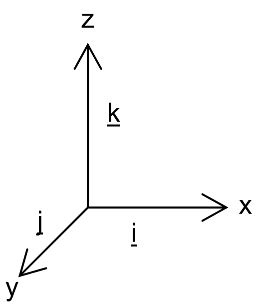
$\left|\underline{j}\right| = 1$

$\underline{i}$ and $\underline{j}$ are base vectors and have unit length.
e.g.



$\tan\theta = \frac{4}{3}$

What is $\underline{V}$ in components?
$\underline{V} = 3\underline{i}+4\underline{j} = (3,4)$
This can be generalised to 3D → need a 3rd basis basic vector parallel to the z-axis.



$\underline{V} = V_x \underline{i} + V_y \underline{j} + V_z \underline{k} = \left(V_x, V_j, V_k\right) = V_1 \underline{c_1} + V_2 \underline{c_2} + V_3 \underline{c_3}$

n.b. we just picked up a right-handed coordinate system (i.e. z-axis points "up")
The magnitude of the vector is obtained by Pythagoras.

$|\underline{A}| = \sqrt{A_x{}^2 + A_y{}^2}$

In 3D:

$|\theta| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$

It is also clear that:

$\underline{A} + \underline{B} = (A_x + B_x)\underline{i} + (A_y + B_y)\underline{j} + (A_z + B_z)\underline{k}$

1.4 Unit Vectors

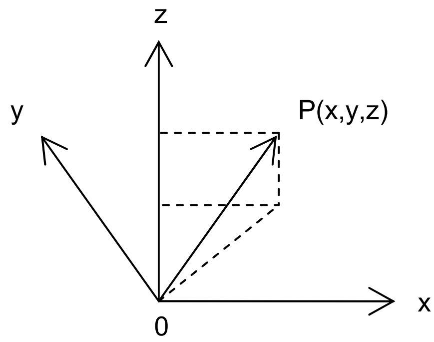
Vectors of magnitude 1 are called unit vectors (i.e. i, j, k)
e.g. to find a unit vector which is parallel to b(1,1,3)

$|b| = \sqrt{11}$

$\hat{\underline{b}} = \frac{(1,1,3)}{\sqrt{11}} = \frac{\underline{i} + \underline{j} + 3\underline{k}}{\sqrt{11}}$

1.5 Position Vectors

Consider a point P. We can represent the location of P relative to an origin O by specifying its' position vector.



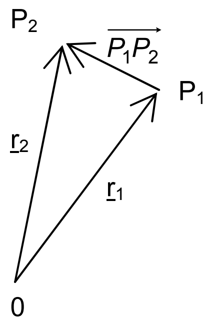
Clearly

$\overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$

$\underline{r} = \overrightarrow{OP}$

Also $|r| = \sqrt{x^2 + y^2 + z^2}$

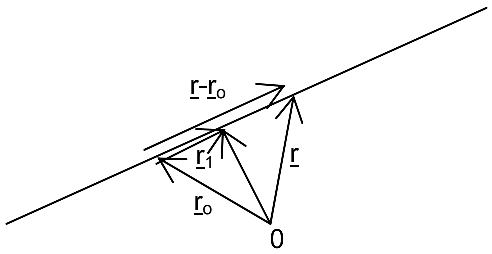
Relational positions are easy to compute.
e.g. consider 2 points P_1 and P_2 at $\underline{r}_1$ and $\underline{r}_2$. What is their relative positions?



$\overrightarrow{P_1P_2} = \underline{r}_2 - \underline{r}_1$

$\overrightarrow{P_2P_1} = \underline{r}_1 - \underline{r}_2$

This last result can be used to derive the vector equation of a line.



$$\boxed{\underline{n} = \underline{r}_o + \lambda(\underline{r}_1 - \underline{r}_o)}$$

λ is some number. It varies from -∞ to +∞ to generate all points.
$\underline{r}_1$ and $\underline{r}_o$ are any two points on the line.

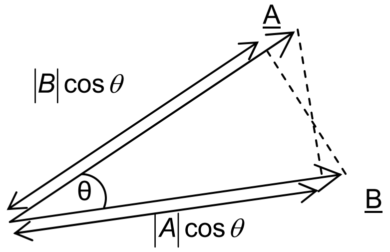
Centre of Mass:
Given a system of particles of mass $m_1, m_2, ..., m_n$ at positions $\underline{r}_1, \underline{r}_2, ... , \underline{r}_n$ the location of the centre of mass is defined to be at:

$$\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2 + ... + m_n\underline{r}_n}{m_1 + m_2 + ... + m_n}$$

1.6 Scalar Product

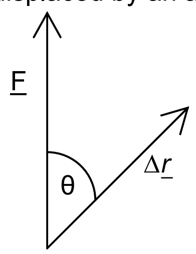
$$\boxed{\underline{A} \cdot \underline{B} = |\underline{A}||\underline{B}|\cos\theta}$$

This is the definition of the scalar product. $\underline{A}\cdot\underline{B}$ is a scalar.
It is often also called the "dot product".

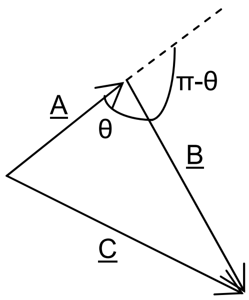


$$\underline{A} \cdot \underline{B} = (|\underline{A}|\cos\theta)|\underline{B}| = (|\underline{B}|\cos\theta)|\underline{A}|$$

e.g. a force $\underline{F}$ acts on a particle. Calculate the work done by $\underline{F}$ when the particle is displaced by an amount $\Delta\underline{r}$.



Assume $\underline{F}$ is constant.
Clearly $\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}$
Not so clear: $\underline{A} \cdot (\underline{B} + \underline{C}) = \underline{A} \cdot \underline{B} + \underline{A} \cdot \underline{C}$ (Prove it!)
For parallel vectors $\underline{A} \cdot \underline{B} = |\underline{A}||\underline{B}|$
For perpendicular $\underline{A} \cdot \underline{B} = 0$
e.g. derive the cosine rule:



$|\underline{a}| = a$

etc...

$\underline{c} = \underline{a} + \underline{b}$

$\underline{c}^2 = \underline{c} \cdot \underline{c} = (\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b})$

$\underline{c}^2 = a^2 + b^2 + 2\underline{ab}$

But $\underline{a} \cdot \underline{b} = ab\cos(\pi - \theta) = -ab\cos\theta$

Therefore $c^2 = a^2 + b^2 - 2ab\cos\theta$.

e.g. calculate $\underline{A} \cdot \underline{B}$ if A = (1,2,3) and B = (-1,2,1)

$\underline{A} \cdot \underline{B} = (\underline{i} + 2\underline{j} + 3\underline{k}) \cdot (-\underline{i} + 2\underline{j} + \underline{k})$

$= -\underline{i} \cdot \underline{i} + 2\underline{i} \cdot \underline{j} + \underline{i} \cdot \underline{k} - 2\underline{j} \cdot \underline{i} + 4\underline{j} \cdot \underline{j} + 2\underline{j} \cdot \underline{k} - 3\underline{i} \cdot \underline{k} + 6\underline{k} \cdot \underline{j} + 2\underline{k} \cdot \underline{k}$

$\underline{i} \cdot \underline{i} = 1$

etc...

$\underline{i} \cdot \underline{j} = 0$

etc...

So $\underline{A} \cdot \underline{B} = -1 + 4 + 3 = 6$

This generalises easily:

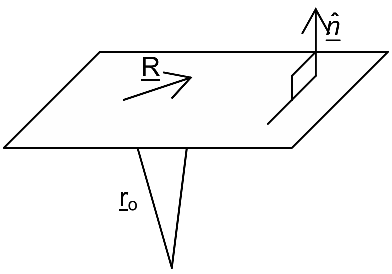
i.e. if $\underline{A} = A_x \underline{i} + A_y \underline{j} + A_z \underline{k}$ and $\underline{B} = B_x \underline{i} + B_y \underline{j} + B_z \underline{k}$

then $\underline{A} \cdot \underline{B} = A_x B_x + A_y B_y + A_z B_z$

This is a scalar quantity i.e. it doesn't depend on the choice of coordinates.

e.g. $A_x B_x - A_y B_y + A_z B_z$ is not a scalar!

1.7 Equation of a Plane



Can specify the plane uniquely by specifying a point in the plane ($\underline{r}_o$) and a vector normal to the plane ($\hat{\underline{n}}$).

$\underline{R} \cdot \hat{\underline{n}} = 0$

$\underline{R} = \underline{r} - \underline{r}_o$

We want an equation for r.

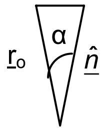
$\Rightarrow (\underline{r} - \underline{r}_o) \cdot \hat{\underline{n}} = 0$

i.e. $\boxed{\underline{r} \cdot \hat{\underline{n}} = \underline{r}_o \cdot \hat{\underline{n}}}$

This is the desired equation.

If $\underline{\hat{n}} = a\underline{i} + b\underline{j} + c\underline{k}$
$\underline{r} = (x, y, z)$

Then $\underline{r}\underline{\hat{n}} = ax + by + cz = \underline{r_o}\underline{\hat{n}}$



$\underline{r}_o \cdot \underline{\hat{n}} = r_o \cos\alpha =$ distance of plane from origin = d.

So we can write:

$\boxed{ax + by + cz = d}$

Equivalent equation of plane.

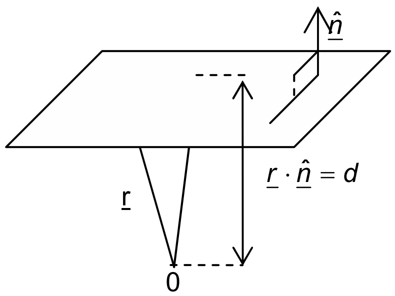
e.g. find the equation of a plane with normal $\underline{N} = (1,2,3)$ passing through the point $(-1,0,1)$.
How far is the plane from the origin?

<u>Equation of a plane</u>

$\underline{r} \cdot \underline{\hat{n}} = d$

ax+by+ca=d

If $\underline{\hat{n}} = a\underline{i} + b\underline{j} + c\underline{k}$ (nb: $a^2 + b^2 + c^2 = 1$)



Equation is $\underline{r} \cdot \underline{\hat{n}} = \underline{r}_o\underline{\hat{n}}$ where $\underline{r}_o = (-1,0,1)$.

$$\underline{\hat{n}} = \frac{(1,2,3)}{\sqrt{14}}$$

$$|\underline{\hat{n}}| = 1$$

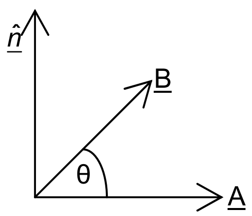
$$r_o \cdot \underline{\hat{n}} = d = \frac{-1+3}{\sqrt{14}} = \frac{2}{\sqrt{14}}$$

$$\frac{x + 2y + 3z}{\sqrt{14}} = \frac{2}{\sqrt{14}}$$

$$\Rightarrow x + 2y + 3z = 2$$

1.8 Vector Product

Definition: $\underline{A}x\underline{B} = |\underline{A}||\underline{B}|\sin\theta\underline{\hat{n}}$

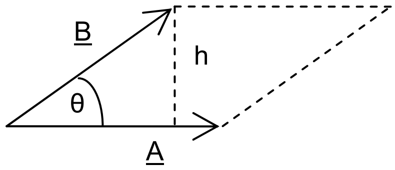


Use right-hand rule to work out which direction $\underline{\hat{n}}$ goes in.

(Twisting anti-clockwise, up is $\underline{\hat{n}}$)

Turn to get correct value.

