1. Defining Space, Time and Motion

1.1 Definitions & Coordinate Systems

What is space?

- 3-Dimensional, soft or like a grid
- Particles can have different positions in space at the same time
- “Distances are measured by rulers”

What is Time?

- Something which flows (?)
- 4th Dimension
- Newton used the word ‘equable’ – equal and uniform
  (We will show later that time intervals are not the same for everybody – it depends on the state of motion of the observer)
- For now, “Time intervals are measured by clocks)

What is Motion?

- A change in position with time

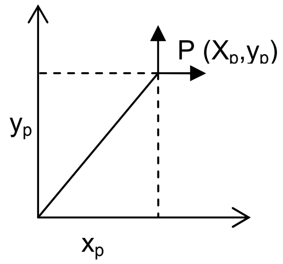
If we have a collection of objects e.g. whose relative positions in space do not change with time, we can use them to define a ‘frame of reference’.
Motion is always measured relative to some frame of reference, e.g. train, plane, earth, this room, …

It will turn out that certain frames of reference are more useful than others.

In a given frame of reference we are free to choose a coordinate system – the “mesh” by which we measure positions and displacements. For most problems motion is confined to a plane – can use 2D coordinate systems. The most familiar is the 2D Cartesian System.

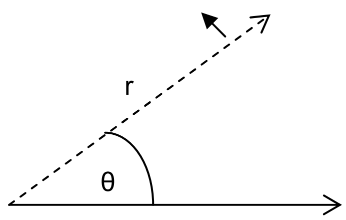
Cartesian system:

Positions are measured using a square grid formed by 2 mutually perpendicular axis (x,y) which meet at 0 (0,0).
The position P relative to O is described by the position vector r̲.
r̲ can be associated with the coordinates (x,y)

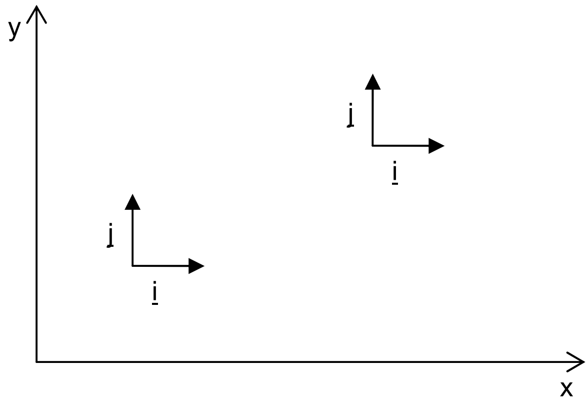


Polar system:

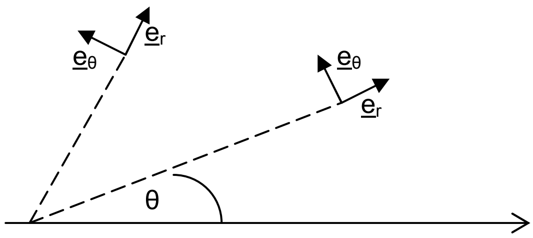
In the same frame of reference we can use Polar Coordinates (r,θ).
Can be related to x and y.



$R^2 = x^2 + y^2$
Tan θ = opp/adj = y/x
and:
x = r.cos θ
y = r.sin θ



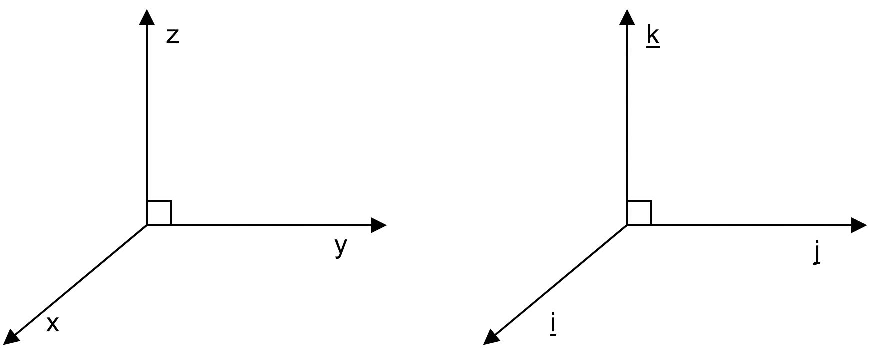
To describe displacement in a given coordinate system, basis vectors are used in the 2D Cartesian system.
e.g. $\underline{i}$ (Direction of increasing x)
$\underline{j}$ (Direction of increasing y)



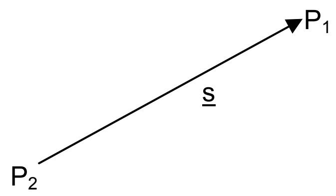
In the polar coordinate system $\underline{e}_r$ and $\underline{e}_\theta$ are used.
Important: basic vectors depend on the position in non-Cartesian systems.
Basic vectors have unit length.