NB: $\underline{A}$x$\underline{B}$ is a vector. $\underline{\hat{n}}$ is perpendicular to $\underline{A}$ and $\underline{B}$.
Also has a geometric interpretation:



Area of parallelogram = |A|h
But $h = |\underline{B}| \sin\theta$
So area is $= |\underline{A}x\underline{B}|$.
$|\underline{A}x\underline{B}| = |\underline{A}||\underline{B}|\sin\theta\underline{\hat{n}}$
As $\underline{\hat{n}}$ =1 this is not commutative!.
Note:
$\underline{A}x\underline{B} \neq \underline{B}x\underline{A}$
$\underline{A}x\underline{B} = -\underline{B}x\underline{A}$
and $\underline{A}x(\underline{B}+\underline{C}) = \underline{A}x\underline{B} + \underline{A}x\underline{C}$
Note that $\underline{A}x\underline{B} = \underline{0}$ for parallel vectors. $(\sin\theta = 0)$

e.g. simplify $(\underline{a}+\underline{b})x(\lambda\underline{a}+\mu\underline{b})$ (λ and μ are scalars)
$= \lambda\underline{a}^2 + \mu\underline{ab} + \lambda\underline{ab} + \mu\underline{b}^2$
$= -\lambda(\underline{a}x\underline{b}) + \mu(\underline{a}x\underline{b})$
$= (\mu-\lambda)(\underline{a}x\underline{b})$

Let us try to figure out a formula for $\underline{A}x\underline{B}$ in components.
$\underline{A}x\underline{B} = (A_x\underline{i} + A_y\underline{j} + A_z\underline{k})x(B_x\underline{i} + B_y\underline{j} + B_z\underline{k})$
We will get things like:
$\underline{i}x\underline{i} = \underline{j}x\underline{j} = \underline{k}x\underline{k} = 0$
$\underline{i}x\underline{j} = \underline{k}$
$\underline{j}x\underline{i} = -\underline{k}$
$\underline{j}x\underline{k} = \underline{i}$
$\underline{k}x\underline{j} = -\underline{i}$
$\underline{i}x\underline{k} = \underline{j}$
$\underline{k}x\underline{i} = -\underline{j}$
$\underline{A}x\underline{B} = A_xB_y\underline{k} + A_xB_z\underline{j}x\underline{k} + A_yB_x\underline{j}x\underline{i} + A_yB_z\underline{j} - A_zB_x\underline{k}x\underline{i} + A_zB_y\underline{k}x\underline{j}$
Collect components:
$\underline{A}x\underline{B} = \underline{j}(A_yB_z - A_zB_y) + \underline{j}(A_zB_x - A_xB_z) + \underline{k}(A_xB_y - A_yB_x)$
Another way to write this is a <u>determinant</u>:

$$\underline{A}x\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1) Select <u>i</u> and cover up column and row it's in
2) Cross-multiply the 4 values which remain uncovered, i.e. $A_yB_z - A_zB_y$
3) Repeat for -<u>j</u> and <u>k</u>.

e.g.:

$\underline{A} = (1,2,3)$
$\underline{B} = (-1,2,1)$
Show that $\underline{A}x\underline{B} = (-4,-4,4)$

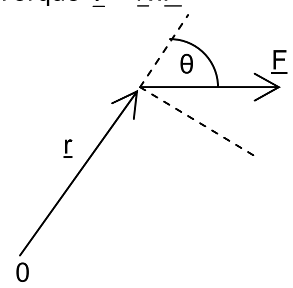
$$\underline{A}x\underline{B} = \underline{i}(A_y B_z - A_z B_y) + \underline{j}(A_z B_x - A_x B_z) + \underline{k}(A_x B_y - A_y B_x)$$
$$= \underline{i}(2x1 - 3x2) + \underline{j}(-1x3 - 1x1) + \underline{k}(1x2 + 2x1)$$
$$= (2-6)\underline{i} + (-3-1)\underline{j} + (2+2)\underline{k}$$
$$= -4\underline{i} - 4\underline{j} + 4\underline{k}$$

Calculate $\underline{A}x\underline{B}$ when $\underline{A} = (1,2,3)$ $\underline{B} = (-1,2,1)$

$$\underline{A}x\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= \underline{i}(2-6) + \underline{j}(3-1) + \underline{k}(2+2)$$
$$= -4\underline{i} - 4\underline{j} + 4\underline{k}$$
$$= (-4,-4,4)$$

1.9 Applications of the Vector Product

Some examples from dynamics:
Torque $\underline{\tau} = \underline{r}x\underline{F}$

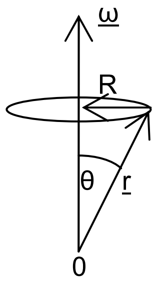


$|\underline{\tau}| = |\underline{r}||\underline{F}|\sin\theta$ which is sensible.
$\underline{\tau}$ is into the paper
→ tells us sense in which torque tends to induce rotation.
Also meet angular moment
$\underline{L} = \underline{r}x\underline{P}$
(Angular momentum relative to some origin)
Key equation in rotational dynamics is $\underline{\tau} = \frac{d\underline{L}}{dt}$ (See later)

$$\underline{F} = \frac{d\underline{P}}{dt}$$

Vector product also appears in electricity and magnetism.
$\underline{F} = q\underline{V}x\underline{B}$ (F=qVB or F=BeV)
q = electric charge
V = velocity
B = magnetic field

e.g.:
A rigid body rotates about an axis through 0. with angular velocity $\underline{\omega}$. Show that the linear velocity $\underline{v}$ of a point P in the body with position $\underline{r}$ is $\underline{v} = \underline{r}x\underline{\omega}$.



$\underline{V}$ is into paper

Body rotates about axis through 0 parallel to $\underline{\omega}$ (= angular velocity).

$|\underline{v}| = R|\underline{\omega}|$

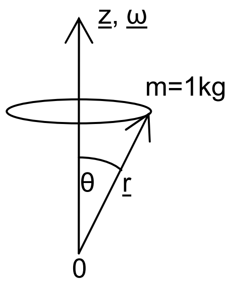
where R = radius of circle = $|\underline{r}|\sin\theta$

So $|\underline{v}| = |\underline{r}x\underline{\omega}|$

From sketch we see that

$\underline{v} = \underline{\omega}x\underline{r}$

e.g. a particle of mass 1kg is rotating about the z-axis at 4 rad.s^{-1} (i.e. $\underline{\omega}$=(0,0,4)rad.s^{-1}) at a fixed distance from 0 and at a fixed angle to the z-axis.



a) What is its' velocity when it is at $\underline{r}$(1,1,1) m?

$$\underline{v} = \underline{\omega}x\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} = -4\underline{i} + 4\underline{j} = (-4,4,0)ms^{-1}$$

b) What is the angular momentum about 0 when it is at this point?

$\underline{L} = \underline{r}x\underline{P}$

$\underline{P} = m\underline{v} = (-4,4,0)$

$$\underline{L} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ -4 & 4 & 0 \end{vmatrix} = (-4\underline{i} - 4\underline{j} + 8\underline{k})$$

n.b. $\underline{L}$ is not parallel to $\underline{\omega}$ (of L=Iω). $L_z=I_z\omega_z$ is true...)

Note: we just evaluated two vector products i.e.:-

$\underline{L} = m\underline{r}x(\underline{r}x\underline{\omega})$

There is a quicker way to evaluate such a "triple vector product".

1.10 Triple Vector Product

Identity: $\boxed{\underline{A}x(\underline{B}x\underline{C}) = (\underline{A}\cdot\underline{C})\underline{B} - (\underline{A}\cdot\underline{B})\underline{C}}$. Proof?

What is $(\underline{A}x\underline{B})x\underline{C}$? $= -\underline{C}x(\underline{A}x\underline{B}) = -(\underline{C}\cdot\underline{B})\underline{A} + (\underline{C}\cdot\underline{A})\underline{B}$

e.g. we just worked out

$\underline{L} = m\underline{r}x(\underline{\omega}x\underline{r}) = -m[(\underline{r}\cdot\underline{\omega})\underline{r} - (\underline{r}\cdot\underline{r})\underline{\omega}]$
$\underline{r}\cdot\underline{r} = r^2$
$r = |\underline{r}|$

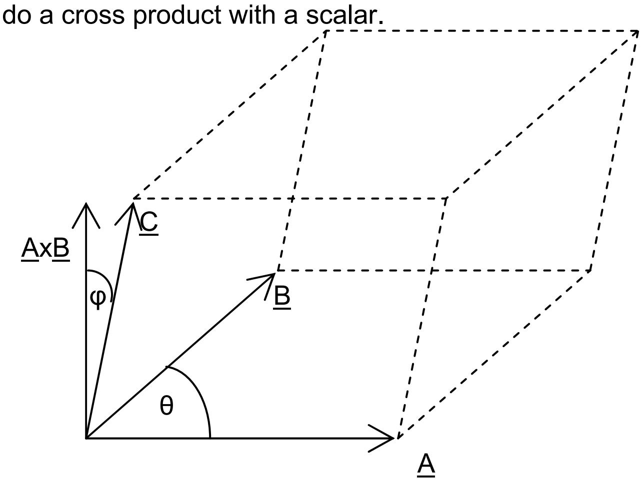
$\underline{r} = (1,1,1)m$
$\underline{\omega} = (0,0,4)rad.s^{-1}$
$m = 1kg$
$\underline{L} = -4(1,1,1) + (3)(0,0,4) = (-4,-4,8)units$
Which is as above.

1.11 Scalar Triple Product

We also compute $(\underline{A}x\underline{B})\cdot\underline{C}$ ← Scalar Triple Product.

Aside → $(\underline{A}x\underline{B})\cdot\underline{C}$ is a third way to multiply by three vectors. It is a vector, and is easy to compute.

Question: is $\underline{A}x(\underline{B}\cdot\underline{C})$ interesting? It is meaningless, as B dot C is a scalar, and you can't do a cross product with a scalar.



The above is a box formed by A, B and C.
Parallelepiped (all forces parallel in pairs)
$(\underline{A}x\underline{B})\cdot\underline{C} = |\underline{A}x\underline{B}||\underline{C}|\cos\varphi$

Volume of box = (area of base)x(perpendicular height) $= |\underline{A}x\underline{B}|\underline{C}\cos\varphi = |(\underline{A}x\underline{B})\cdot\underline{C}|$

This interpretation allows us to see that:
$(\underline{A}x\underline{B})\cdot\underline{C} = \underline{A}\cdot(\underline{B}x\underline{C}) = \underline{B}\cdot(\underline{C}x\underline{A})$
$\underline{A}\cdot(\underline{B}x\underline{C}) = -\underline{A}\cdot(\underline{C}x\underline{B})etc...$
e.g show that:

$$(\underline{A}x\underline{B})\cdot\underline{C} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z) + C_z(A_xB_y - A_yB_x)$$
$$\underline{A}x\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \cdot (C_x\underline{i} + C_y\underline{j} + C_z\underline{k})$$
$$= [\underline{i}(A_yB_z - A_zB_y) + ...](C_x\underline{i} + ...)$$
$$= C_x\underline{i}.\underline{i}\underline{(A_yB_z - A_zB_y)} = C_x(A_yB_z - A_zB_y)$$

1.12 Intersecting Planes

e.g. find the point of intersection of the 3 planes:

$3x + 3y + 2z = 3 - (1)$

$x + y + 3z = 4 - (2)$

$x + y + z = 1 - (3)$

Can solve by "brute force":

Using (3): $z = 1 - x - y$

$$x + y + 3 - 3x - 3y = 4$$

Into (2): $-2x - 2y = 1$

$$y = \frac{-1 - 2x}{2} = -x - \frac{1}{2}$$

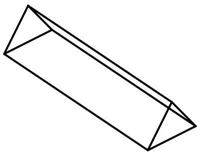
Put these into (1): $3x + 3\left(-x - \frac{1}{2}\right) + 2\left(1 - x - \left(-x - \frac{1}{2}\right)\right) - \frac{3}{2} + 2 + 1 \neq 3!$

There is no solution! The planes do not meet in a point.

$\underline{n}_1 = 3\underline{i} + 3\underline{j} + 2\underline{k}$

$\underline{n}_2 = \underline{i} + \underline{j} + 3\underline{k}$

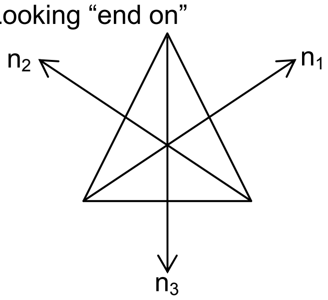
$\underline{n}_3 = \underline{i} + \underline{j} + \underline{k}$



Must be in this "Toblerone" configuration because $\underline{n}_1$, $\underline{n}_2$ and $\underline{n}_3$ are not parallel.

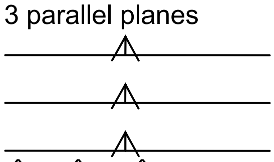
When do 3 planes not meet in a point?

Looking "end on"



The other cases which have no solution also have the property that the normal vectors lie in a plane.

I. 3 parallel planes



$\hat{\underline{n}}_1 = \hat{\underline{n}}_2 = \hat{\underline{n}}_3$

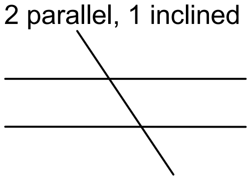
No solution at all.

II. 2 planes the same:

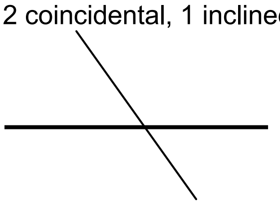
III. 3 planes the same.

All points are solutions to the original equation.

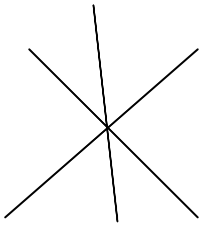
IV. 2 parallel, 1 inclined → no solution.



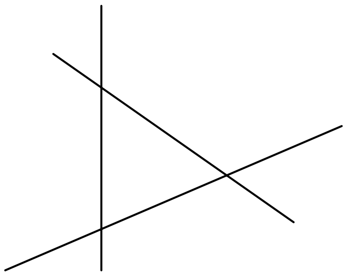
V. 2 coincidental, 1 inclined. Solution is a line.



VI. All meet in a line.



VII. Toblerone → no solution.



All have the property that normals lie in the plane of the blackboard / page.

→ $\underline{n}_1(\underline{n}_2 \cdot \underline{n}_3) = 0$

$(\underline{n}_2 \cdot \underline{n}_3)$ is the vector out of (or into) the plane. i.e. 90° to $\underline{n}_1$.

So we could have checked to see if a solution existed by calculating:

$$\underline{n}_1 \cdot (\underline{n}_2 x \underline{n}_3) = (3,3,2) \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0 \text{ therefore no unique solution.}$$

1.14 Differentiation of Vectors

It is straight forward to differentiate vectors. Generally if $\underline{A}$(s) is a vector which depends on s we can define:

$$\frac{d\underline{A}}{ds} = \lim_{\Delta s \to 0}\left[\frac{\underline{A}(s+\Delta s) - \underline{A}(s)}{\Delta s}\right]$$

In components this reduces to:

$$\frac{d\underline{A}}{ds} = \frac{dA_x}{ds}\underline{i} + \frac{dA_y}{ds}\underline{j} + \frac{dA_z}{ds}\underline{k}$$

NB: $\underline{i}$, $\underline{j}$ and $\underline{k}$ are fixed.

e.g.:-

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

is the velocity vector for a particle at position $\underline{r} = (x, y, z)$.

Newton's Second Law becomes:

$$\underline{F} = \frac{d\underline{P}dt}{\;}$$

$$\underline{P} = \frac{md\underline{r}}{dt} = m\underline{v}$$

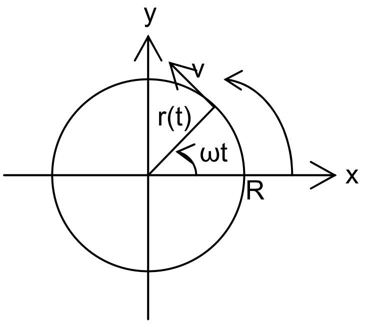
Clearly $\frac{d}{ds}(\underline{A} + \underline{B}) = \frac{d\underline{A}}{ds} + \frac{d\underline{B}}{ds}$ and $\frac{d(\lambda\underline{A})}{ds} = \lambda\frac{d\underline{A}}{ds} + \underline{A}\frac{d\lambda}{ds}$.

$$\boxed{\begin{array}{l}\dfrac{d}{ds}\left(\underline{A}\cdot\underline{B}\right)=\underline{A}\dfrac{d\underline{B}}{ds}+\underline{B}\dfrac{d\underline{A}}{ds}\\ \dfrac{d}{ds}\left(\underline{A}x\underline{B}\right)=\underline{A}x\dfrac{d\underline{B}}{ds}+\dfrac{d\underline{A}}{ds}x\underline{B}\end{array}}$$

(Exercise: prove these)

Note that the order matters in vector products.

e.g. calculate the velocity and acceleration for a body whose position is
$\underline{r}=\left(R\cos\omega t, R\sin\omega t, 0\right)$ (R, ω are positive constants)



We expect:

$|\underline{v}|=\omega R$

$\underline{v}\cdot\underline{r}=0$

$|\underline{a}|=\omega^2 R$

$\underline{a}=-\omega^2 R\hat{\underline{r}}$

$\hat{\underline{r}}$ is the unit vector pointing radially outwards.

$$\underline{v}=\frac{d\underline{r}}{dt}=\dot{\underline{r}}=R\left(-\omega\sin\omega t,\omega\cos\omega t,0\right)$$

$$\underline{a}=\frac{d^2\underline{r}}{dt^2}=\ddot{\underline{r}}=R\left(-\omega^2\cos\omega t,-\omega^2\sin\omega t,0\right)=-\omega^2R\left(\cos\omega t,\sin\omega t,0\right)=-\omega^2R\hat{\underline{r}}=-\omega^2\underline{r}$$

e.g. $\underline{L}=\underline{r}x\underline{P}$

Basic equation of rotational dynamics is:

$$\frac{d\underline{L}}{dt}=\underline{n}x\underline{F}$$

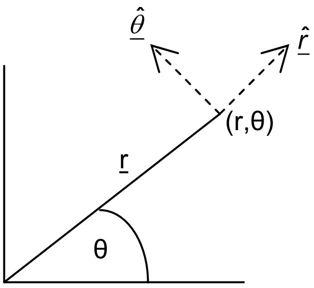
Derive this for a particle at position $\underline{r}$ and momentum $\underline{P}$.

$$\frac{d\underline{L}}{dt}=\frac{d\left(\underline{r}x\underline{P}\right)}{dt}=\frac{d\underline{r}}{dt}x\underline{P}+\underline{r}x\frac{d\underline{P}}{dt}=\underline{r}x\underline{F}$$

S far basis vectors have been fixed – sometimes it is useful to use a moving basis (!)

e.g. plane polar cords:

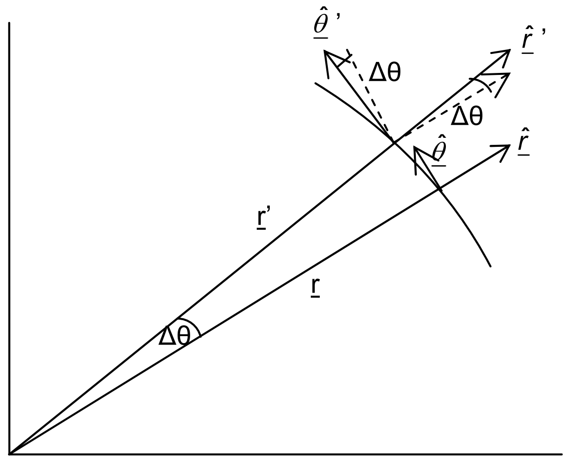
Consider a particle moving in a plane. Its' position at time t is (r,θ). Can use $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ as basis vectors.

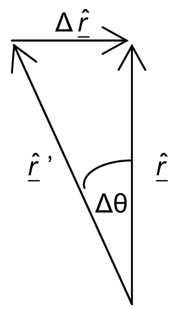


$\hat{\underline{\theta}}$, $\hat{\underline{r}}$ are unit vectors and are orthogonal $\left(\hat{\underline{\theta}} \cdot \hat{\underline{r}} = 0\right)$. But they depend upon time, t.

$\underline{r} = r\hat{\underline{r}}$

$$\frac{d\underline{r}}{dt} = \frac{dr}{dt}\hat{\underline{r}} + r\frac{d\hat{\underline{r}}}{dt}$$





$\Delta\hat{\underline{r}} = \left|\hat{\underline{r}}\right|\Delta\theta\hat{\underline{\theta}} = \Delta\theta\hat{\underline{\theta}}$

So:

$$\frac{d\underline{r}}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta\hat{\underline{r}}}{dt}\right) = \frac{d\theta}{dt}\hat{\underline{\theta}}$$

$\dot{\hat{\underline{r}}} = \dot{\theta}\hat{\underline{\theta}}$

Hence:

$\boxed{\dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}}$ Velocity in plane polar coordinates.

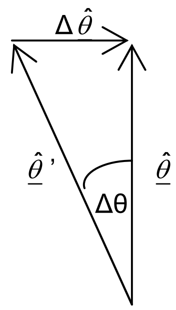
e.g. uniform circular motion:

$\dot{\underline{r}} = r\dot{\underline{\theta}}\hat{\underline{\theta}} \rightarrow "v = r\omega"$

$accel : \ddot{\underline{r}} = \ddot{r}\hat{\underline{r}} + r\hat{\underline{r}} + r\dot{\hat{\underline{r}}} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}\dot{\hat{\underline{\theta}}}$

$\dot{\hat{\underline{\theta}}}$ is nasty!

Look back at diagram:



$\Delta\underline{\hat{\theta}} = \left|\underline{\hat{\theta}}\right|\Delta\theta\left(-\underline{\hat{r}}\right) \Rightarrow \dot{\underline{\hat{\theta}}} = -\dot{\theta}\underline{\hat{r}}$

Substitute into $\ddot{\underline{r}}$ gives:

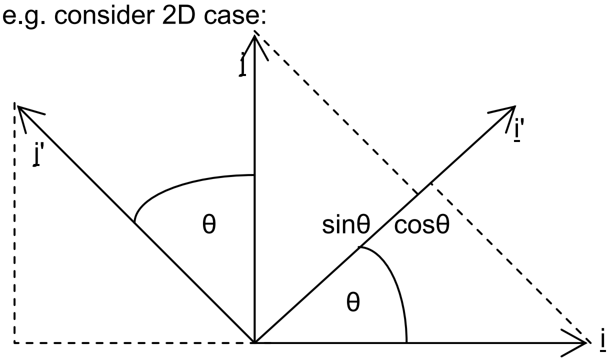
$\ddot{\underline{r}} = \ddot{r}\underline{\hat{r}} + \dot{r}\left(\dot{\theta}\underline{\hat{\theta}}\right) + \dot{r}\dot{\theta}\underline{\hat{\theta}} + r\ddot{\theta}\underline{\hat{\theta}} - r\dot{\theta}\left(\dot{\theta}\underline{\hat{\theta}}\right)$

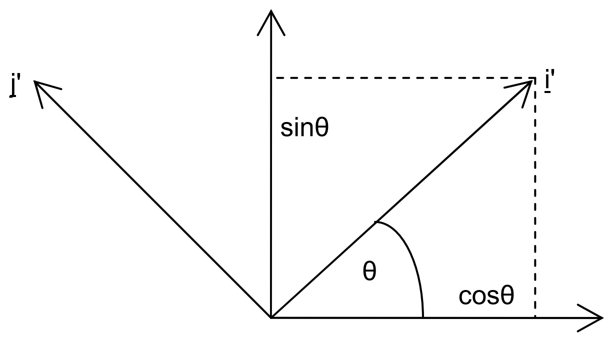
$\ddot{\underline{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{\hat{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{\hat{\theta}}$

For uniform circular motion, $\ddot{\underline{r}} = -r\dot{\theta}^2\underline{\hat{r}}$.

1.14 Rotations

We might want to convert from one basis (i, j, k) to another (i', j', k'). How do we do it?
e.g. consider 2D case:





$\underline{i}' = \cos\theta\underline{i} + \sin\theta\underline{j}$

$\underline{j}' = -\sin\theta\underline{i} + \cos\theta\underline{j}$

$\underline{V} = V_x\underline{i} + V_y\underline{j} = V_x'\underline{i}' + V_y'\underline{j}'$

$V_x' = \underline{V}\cdot\underline{i}' = V_x\underline{i}\cdot\underline{i}' + V_y\underline{j}\cdot\underline{i}' = V_x\cos\theta + V_y\sin\theta$

$V_y' = \underline{V}\cdot\underline{j}' = V_x\underline{i}\cdot\underline{j}' + V_y\underline{j}\cdot\underline{j}' = -V_x\sin\theta + V_y\cos\theta$

e.g. show that $\underline{A}\cdot\underline{B}$ is <u>invariant</u> under rotation of coordinate axis (Stick to 2D)

$\underline{A}\cdot\underline{B} = A_xB_x + A_yB_y = A_x'B_x' + A_y'B_y'$

$= \left(A_x\cos\theta + A_y sn\theta\right)\left(B_x\cos\theta + B_y\sin\theta\right) + \left(-A_x\sin\theta + A_y\cos\theta\right)\left(-B_x\sin\theta + B_y\cos\theta\right)$

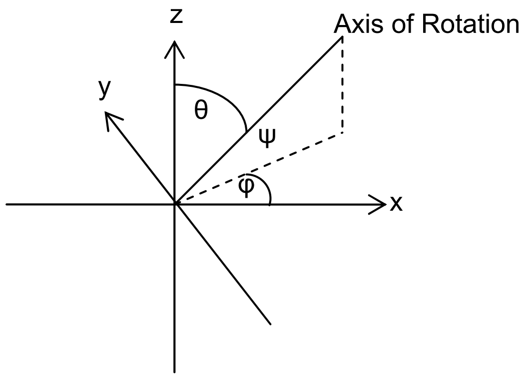
$= A_xB_x + A_yB_y = \underline{A}\cdot\underline{B} = \underline{A}'\cdot\underline{B}'$

So $\underline{A} \cdot \underline{B}$ is a <u>scalar</u>.

e.g. $A_x B_x + 2A_y B_y \neq A_x' B_x' + 2A_y' B_y'$

So $A_x B_x + 2A_y B_y$ is not a scalar.