$\underline{r} = x\underline{i} + y\underline{j}$
or:
$\underline{r} = r\underline{e}_r$

In 3D the Cartesian basis is $\underline{i}$ $\underline{j}$ $\underline{k}$



Displacements in space may be described by vectors:

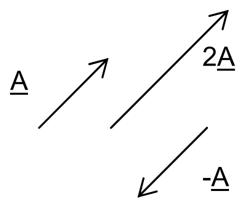


$\underline{s} = s_x\underline{i} + s_y\underline{j}$
s_x = distance s_x in x direction
s_y = distance s_y in y direction

True in flat spaces.

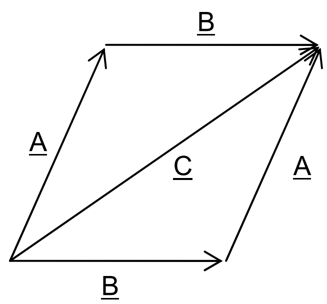
$\underline{A} + \underline{B} = \underline{B} + \underline{A}$

Multiply by a positive number changes the length of the vector but not its' direction. Multiplication by a negative number results in a vector in the opposite direction.



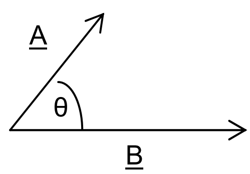
$\underline{A} - \underline{A} = \underline{0}$ (the 'null vector' – a vector of 0 length and no defined direction)

$\underline{A} + \underline{B} = \underline{C}$



$C_x\underline{i} + C_y\underline{j} = (A_x + B_x)\underline{i} + (A_y + B_y)\underline{j}$
$C_x = A_x + B_x$
$C_y = A_y + B_y$

Scalar product of two vectors (Dot product)



$\underline{A}.\underline{B} = |A|\,|B| \cos\theta$

$|A|$ is magnitude (length) of vector $\underline{A}$
$|B|$ is magnitude (length) of vector $\underline{B}$

Can be written in terms of components in a Cartesian basis:
$\underline{A}.\underline{B} = A_xB_x + A_yB_y + A_zB_z$
Note that i.i = $\underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$

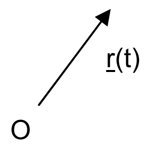
1.2 Standards for the measurement of length and time

The metre was originally defined to be 10^{-7} of the distance from the pole to the equator.
This was not precise enough (Late 18th Century)
Defined as the length of a particular metal bar (Sévres)
1960 – defined as a multiple of orange/red lines in ^{86}Kr
1983 – c was defined to be 299,792,458 ms^{-1}
→ define the second, define the metre.

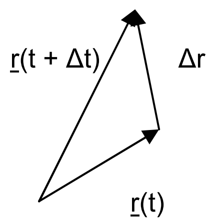
Definition of a second: 1/86,400 of a mean day.
→ Variations due to changes in the Earth's speed and distance from the sun meant this was no good.
In 1967, Cs Atomic Clock introduced.
1s = 9,192,631,770 cycles of vibration in ^{133}Cs
Gains/Looses 0.1ns in one day.

1.3 Velocity and Acceleration

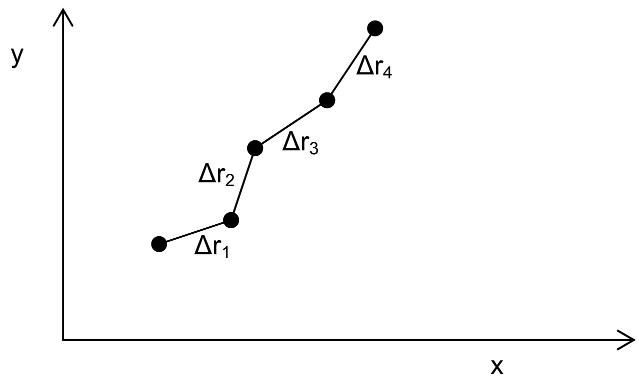
Consider an object in motion.
Position vector is a function of time $\underline{r}(t)$