The generalisation to 3D is in principle straight forward.
Need 3 angles to specify a general rotation.



θ and φ specify axis of rotation.
Ψ specifies angle of rotation about that axis.
Quite a bit more involved than the 2D case.

There is a neat way to express 2D rotations:

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

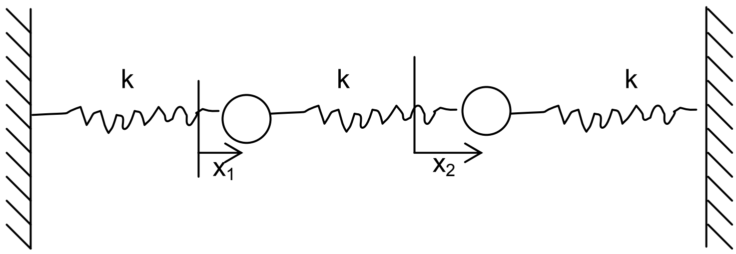
This is a “rotational axis”. $\underline{V}' = \underline{\underline{R}}\,\underline{V}$

Matrices are the subject of Section 4 of the course.

2. Determinants

2.1 Linear Equations

In physics we often encounter quantities which depend linearly on several others.
e.g. rotations in 2D:
$x' = x\cos\theta + y\sin\theta$
"x' is a combination of x and y"
e.g. Hooke's Law
"F=-kx"
Consider the case of 2 masses connected by springs.



Let F_c = Force on left hand mass (Convention, $F_c>0$ if force acts to right)
Let F_r = Force on right hand mass.

$F_c = -kx_1 + k(x_2 - x_1)$

$F_r = -kx_2 - l(x_1 - x_2)$

These are linear equations in x_1 and x_2.
In general (i.e. for unequal masses and springs) we would have:

$F_c = -(K_{11}x_1 + K_{12}x_2)$

$F_r = -(K_{21}x_1 + K_{22}x_2)$

In our case:

$K_{11} = 2k$

$K_{12} = -k$

$K_{21} = -k$

$K_{22} = 2k$

We can write these equations as:

$\underline{F} = -\underline{\underline{K}}\,\underline{x}$ ← Matrix Equation

$\underline{F} = \begin{pmatrix} F_c \\ F_r \end{pmatrix}$

$\underline{\underline{K}} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Now let's ask the following question:
Start from equilibrium and apply external forces F_1 and F_2 to the left and right masses respectively. How far do the masses move before they again come to rest?
In static equilibrium, the external forces must balance those from Hooke's law, i.e.

$F_1 + F_2 = 0, F_c + F_r = 0$

Equivalently

$F_{ext} + F = 0$

$F_{ext} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is a column vector.

→ $F_{ext}\underline{\underline{K}}\,\underline{x}$

Can't solve for $\underline{x}$ just by dividing by $\underline{\underline{K}}$ since it is a matrix.

i.e. we must solve:

$K_{11}x_1 + K_{12}x_2 = F_1 - (1)$

$F_{21}x_1 + K_{22}x_2 = F_2 - (2)$

Solving gives:

$$x_1 = \frac{K_{22}F_1 - k_{12}F_2}{K_{11}K_{22} - K_{12}K_{21}} \left(= \frac{2F_1 + F_2}{3k} \right)$$

$$x_2 = \frac{k_{11}F_2 - K_{21}F_1}{K_{11}K_{22} - K_{12}K_{21}} \left(= \frac{2F_2 + F_1}{3k} \right)$$

Solution exists providing the denominator is not zero. i.e. $K_{11}K_{22} - K_{12}K_{21} \neq 0$.

But $K_{11}K_{22} - K_{12}K_{21} = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = Det\underline{\underline{K}} = \left|\underline{\underline{K}}\right|$

This is reminiscent of the intersecting planes problem.

i.e.:

$3x + 3y + 2z = 3$

$x + y + 3z = 4$

$x + y + z = 1$

There is a unique solution if $\begin{vmatrix} 3 & 3 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$.

2.2 Determinants

Let's start with the general rule for evaluating determinants:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The cofactor of a_{11} is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, etc.

The sign of the cofactor is determined by the pattern:

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix} \text{etc...}$$

What is the cofactor of a_{21}?

$$-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Can choose <u>ANY</u> row or column to evaluate a determinant.

e.g. suppose A_{ij} is the element of a matrix at row i and column j, then detA (i.e. the determinant of A_{ij}) is:

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$.

i labels any row.

$= a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{Nj}C_{Nj}$

j is any column.

e.g. Calculate the determinant of the matrix:

$$\underline{\underline{M}} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

Choose the first row:

$$\det \underline{\underline{M}} = 1\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -2 + 15 + 1 = 14$$

Do it again for the second row:

$$\det \underline{\underline{M}} = -1\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 4 + 10 = 14$$

2.3 Uses of Determinants

We have seen 2 examples (Spring and planes) of uses already.
In general we can test to see if a system of linear equations has a solution by evaluating the appropriate determinant.

$$\sum_{j=1}^{N} a_{ij} x_j = y_i$$

$$i = 1,2,\ldots,n$$

This is a set of linear equations which we might want to solve for the x_i.
e.g. pick n=2

$$a_{11}x_1 + a_{12}x_2 = y_1$$

$$a_{21}x_1 + a_{22}x_2 = y_2$$

These n equations have a solution if:

$$\det \underline{\underline{A}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\ a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \neq 0$$

If $\det \underline{\underline{A}} = 0$ then either the equations are <u>inconsistent</u> or there is an <u>infinity</u> of solutions.

e.g. does

$$\begin{vmatrix} 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 2 & 2 & 4 \\ 2 & 1 & 0 & -1 \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$$

have a unique solution?

Will if $\begin{vmatrix} 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 2 & 2 & 4 \\ 2 & 1 & 0 & -1 \end{vmatrix} \neq 0 = 28$

3. Fields

3.1 Introduction

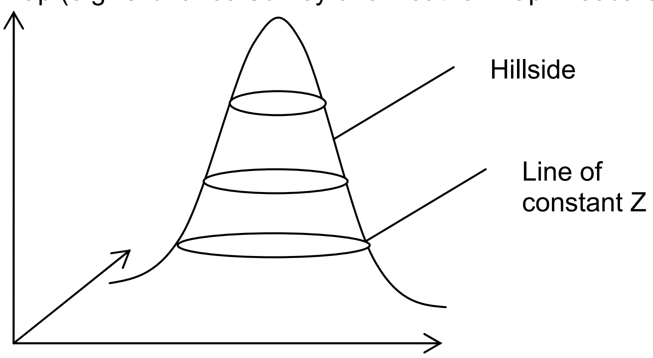
A field is used to describe a physical quantity whose value depends upon position. If the value is just a number then it is a scalar field. If it is a vector then it is a vector field.

e.g. classify the following:

- Temperature in this room – scalar field. $T(\underline{r})$
- Gravitational field in this room – vector field. $\underline{g}(\underline{r})$
- Magnetic field around a bar magnet – vector field. $\underline{B}(\underline{r})$

3.2 Functions of several variables

Functions of only one variable are pretty rare in physics. e.g. T(x,y,z), U(p,V) etc…
Functions of two variables can be represented as surfaces in 3 dimensions, or a contour map (e.g. Ordnance Survey or a weather map – isobars).

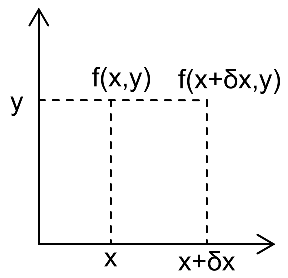


3.3 Partial Derivatives

Rates of change are crucial in physics. Need to understand how to do calculus with functions of >1 variable.

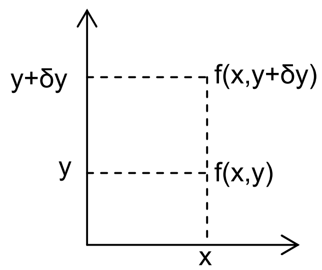
For z=f(x,y):

Define $\left(\frac{\partial f}{\partial x}\right)_y = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right]$



Clearly $\left(\frac{\partial f}{\partial x}\right)_y$ tells us the rate of change of f(x,y) in the x-direction.

Define $\left(\frac{\partial f}{\partial y}\right)_x = \lim_{\delta y \to 0}\left[\frac{f(x, y+\delta y) - f(x,y)}{\delta y}\right]$



Clearly $\left(\frac{\partial f}{\partial y}\right)_x$ tells us the rate of change of f(x,y) in the y-direction.

e.g. Given f(x,y)=x^2+xy+4y^2, calculate the slope of f(x,y) at a general point
a) in the x-direction;
b) in the y-direction.

a) In the x-direction the slope is:

$$\left(\frac{\partial f}{\partial x}\right)_y = 2x + y\left(= f_x\right)$$

Small y reminds us to keep y fixed.
"Partial df by dx"
f_x is an alternative notation.

b) In the y-direction the slope is:

$$\left(\frac{\partial f}{\partial y}\right)_x = 8y + x\left(f_y\right)$$

Higher derivatives can be computed

e.g. $\frac{\partial}{\partial x}\left[\left(\frac{\partial f}{\partial x}\right)_y\right]_y = \left(\frac{\partial^2 f}{\partial x^2}\right)_y$

In our case $\left(\frac{\partial^2 f}{\partial x^2}\right)_y = 2 = f_{xx}$

Similarly $\left(\frac{\partial^2 f}{\partial y^2}\right)_x = 8 = f_{yy}$

Also $\frac{\partial}{\partial x}\left[\left(\frac{\partial f}{\partial y}\right)_x\right]_y = \frac{\partial}{\partial x}\left(x + 8y\right)_y = 1 = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

$$\frac{\partial}{\partial y}\left[\left(\frac{\partial f}{\partial x}\right)_y\right]_x = \frac{\partial}{\partial y}\left(2x + y\right)_x = 1 = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

i.e. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

This is generally true, i.e. order of differentiation is unimportant.

e.g. for 1 mole of an ideal gas pV=RT.

What are $\left.\frac{\partial T}{\partial V}\right)_p$ and $\left.\frac{\partial T}{\partial P}\right)_v$?

$$\left.\frac{\partial T}{\partial V}\right)_p = \left.\frac{\partial}{\partial V} T(p,V)\right)_p = \frac{\partial}{\partial V}\left(\frac{pV}{R}\right)_p = \frac{p}{R}$$

$$\left.\frac{\partial T}{\partial p}\right)_V = \frac{\partial}{\partial p}\left(\frac{pV}{R}\right)_V = \frac{V}{R}$$

3.4 Total Differentials

How does f(x,y) change as we go from (x,y) to $\left(x + \delta x, y + \delta y\right)$?
Recall, for a function of 1 variable g(x)

$\delta g = g\left(x + \delta x\right) - g(x) \approx \frac{dg}{dx}\delta x$ (Taylor's theorem)

$$\delta f \approx \left(\frac{\partial f}{\partial x}\right)_y \delta x + \left(\frac{\partial f}{\partial y}\right)_x \delta y$$

In the limit $\delta x, \delta x \to 0$ we can write $\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$

df is called the "<u>total differential</u>" of f [NB generalizes to 3 dimensions

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$]

$\frac{\partial f}{\partial y}$ implies $\left(\frac{\partial f}{\partial y}\right)_{x,z}$ i.e. all other values remain constant.

e.g. $f(x,y)=x^2-xy+4y^2$. Estimate f(1.00001,1.00002)
f(1,1)=4 ← wanted more accurately.

$\delta f = (2x - y)\delta x + (-x + 8y)\delta y$

$\delta x = 10^{-5}$

$\delta y = 2x10^{-5}$

$\delta f = 10^{-5} + 14x10^{-5} = 1.5x10^{-4}$

$f(1.00001,1.00002) \approx 4.00015$

e.g. Suppose we walk along a path (x(t), y(t), z(t)) where z(t) is height above sea level. What is the rate at which we gain height?

$= \frac{dz(t)}{dt}$

What if you were given z(x,y) and x(t) and y(t)?

Can substitute for x(t) and y(t) into z(x,y) then do $\frac{dz(t)}{dt}$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

We can work this out without substituting.

Generalization of the chain rule $\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$

Suppose we now ask for the change in altitude for an infinitesimal displacement in the x-direction <u>along the path</u>.

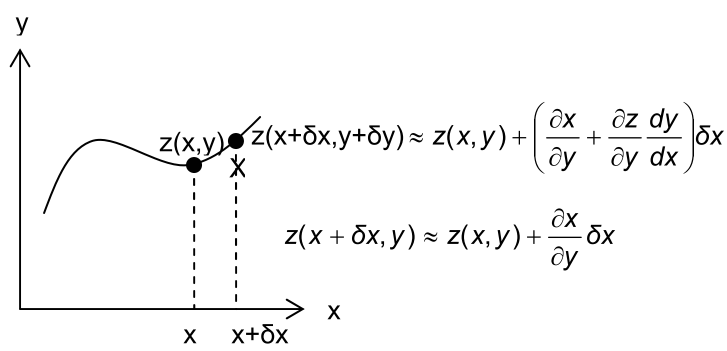
δz =(rate of change of altitude with x) δx

$\frac{\partial z}{\partial x}$ ← it is <u>not</u> this since y varies as we move along the path!

$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}$

$\boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}}$ ← "total derivative of z with respect to x"

View from above:



e.g. Suppose f(x,y) is such that x=x(u,v) and y=y(u,v). What is $\left(\frac{\partial f}{\partial v}\right)_u$?

[Again, it is possible to substitute back.]
But much easier (usually) is to do:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\left(\frac{\partial f}{\partial v}\right)_u = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial v}\right)_u + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial v}\right)_u$$

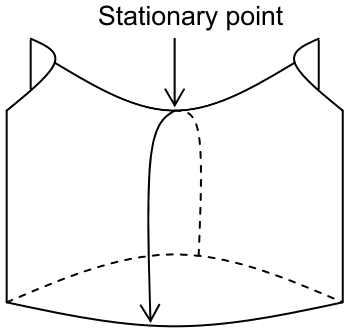
$$df = \lim_{all.changes.\inf itesimal} (\delta f)$$

3.5 Stationary Points

Are points where "all slopes vanish".

i.e. for g(x,y) stationary points are at $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$

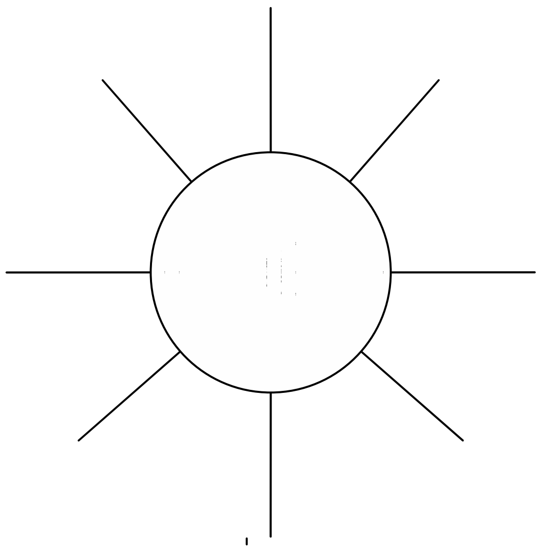
See handout for test of nature. For a function of 2 variables there is a new type of turning point. It is called a "saddle point".

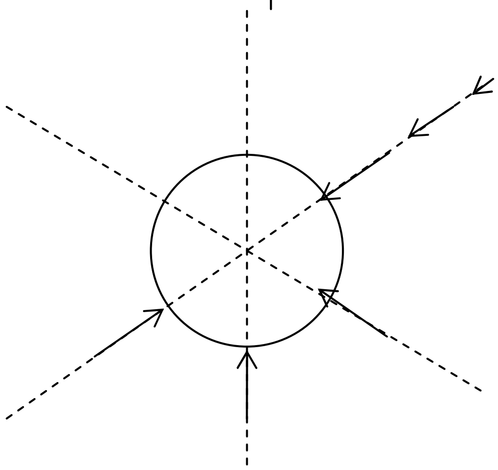


3.6 Vector Fields

How would you sketch the earth's gravitational field?

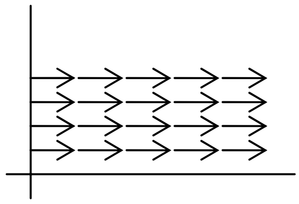
$$\underline{g}(\underline{r}) = \frac{-Gm_E}{r^2}\hat{\underline{r}}$$



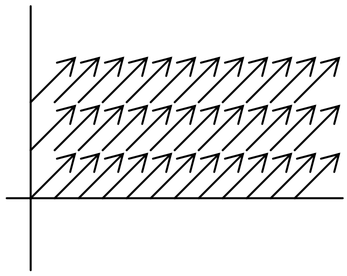


Use length of the arrow to specify strength of field.

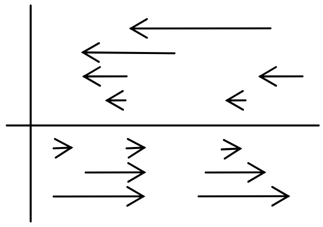
e.g. Sketch the field $\underline{A}(\underline{r}) = \underline{i}$



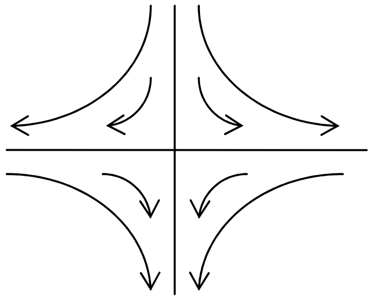
e.g. Sketch $\underline{C}(\underline{r}) = \underline{i} + \underline{j}$



e.g. sketch $\underline{D}(\underline{r}) = -y\underline{i}$



e.g. Sketch $\underline{E}(\underline{r}) = \frac{1}{2}(x\underline{i} - y\underline{j})$

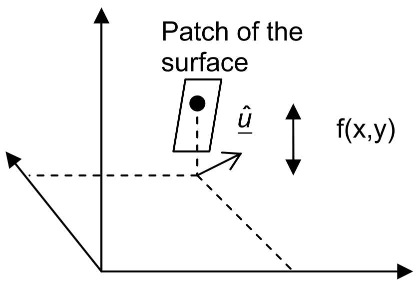


3.7 Gradient Vector

Typically we will want the rate of change of a function in some particular direction. (ie. not just $\frac{\partial f}{\partial x}$ etc)

Start off in 2 dimensions.

$f(x, y)$.



what is the rate of change of f(x,y) in the $\hat{\underline{u}}$ direction?

We want:

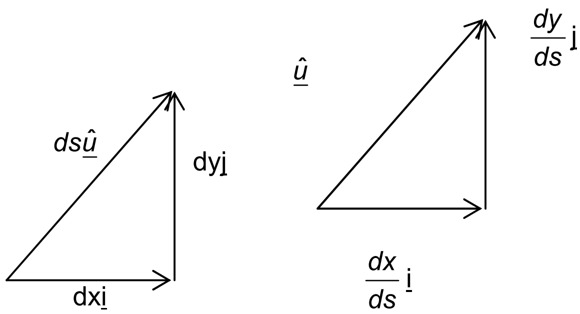
$$\left.\frac{df}{ds}\right|_{\hat{\underline{u}}} = \lim_{\delta s \to 0}\left[\frac{f(\underline{r} + \hat{\underline{u}}\delta s) - f(\underline{r})}{\delta s}\right]$$

$$\left.\frac{df}{ds}\right|_{\underline{i}} = \frac{\partial f}{\partial x}$$

$$\left.\frac{df}{ds}\right|_{\underline{j}} = \frac{\partial f}{\partial y}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\left.\frac{df}{ds}\right|_{\hat{\underline{u}}} = \frac{\partial f}{\partial x}\left.\frac{dx}{ds}\right|_{\hat{\underline{u}}} + \frac{\partial f}{\partial y}\left.\frac{dy}{ds}\right|_{\hat{\underline{u}}}$$



$$\underline{\hat{u}} = \left.\frac{dx}{ds}\right|_{\underline{\hat{u}}} \underline{i} + \left.\frac{dy}{ds}\right|_{\underline{\hat{u}}} \underline{j}$$

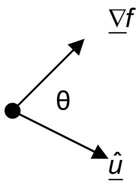
So $\left.\dfrac{df}{ds}\right|_{\underline{\hat{u}}} = \left(\underline{i}\dfrac{\partial f}{\partial x} + \underline{j}\dfrac{\partial f}{\partial y} \right).\underline{\hat{u}}$

$\left(\underline{i}\dfrac{\partial f}{\partial x} + \underline{j}\dfrac{\partial f}{\partial y} \right)$ is the gradient vector.

$$\underline{\nabla} f(x, y) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}$$

$$\underline{\nabla} f(x, y, z) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}\frac{\partial f}{\partial z}\underline{k}$$

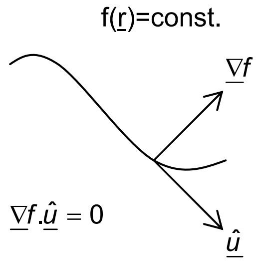
Directional derivative can be written as $\left.\dfrac{df}{ds}\right|_{\underline{\hat{u}}} = \underline{\nabla} f.\underline{\hat{u}}$ = slope of the function f in the direction $\underline{\hat{u}}$.



$\underline{\nabla} f.\underline{\hat{u}} = |\, \underline{\nabla} f \,| \cos\theta$

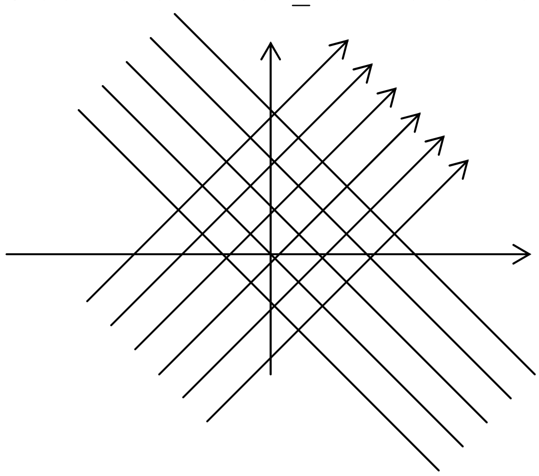
Hence $\underline{\nabla} f$ points in the "steepest uphill direction" and $|\underline{\nabla} f|$ is the slope in that direction. It is also easy to show that $\underline{\nabla} f$ evaluated at some point is perpendicular to the contour lines f($\underline{r}$)=constant at that point.

Proof:



e.g. f($\underline{r}$)=x+y

Sketch the field lines of $\underline{\nabla} f$ and the contours of constant f. x+y=C → y=C-x



$$\underline{\nabla} f = \frac{\partial f}{\partial x}\hat{\underline{i}} + \frac{\partial f}{\partial y}\hat{\underline{j}}$$

$$\underline{\nabla} f = \hat{\underline{i}} + \hat{\underline{j}}$$

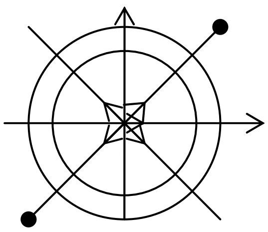
e.g. Find the gradient vector of the scalar field $g(\underline{r}) = \frac{1}{2}\left(x^2 + y^2\right)\left[= \frac{1}{2}r^2\right]$

Sketch $\underline{\nabla} g$ and the contour lines.

$$\underline{\nabla} f = \frac{\partial g}{\partial x}\hat{\underline{i}} + \frac{\partial g}{\partial y}\hat{\underline{j}}$$

$$\underline{\nabla} f = x\hat{\underline{i}} + y\hat{\underline{j}}$$

$$= \underline{r}$$



Summarize properties of $\underline{\nabla} f$:

– Slope in direction $\hat{\underline{u}}$ is $\left.\frac{df}{ds}\right|_{\hat{\underline{u}}} = \underline{\nabla} f.\hat{\underline{u}}$

– $\underline{\nabla} f$ points steepest uphill

– $\left|\underline{\nabla} f\right|$ slope in steepest uphill direction.