Consider a slightly later time $\underline{r}(t + \Delta t)$



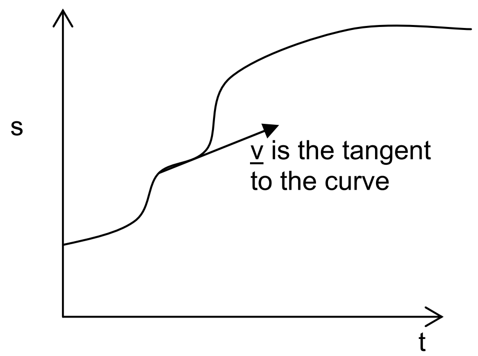
$\underline{r}(t) + \Delta r = \underline{r}(t + \Delta t)$

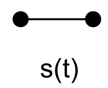


Consider shorter and shorter intervals of time.
$\Delta\underline{r}/ \Delta\underline{t}$ approaches a limit.

$\underline{V}(t) = \lim(\Delta t \rightarrow 0) (\Delta\underline{r}/\Delta\underline{t}) = d\underline{r}/dt$

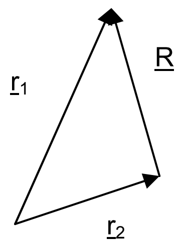
For motion in 1D we can plot a space-time graph.





On this graph, v = ds/dt = gradient

In problems with 2 objects in motion it is often convenient to use a <u>relative</u> velocity.



As $\underline{r}_1$ and $\underline{r}_2$ move, $\underline{R}$ will change.

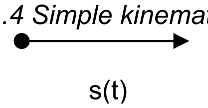
$\underline{R} = \underline{r}_1 - \underline{r}_1$
So $d\underline{R}/dt = (d\underline{r}_2/dt) - (d\underline{r}_1/dt)$
Or $\underline{V} = \underline{v}_1 - \underline{v}_1$

The relative velocity is often useful to solve collision problems, for example two aeroplanes which are travelling at different velocities crossing paths. It is possible to use vector diagrams to confirm that these will not collide.

Acceleration:
Definition: $\underline{a} = d\underline{v} / dt = \lim(\Delta t \rightarrow 0)\ (\underline{v}(t + \Delta t) - \underline{v}(t))/ \Delta t$

So $\underline{a} = d^2\underline{r}(t) / dt^2$
Higher order derivations of $\underline{r}(t)$ are not so important => equations of motion will turn out to be second order.

1.4 Simple kinematics in 1D



Suppose that the acceleration a(t) is known.
To get v(t) we integrate a(t)

$$v(t_2) - v(t_1) = \int_0^t a(t)dt + v_o$$

We can do a similar thing to get s(t)

$$s(t) = \int_0^t v(t)dt + s_0$$

If a(t) = 0 we get $v(t) = v_o$ and $s(t) = v_o t + s_o$

If a(t) = a = constant

Then $v(t) = at + v_o$ –(1)
$s(t) = \frac{1}{2} at^2 + v_o t + s_o$ –(2)

Square (1) and substitute into (2)
→ $v^2 = v_o^2 + 2a(s-s_o)$

2. Newton's Laws of Motion

2.1 Newton's Laws

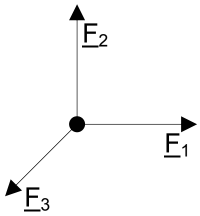
What is a force?

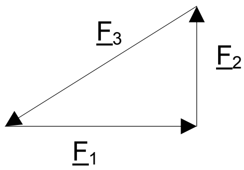
Something which pushes or pulls

What type of quantity is it?

A vector (Magnitude and direction)

Forces were known long before Newton.
A body is said to be in static equilibrium if the sum of the forces is null.
i.e. $\underline{F}_1 + \underline{F}_2 + ... = \sum_i \underline{F}_i = \underline{0}$





i.e. in static equilibrium a closed polygon is formed by the vectors.
$F_{1x}+F_{2x}+F_{3x}+\ldots=0$
$F_{1y}+F_{2y}+F_{3y}+\ldots=0$
$F_{1z}+F_{2z}+F_{3z}+\ldots=0$

What types of forces are there?