– $\underline{\nabla} f$ is normal to surfaces of constant f.

NB: All these statements refer to a particular point in space.

$$\underline{\nabla} f = \frac{\partial f}{\partial x}\hat{\underline{i}} + \frac{\partial f}{\partial y}\hat{\underline{j}} + \frac{\partial f}{\partial z}\hat{\underline{k}}$$

Potential energy is a scalar function U($\underline{r}$)
e.g. uniform gravitational field U($\underline{r}$)=mgz
Feel the effect of the gravitational field as a force $\underline{F}$.
$\underline{F}$=-mg $\hat{\underline{k}}$
In general: $\underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r})$
In this case $\underline{F} = -\frac{\partial U}{\partial x}\hat{\underline{i}} - \frac{\partial U}{\partial y}\hat{\underline{j}} - \frac{\partial U}{\partial z}\hat{\underline{k}} = -mg\hat{\underline{k}}$

e.g. The PE of a particle at some point $\underline{r}$ is:

$$U(\underline{r}) = \frac{-\alpha}{r}$$

What is the force acting on this particle at point $\underline{r}$?

$$\underline{F} = -\underline{\nabla} U = -\frac{\partial U}{\partial x}\hat{\underline{i}} - \frac{\partial U}{\partial y}\hat{\underline{j}} - \frac{\partial U}{\partial z}\hat{\underline{k}}$$

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} = \frac{\alpha}{r^2}\frac{x}{r}$$

Similarly for $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$.

$$\therefore \underline{F} = -\frac{\alpha x}{r^3}\underline{\hat{i}} - \frac{\alpha x}{r^3}\underline{\hat{j}} - \frac{\alpha x}{r^3}\underline{\hat{k}}$$

$$\underline{F} = -\frac{\alpha}{r^3}(x\underline{\hat{i}} + y\underline{\hat{j}} + z\underline{\hat{k}})$$

$$\underline{F} = -\frac{\alpha}{r^3}\underline{r} = -\frac{\alpha}{r^2}\underline{\hat{r}}$$

This is the 3D generalization of Coulomb's Law: $\frac{1}{r}$ potential → $\frac{1}{r^2}$ via $\underline{F} = -\underline{\nabla}U$

Exercise:

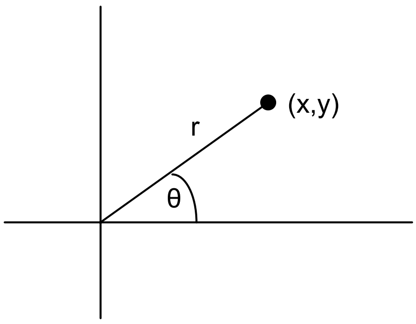
Find the gradient of $u(\underline{r}) = 3x - 2y - z$

$\underline{\nabla}u = 3\underline{i} - 2\underline{j} - \underline{k}$

3.8 Changing variables

e.g. Supposing you are given a function $f(x, y)$ and decide that you want to work in Polar coordinates (rather than Cartesian)

i.e.



$x = r\cos\theta$

$y = r\sin\theta$

Might want (e.g.)

$$\left(\frac{\partial f}{\partial r}\right)_\theta$$

Can get it by substituting back <u>or</u> can use chain rule as before.

Chain rule tells us that:

$$\left(\frac{\partial f}{\partial r}\right)_\theta = \left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial r}\right)_\theta + \left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial r}\right)_\theta$$

Can calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ directly from f(x,y)

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

$$\left(\frac{\partial y}{\partial r}\right)_\theta = \sin\theta$$

May also need (for other derivatives e.g. $\frac{\partial f}{\partial \theta}$)

$$\left(\frac{\partial x}{\partial \theta}\right)_r = -r\sin\theta$$

$$\left(\frac{\partial y}{\partial \theta}\right)_r = r\cos\theta$$

Could go "the other way", i.e. from a function of (r,θ). Then we might need $\left(\frac{\partial r}{\partial x}\right)_y$, $\left(\frac{\partial \theta}{\partial x}\right)_y$, $\left(\frac{\partial r}{\partial y}\right)_x$, $\left(\frac{\partial \theta}{\partial y}\right)_x$.

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Let's evaluate them.

$\left(\frac{\partial r}{\partial x}\right)_y$ Need to write r(x,y)

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}}$$
$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{2}\left(x^2 + y^2\right)^{\frac{1}{2}} 2x = \frac{x}{r}$$
$$\left(\frac{\partial r}{\partial y}\right)_x = \frac{1}{2}\left(x^2 + y^2\right)^{\frac{1}{2}} 2y = \frac{y}{r}$$
$$\left(\frac{\partial \theta}{\partial x}\right)_y =$$

Need

$$\theta(x,y) = \arctan\left(\frac{x}{y}\right)$$
$$\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{\partial}{\partial x}\left[\arctan\left(\frac{x}{y}\right)\right] = \frac{-y}{r^2}$$

<u>Note</u>:

$$\left(\frac{\partial r}{\partial x}\right)_y \neq \frac{1}{\left(\frac{\partial x}{\partial r}\right)_\theta}$$

Because different variables are being held fixed.

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial r}\right)_y}$$

e.g. f(x,y)=xy

Evaluate $\frac{\partial f}{\partial \theta}$ (Implied that r is fixed).

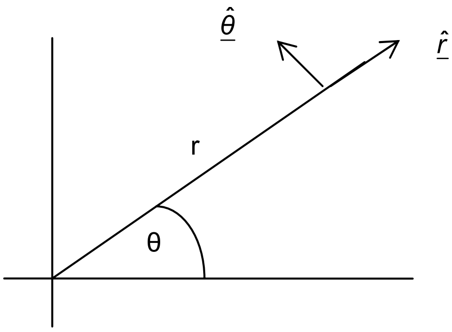
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$
$$\frac{\partial f}{\partial \theta} = yr\sin\theta + xr\cos\theta$$
$$\frac{\partial f}{\partial \theta} = -r^2\sin^2\theta + r^2\cos^2\theta = r^2\left(\cos^2\theta - \sin^2\theta\right)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

<u>check</u> by substitution.

$f = xy = r^2 \sin\theta\cos\theta$

$\frac{\partial f}{\partial\theta} = r^2\left(\cos^2\theta - \sin^2\theta\right)$

How do we compute the gradient in polar coordinates?

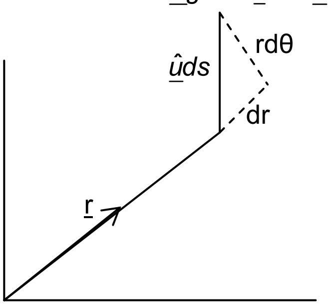


Given g(r,θ), what is $\underline{\nabla} g$?

$\underline{\nabla} g = \frac{\partial g}{\partial r}\underline{\hat{r}} + \frac{\partial g}{\partial\theta}\underline{\hat{\theta}}$ Tempting, but wrong.

To derive it properly we'll consider the directional derivative $\underline{\nabla} g.\underline{\hat{u}} = \left.\frac{dg}{ds}\right|_{\underline{\hat{u}}}$.

We know that $\underline{\nabla} g = A\underline{\hat{r}} + B\underline{\hat{\theta}}$. Goal is to figure out A and B.



$\underline{u}ds = dr\underline{\hat{r}} + rd\theta\underline{\hat{\theta}}$

$\therefore \underline{\hat{u}} = \frac{dr}{ds}\underline{\hat{r}} + r\frac{d\theta}{ds}\underline{\hat{\theta}}$

So

$\underline{\nabla} g.\underline{\hat{u}} = \left.\frac{dg}{ds}\right|_{\underline{\hat{u}}}$

LHS is $A\frac{dr}{ds} + Br\frac{d\theta}{ds}$

But we can use the chain rule to simplify RHS.

$dg = \frac{\partial g}{dr}dr + \frac{\partial g}{\partial\theta}d\theta$

$\therefore \frac{dg}{ds} = \frac{\partial g}{dr}\frac{dr}{ds} + \frac{\partial g}{\partial\theta}\frac{d\theta}{ds} = A\frac{dr}{ds} + Br\frac{d\theta}{ds}$

$\therefore A = \frac{\partial g}{dr}, B = \frac{\partial g}{\partial\theta}$

Hence

$$\boxed{\underline{\nabla} g = \frac{\partial g}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial g}{\partial\theta}\underline{\hat{\theta}}}$$

e.g. $\underline{\nabla} r = \underline{\hat{r}}$ from above.

$$\left[\frac{\partial r}{\partial x}\underline{i}+\frac{\partial r}{\partial y}\underline{j}+\frac{\partial r}{\partial z}\underline{k}=\underline{\nabla}r=\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)\underline{i}+\ldots=\frac{x}{r}\underline{i}+\frac{y}{r}\underline{j}+\frac{z}{r}\underline{k}=\frac{\underline{r}}{r}=\underline{\hat{r}}\right]$$

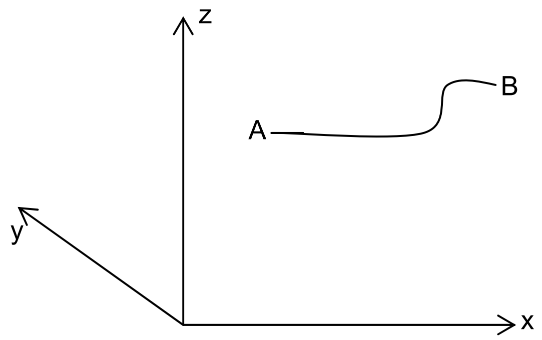
e.g. Coulomb's law $U=-\frac{\alpha}{r}$, α constant.

$$\underline{F}=-\underline{\nabla}U=\frac{\partial u}{\partial r}\underline{\hat{r}}=-\frac{\alpha}{r^2}\underline{\hat{r}}$$

3.9 Line Integrals

How do you calculate the length of a curve in 3 dimensions?

$$d\underline{l}=dx\underline{i}+dy\underline{j}+dz\underline{k}$$



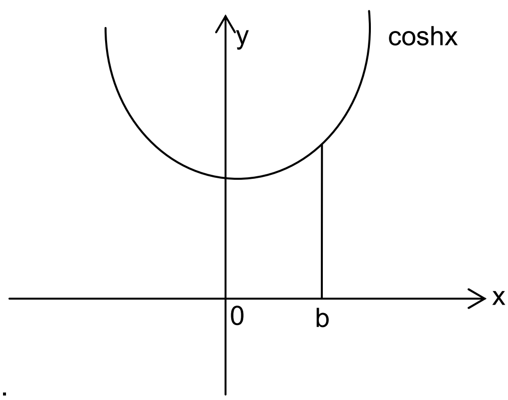
Length of curve from A to B $=\lim_{\delta l\to 0}\sum_{all.bits}\delta l=\int_A^B dl=\int_C dl$

(C denotes the curve from A to B)

$$dl=\sqrt{(dx)^2+(dy)^2+(dz)^2}=dx\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}$$

So length $=\int_{x_A}^{x_B}dx\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}$

e.g. Calculate the length of the curve defined by $\begin{matrix}y(x)=\cosh x\\ z=0\end{matrix}$ as x varies from 0 to



b.

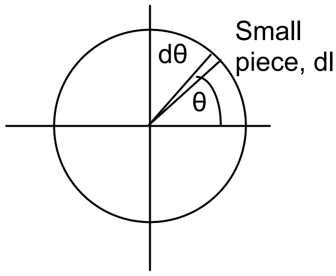
Length $=\int_0^b dx\sqrt{1+\sinh^2 x}=\int_0^b dx\cosh x=\left[\sinh x\right]_0^b=\sinh b$

Curve may be defined parametrically, i.e. $x=x(s),y=y(s),z=z(s)$ (eg. s could be time)

$$dl = \sqrt{dx^2 + dy^2 + dz^2}$$

$$\int_C dl = \int_{s_A}^{s_B} ds \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

Example: Calculate the mass of a circular hoop of radius a whose mass per unit length $p(\theta) = \theta$



$$dl = \sqrt{(dx)^2 + (dy)^2}$$

$$\int dl\rho = \int dx\rho(\theta)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = horrible$$

$dl\rho$ = mass of element

$$\int dl\rho = \int d\theta\rho(\theta)\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$x = a\cos\theta, \therefore \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = a\cos\theta$$

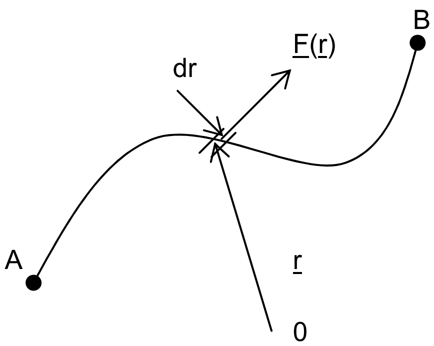
$$\int dl\rho = \int_0^{2\pi} d\theta.\theta\sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a\int_0^{2\pi}\theta d\theta = a\left[\frac{1}{2}\theta^2\right]_0^{2\pi} = 2\pi^2 a$$

$$dl = ad\theta$$

$$\therefore Mass = \int dlp(\theta) = \int_0^{2\pi} ad\theta\theta = 2\pi^2 a$$

3.10 Line Integrals Involving vector fields

Example: Write down a formula for the work done by a force $\underline{F}(\underline{r})$ which acts on a particle that moves from A to B along some curve.



dW = work done by $\underline{F}$ as particle moves from $\underline{r}$ to $\underline{r}$+d$\underline{r}$.

Total work done by $\underline{F} = \int_A^B dW$

$dW = \underline{F}.d\underline{r}$

Therefore Total Work $= \int_A^B \underline{F}.d\underline{r}$

Example: What is the work done by $\underline{F} = -k\underline{r}$ in going from (0,0) to (1,1)

a)along the curve $y = x^3$

$$W = \int \left(-k\underline{r}\right).\left(dx\underline{i} + dy\underline{j}\right)$$

$$\underline{r} = x\underline{i} + y\underline{j}$$

$$d\underline{r} = dx\underline{i} + dy\underline{j}$$

$$W = -k\int \left(xdx + ydy\right) = -k\left(\int_0^1 xdx + \int_0^1 ydy\right) = -k$$

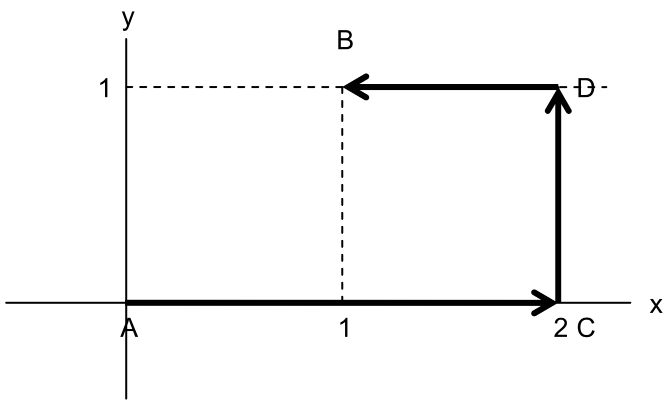
Alternative way is to write everything in terms of either x or y.
Since $y=x^3$:

$$\int dW = -k\int \left(xdx + x^3 d\left(x^3\right)\right)$$

$$d\left(x^3\right) = 3x^2 dx$$

$$W = -k\int \left(xdx + 3x^5 dx\right) = -k\int_0^1 \left(x + 3x^5\right)dx = -k$$

b)along the path shown in the figure.



$$W = -k\int \left(xdx + ydy\right)$$

How do we handle the limits?
Need to consider each of the 3 parts of the path separately.
i.e. $W = W_{AC} + W_{CD} + W_{DA}$

$$W_{AC} = -k\int \left(xdx + ydy\right)$$

ydy=0 since y=0

$$W_{AC} = -k\int_0^2 xdx = -2k$$

$$W_{CD} = -k\int \left(xdx + ydy\right)$$

x=2, therefore dx=0

$$W_{CD} = -k\int_0^1 ydy = -\frac{1}{2}k$$

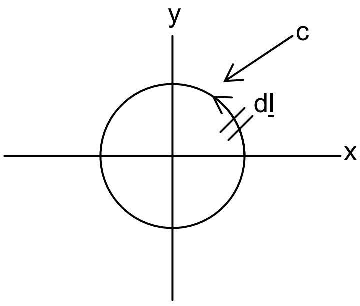
$$W_{DB} = -k\int \left(xdx + ydy\right)$$

y=1, therefore dy=0

$$W_{DB} = -k\int_2^1 xdx = -k\left[\frac{1}{2}x^2\right]_2^1 = \frac{-k}{2}\left[1-4\right] = \frac{3}{2}k$$

So W=-k as before….

Example: Calculate the integral $\int_C \underline{B} d\underline{l}$ where C is a circle of radius a centred on the origin and $\underline{B} = \frac{\beta(-y, x)}{r^2}$ (β = constant)



$x = a\cos\theta$
$y = a\sin\theta$
$dx = -a\sin\theta d\theta$
$dy = a\cos\theta d\theta$
$d\underline{l} = ad\theta\underline{\hat{\theta}} = a\left(-\sin\theta\underline{i} + \cos\theta\underline{j}\right)d\theta$

$$\underline{B}d\underline{l} = \frac{\beta}{r^2}\left(a^2\sin^2\theta + a^2\cos^2\theta\right)d\theta = \frac{a^2\beta}{r^2}d\theta = \beta d\theta$$

$$\int_C \underline{B}d\underline{l} = \beta\int_0^{2\pi} d\theta = 2\pi\beta$$

Quick way:

$$\underline{B} = \beta\frac{r\underline{\hat{\theta}}}{r^2}$$

$$\underline{B}d\underline{l} = \int_0^{2\pi}\beta d\theta = 2\pi\beta$$

4. Matrices

4.1 Introduction

We already met matrices.

Example: linear equations

$$y_i = \sum_{j=1}^{n} a_{ij} x_j , \underline{i} = 1,2,3,..,m$$

$$[y_1 = a_{ll} x_l + a_{l2} x_2 + ... + a_{1n} x_n , etc...]$$

Can be written $\underline{y} = \underline{\underline{A}}\underline{x}$ where

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

(An (mxn) matrix)

Matrices are used

1. In quantum mechanics
2. To describe symmetry

Matrix algebra is like vector algebra for addition and subtraction.

i.e. $\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{C}}, \therefore c_{ij} = a_{ij} + b_{ij}$

e.g. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 6 & 2 \end{pmatrix}$

4.2 Matrix Multiplication

As an example, suppose

$\underline{y} = \underline{\underline{B}}\underline{x}, (\underline{x} = (x_1, x_2, x_3, ..., x_p), \underline{y} = (y_1, y_2, y_3, ..., y_n))$

So $\underline{B}$ is an nxp matrix

and $\underline{z} = \underline{\underline{A}}\underline{y}, (\underline{z} = (z_1, z_2, z_3, ..., z_m))$

$\underline{A}$ is a mxn matrix.

What is $\underline{\underline{C}}$ in $\underline{z} = \underline{\underline{C}}\underline{x}$?

$$\underline{z} = \underline{\underline{A}}\underline{y}$$

$$= \underline{\underline{A}}\underline{\underline{B}}x$$

$$\therefore \underline{\underline{C}} = \underline{\underline{A}}\underline{\underline{B}}$$

But what does it mean?

In components

$$z_i = \sum_{k=1}^{n} a_{ik} \sum_{j=1}^{p} b_{kj} x_j$$

$$\sum_{j=1}^{p} b_{kj} x_j = y_k$$

$$\left(z_i = \sum_{k=1}^{n} a_{ik} y_k \right)$$

So $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Let's write $\underline{\underline{C}}$ out explicitly.

$$C_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + ... + a_{1n}b_{n1}$$

$$C_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + ... + a_{1n}b_{n2}$$

$$\begin{pmatrix} a_{11} & a_{12} & ... \\ a_{21} & a_{22} & ... \\ \vdots & \vdots & ... \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & ... \\ b_{21} & b_{22} & ... \\ \vdots & \vdots & ... \end{pmatrix} = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + ... + a_{1n}b_{n1})$$

The cell in C is the row in A times the column in B where the row and column meet at the cell in C.
NB: Multiplication only defined if the number of rows in B is equal to the number of columns in A.

e.g.:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\underline{\underline{AB}} = \begin{pmatrix} 1x1 + 2x3 & 1x2 + 2x4 \\ 2x1 - 1x3 & 2x2 - 1x4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ -1 & 0 \end{pmatrix}$$

$$\underline{\underline{BA}} = \begin{pmatrix} 1x1 + 2x2 & 1x2 - 1x2 \\ 3x1 + 4x2 & 3x2 - 4x1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 11 & 2 \end{pmatrix} \neq \underline{\underline{AB}}$$

Multiplication is:
<u>non-commutative</u>
associative: $(\underline{\underline{AB}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{BC}})$
Distributive: $\underline{\underline{A}}(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{AB}} + \underline{\underline{AC}}$

e.g. which matrix acts on a vector to give the same vector?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

This is the unit matrix $\delta_{ij} = \underline{1}$ (1 is a stylised 1)

$$\underline{1} = 1.(i = j)$$
$$\underline{1} = 0.(i \neq j)$$

e.g. useful in quantum mechanics is the <u>commutator</u> of two matrices, $\underline{\underline{A}}$ and $\underline{\underline{B}}$

$$\underline{\underline{AB}} - \underline{\underline{BA}} = \lfloor \underline{\underline{A}}, \underline{\underline{B}} \rfloor$$

4.3 Transpose

Transpose is defined to be the matrix which is obtained by swapping rows and columns.
e.g.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\underline{\underline{A}}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

In component notation:

$$A_{ij}^T = A_{ji}$$

e.g. transpose of a “column vector” is a “row vector”.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underline{x}^T = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$$

Note $\underline{x}^T \underline{x} = x_1^2 + x_2^2$

$\underline{x}^T \underline{y}$ is a way of taking the vector product

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$$

[What is $\underline{x}\underline{y}^T$?]

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$$ - Tenser product

If $\underline{\underline{C}} = \underline{\underline{A}}\underline{\underline{B}}$

$\underline{\underline{C}}^T = \left(\underline{\underline{A}}\underline{\underline{B}}\right)^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$

Proof:

$\underline{\underline{C}}^T = \left(\underline{\underline{A}}\underline{\underline{B}}\right)^T$

$\left(C^T\right)_{ij} = \left(A_{ik} B_{kj}\right)^T = A_{jk} B_{ki} = A^T_{kj} B^T_{ik} = B^T_{ik} A^T_{kj}$

[If indices are repeated then summation implied]

4.4 Inverse

We have met $\underline{\underline{A}}\underline{x} = \underline{y}$.

Let us introduce the matrix

$\underline{\underline{A}}^{-1}$ such that $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{1}$

$\underline{\underline{A}}^{-1}\underline{\underline{A}}\underline{x} = \underline{\underline{A}}^{-1}\underline{y}$

$\underline{x} = \underline{\underline{A}}^{-1}\underline{y}$

Note

$\underline{\underline{A}}^{-1}$ does NOT always exist.

How do we find $\underline{\underline{A}}^{-1}$?

We use the result

$$\sum_{j=1}^{n} a_{ij} C_{kj} = \left(\det \underline{\underline{A}}\right)\delta_{ik}$$

C is the co-factor (kj^{th}) of $\underline{\underline{A}}$

Proof:

Consider i=k, $\delta_{ik} = 1$

$$\sum_{j=1}^{n} a_{ij} C_{ij} = \det \underline{\underline{A}}$$ ← definition of determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det \underline{\underline{A}}$$

For any i

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$ etc

i.e. just the definition of $\det \underline{\underline{A}}$

Now consider i≠k, $\delta_{ik} = 0$

$$\sum_{j=1}^{n} a_{ij} C_{kj} = 0$$

LHS is just the determinant of a matrix with 2 equivalent rows which we earlier saw to be zero.

e.g. if i=1, k=2. (n=3 case)

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Hence:

$$\sum_{j=1}^{n} a_{ij} C_{jk}{}^{T} = \left(\det \underline{\underline{A}}\right)\delta_{ik}$$

$$\underline{\underline{A}}\,\underline{\underline{C}}^{T} = \left(\det \underline{\underline{A}}\right)\underline{\underline{1}}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{1}}$$

$$\underline{\underline{A}}\left[\frac{\underline{\underline{C}}^{T}}{\left(\det \underline{\underline{A}}\right)}\right] = \underline{\underline{1}}$$

$$\underline{\underline{A}}^{-1} = \frac{\underline{\underline{C}}^{T}}{\det \underline{\underline{A}}}$$

Inverse = "transpose of the matrix of cofactors (divided by the determinant)"
NB: Inverse does not exist if $\det \underline{\underline{A}} = 0$ "singular matrix".

Example: Find the inverse of

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\underline{\underline{C}} = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\underline{\underline{C}}^{T} = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$\det \underline{\underline{A}} = 4 - 6 = -2$$

$$\underline{\underline{A}}^{-1} = \frac{\underline{\underline{C}}^{T}}{\det \underline{\underline{A}}} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Check:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2+3 & 1-2x\frac{1}{2} \\ -6+4x\frac{3}{2} & 3-\frac{4}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\det \underline{\underline{A}} = 14$$

$$\underline{\underline{C}} = \begin{pmatrix} -2 & 5 & 1 \\ 4 & -3 & 5 \\ 6 & -1 & -3 \end{pmatrix}$$

$$\underline{\underline{A}}^{-1} = \frac{1}{14}\begin{pmatrix} -2 & 4 & 6 \\ 5 & -3 & -1 \\ 1 & 5 & -3 \end{pmatrix}$$

4.5 Special Matrices

Symmetric:

Has to be symmetric about the diagonal.

$A_{ij} = A_{ji}$

$\underline{\underline{A}} = \underline{\underline{A}}^T$

$\left(\underline{\underline{A}} = \left(\underline{\underline{A}}^*\right)^T = A^{dagger.symbol}\right)$

$\underline{\underline{A}}$ is hermitian. $A^{dagger.symbol}$ is the hermitian conjugate.

Orthogonal:

$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

i.e. $\underline{\underline{A}}\underline{\underline{A}}^T = \underline{\underline{1}}$

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

e.g. $\underline{v}^T\underline{v} = |\underline{v}|^2 = \underline{v}.\underline{v}$

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = {v_1}^2 + {v_2}^2 + {v_3}^2$$

$\underline{v}' = \underline{\underline{R}}\underline{v}$

(Orthogonal matrix)

$\underline{v}'.\underline{v}' = \underline{v}'^T\underline{v}' = \left(\underline{\underline{R}}\underline{v}\right)^T\left(\underline{\underline{R}}\underline{v}\right) = \underline{v}^T\underline{\underline{R}}^T\underline{\underline{R}}\underline{v} = \underline{v}^T\underline{v}$

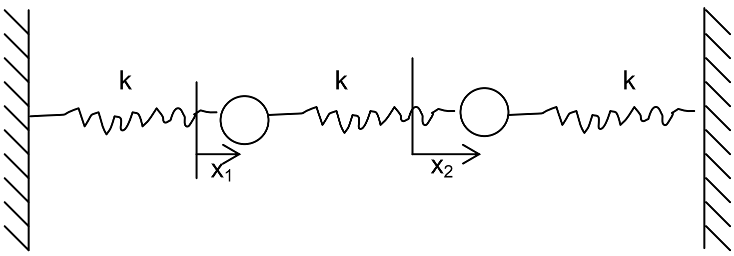
→ Length of $\underline{v}$ is invariant under an orthogonal transformation.

If $\underline{\underline{A}}^{dagger}\underline{\underline{A}} = 1$

$\underline{\underline{A}}$ is <u>unitary</u>.

4.6 Eigenvalues and Eigenvectors

Back to system of masses and strings to look at dynamics.



Recall

$F_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$

$F_2 = k(x_1 - x_2) - kx_2 = kx_1 - 2kx_2$

i.e.:

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

by Newton's 2nd law

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}$$

Generally motion is complicated. But we can look for solutions of definite frequency ("normal modes").
Try to find solutions of the form $x_1(t) = X_1 \cos \omega t, x_2(t) = X_2 \cos \omega t$ (X_1, X_2 constants)

$$\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}\begin{pmatrix} X_1 \cos \omega t \\ X_2 \cos \omega t \end{pmatrix} = m\begin{pmatrix} -X_1\omega^2 \cos \omega t \\ -X_2\omega^2 \cos \omega t \end{pmatrix}$$

$$\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = m\omega^2\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Can write as:

$$\boxed{\underline{\underline{K}}\,\underline{X} = m\omega^2 \underline{X}}$$

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\underline{\underline{K}} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

This is an EIGENVALUE equation.
$\underline{X}$ is the eigenvector, while $m\omega^2$ is the eigenvalue.
Let's solve the general case.

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

where λ is a number.

$$\left(\underline{\underline{A}} - \lambda \underline{\underline{1}}\right)\underline{x} = 0$$

Homogenous linear equation.
For interesting (not unique) solutions we require $\det\left(\underline{\underline{A}} - \lambda \underline{\underline{1}}\right)$ ← Gives us the eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$, (if $\underline{\underline{A}}$ is a NxN matrix)

$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{1}}\right) \neq 0$ leads to unique but trivial solution $\underline{x}$=0.

For each eigenvalue λ_i we need the corresponding vector $\underline{X}_i$. i.e. we must solve

$$\left(\underline{\underline{A}} - \lambda_i\right)\underline{X}_i = \underline{0}$$

$\underline{X}_i$ is called an EIGENVECTOR. Note: since RHS $\underline{X}_i$ is defined only up to an overall factor.

e.g.: $\underline{\underline{A}} = \begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix}$ - find the eigenvalues and eigenvectors of $\underline{\underline{A}}$.

To get eigenvalues we must solve $\det\left(\underline{\underline{A}} - \lambda \underline{\underline{1}}\right) = 0$. i.e. $\begin{pmatrix} 6-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = 0$

$$(6-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda = 4 \pm \sqrt{5}$$

Eigenvalues are $\lambda_1 = 4 - \sqrt{5}, \lambda_2 = 4 + \sqrt{5}$.
Check: "sum of eigenvalues = trace of matrix"
Add the λ up, and check that they equal to the diagonal elements of the matrix (i.e. 6+2 in $\underline{\underline{A}}$.
To get eigenvector corresponding to λ_1, $\left(\underline{\underline{A}} - \lambda_1 \underline{\underline{1}}\right)\underline{X}_1 = 0$

$$\begin{pmatrix} 6-\lambda_1 & -1 \\ -1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{X}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = b\begin{pmatrix} a/b \\ 1 \end{pmatrix} (b \neq 0)$$

$$\begin{pmatrix} 2+\sqrt{5} & -1 \\ -1 & -2+\sqrt{5} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the first equation:

$\left(2+\sqrt{5}\right)a - b = 0$

$\underline{X}_1 = \begin{pmatrix} a \\ \left(2+\sqrt{5}\right)a \end{pmatrix}$

a can be anything i.e. norm of $\underline{X}_1$ is not fixed.

$\underline{X}_1 = \begin{pmatrix} 1 \\ 2+\sqrt{5} \end{pmatrix}$

Note, could get $\underline{X}_1$ from the 2[nd] equation i.e. $-a + \left(-2+\sqrt{5}\right)b = 0$ - it givees the same answer.

For λ_2 $\begin{pmatrix} 6-\lambda_2 & -1 \\ -1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2-\sqrt{5} & -1 \\ -1 & -1-\sqrt{5} \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $\left(2-\sqrt{5}\right)c - d = 0 \Rightarrow \underline{X}_2 = \begin{pmatrix} c \\ c\left(2-\sqrt{5}\right) \end{pmatrix}$

Can check:
Substitute back in:

$\underline{\underline{A}}\underline{x}_1 = \lambda_1 \underline{x}_1$

$$\begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2+\sqrt{5} \end{pmatrix} = \begin{pmatrix} 4-\sqrt{5} \\ 2+2\sqrt{5} \end{pmatrix} = \left(4-\sqrt{5}\right)\begin{pmatrix} 1 \\ 2+\sqrt{5} \end{pmatrix}$$

e.g. calculate the eigenvalues and eigenvectors of:

$$\underline{\underline{B}} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Want $\underline{\underline{B}}\underline{x} = \lambda \underline{x}$, i.e. $\left(\underline{\underline{B}} - \lambda \underline{\underline{1}}\right)\underline{x} = \underline{0}$.

To get λ:

$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix}(1-\lambda) = \left[(2-\lambda)^2 - 9\right](1-\lambda) = 0$$

Solutions:

$\lambda = \pm 1{,}5$ ← Eigenvalues.

Check works. (5=5)

To get eigenvectors:

$\lambda_1 = -1$

$\left(\underline{\underline{B}} - \lambda_1 \underline{\underline{1}}\right)\underline{X}_1 = 0$

$$\begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $\underline{X}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Therefore $a = -b$ and $c = 0$

S:

$$\underline{X}_1 = \begin{pmatrix} a \\ -a \\ 0 \end{pmatrix}$$

Unit eigenvectors:

$$\underline{\hat{X}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix}$$

$$\underline{\hat{X}}_1^T \underline{X}_1 = 1$$

$$\lambda_1 = 1$$

$$(\underline{\underline{B}} - \lambda_1 \underline{\underline{1}})\underline{X}_1 = 0$$

$$\begin{pmatrix}1&3&0\\3&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$a + 3b = 0$$

$$3a + b = 0$$

$$0 = 0$$

→ C can be anything!

$$a = b = 0$$

$$\underline{X}_2 = \begin{pmatrix}0\\0\\C\end{pmatrix}$$

$$\underline{\hat{X}}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\lambda_1 = 5$$

$$(\underline{\underline{B}} - \lambda_1 \underline{\underline{1}})\underline{X}_1 = 0$$

$$\begin{pmatrix}-3&3&0\\3&-3&0\\0&0&-4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-3a + 3b = 0$$

$$3a - 3b = 0$$

$$-4c = 0$$

$$a = b$$

$$c = 0$$

$$\underline{X}_3 = \begin{pmatrix}a\\a\\0\end{pmatrix}$$

$$\underline{\hat{X}}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

So:

$$\underline{\hat{X}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} \underline{\hat{X}}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} \underline{\hat{X}}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

These are all orthogonal.

4.7 Real Symmetric Matrices

Note that for both $\underline{\underline{A}}$ and $\underline{\underline{B}}$ in the previous exams

(a) Eigenvalues were <u>REAL</u>.

(b) Eigenvectors were <u>ORTHOGONAL</u> (e.g. $\hat{\underline{X}}_1^T \underline{X}_2 = 0$

(a) and (b) occurred because $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are <u>real</u> (i.e. $\underline{\underline{A}} = \underline{\underline{A}}^*$) and <u>symmetric</u> (i.e. $\underline{\underline{A}} = \underline{\underline{A}}^T$)

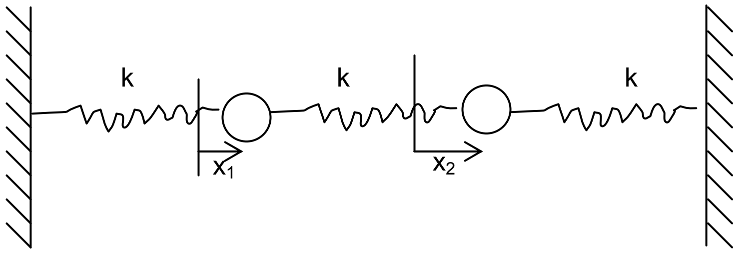
(a) and (b) hold for <u>all</u> real and symmetric matrices.

4.8 Normal Modes

We can now finish off solving the coupled springs problem.
We had:

$$\underline{\underline{K}}\,\underline{X} = m\omega^2 \underline{X}$$

$$\underline{x} = \underline{X}\cos\omega t, \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\underline{\underline{K}} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

→ to get eigenvalues

$$\begin{vmatrix} 2k-\lambda & -k \\ -k & 2k-\lambda \end{vmatrix} = 0$$

$$\lambda = m\omega^2$$

$$(2k-\lambda)^2 - k^2 = 0$$

$$\lambda_1 = k = m\omega_1^{\;2} \Rightarrow \omega_1 = \sqrt{\frac{k}{m}}$$

$$,\lambda_2 = 3k = m\omega_2^{\;2} \Rightarrow \omega_2 = \sqrt{\frac{3k}{m}}$$

These are the angular frequencies of the <u>normal modes</u>.
The corresponding eigenvectors are:

$$\underline{X}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow x_1 = x_2$$

$$\underline{X}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow x_1 = -x_2$$

$\underline{x}_1$ corresponds to the masses moving in phase.

$\underline{x}_2$ corresponds to the masses moving in antiphase.