Contact forces between two bodies
Elastic forces $F=-k(x-x_0)$
Electrostatic forces $F_{12} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}^2}$
Frictional forces F=nN where n=coefficient and N=Normal Force.
etc…

Newton 1

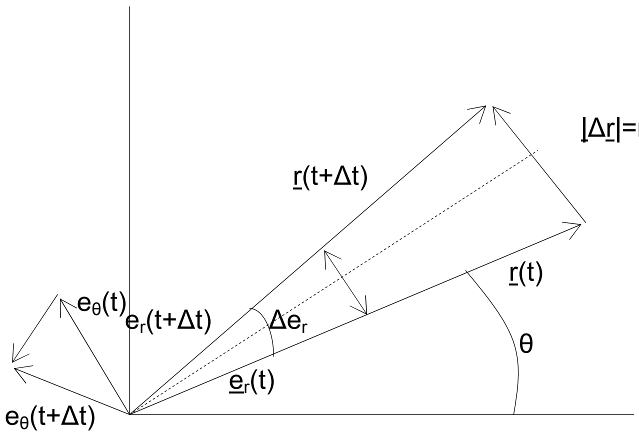
The Greeks thought that a force was needed in order to keep an object moving.
In order to see the motion of an object with no applied force you have to try hard to overcome friction The linear air track has a layer of air with viscosity ~1/5000 of a layer of oil.
With no net force an object either

1. Remains at rest
2. Remains in a state of constant velocity

With an applied force the object accelerates.
It is always possible to find a coordinate system in which an isolated body moves uniformly i.e. with constant velocity. Such a coordinate system is known as "<u>inertial</u>".
Note that it is always possible to choose a non-inertial coordinate system.

Circular Motion

This type of motion is best handled in polar coordinates.
Position vector $\underline{n}=r\underline{e}_r$
Recall that $\underline{e}_r$ depends on the position of the object and may change with time.



$$\left|\Delta r\right| = 2r\sin\frac{\Delta\theta}{2}$$

$$\left|\Delta r\right| = 2r\left[\frac{\Delta\theta}{2} + ...\right]$$

$|\Delta r| = r\Delta\theta$ as $\Delta\theta \rightarrow 0$

$\underline{\Delta r} = r\Delta\theta\underline{e}_\theta$

so $\lim(\Delta t \triangleright 0)\frac{\underline{\Delta r}}{\Delta t} = r\omega\underline{e}_\theta$ where $\omega = \frac{d\theta}{dt}$

So $\boxed{\underline{v} = r\omega\underline{e}_\theta}$ –(1)

$|\underline{v}| = r\omega$

Take the case where r=1

$\boxed{\frac{d}{dt}\underline{e}_r = \omega\underline{e}_\theta}$ –(2)

By similar argument

$\boxed{\frac{d}{dt}\underline{e}_\theta = -\omega\underline{e}_r}$ –(3)

(2) and (3) are important when working in polar coordinates.

(1) $\rightarrow$

$$\underline{a} = \frac{d}{dt}\left(r\omega\underline{e}_\theta\right)$$

$$\underline{a} = r\omega\left(-\omega\underline{e}_r\right)$$

r $\rightarrow$ constant because this is circular motion

ω $\rightarrow$ constant (Uniform circular motion)

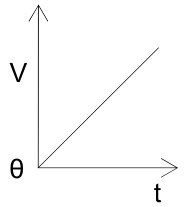
$\boxed{\underline{a} = -r\omega^2\underline{e}_r}$ using (3)

1) The force providing this acceleration is the contact force between the bearing and the ring (in the demo)
2) As soon as that force is removed, the velocity is constant and in a tangent to the circle.

Newton Continued

1st law: mentions the "isolated object". It is difficult to imaging how you identify such an object. e.g. if you see your isolated system accelerating, is this due to some force or are we in a non-inertial frame of reference?

With the Air Track we can apply a force using a stretched spring (constant extension, i.e. constant force)



a_0=dv/dt

If we use the same mass with two additional isolated springs then we get $2a_0$.
One spring, double the mass = $0.5a_0$, etc…
Experiments like these would lead us two Newton's 2nd law of motion.

$$F = m\frac{dv}{dt} = ma$$

(1) $\boxed{\underline{F} = m\underline{a}}$ Newton's 2nd law

$\underline{F}$ is the net force on the body, i.e. $\underline{F} = \sum_i \underline{F}_i$

e.g if we are in static equilibrium
$\sum \underline{F}_i = 0$ therefore $\underline{a}$=0

We can use (1) to define a mass scale.
Apply same force to objects of different mass:

$$\frac{M_2}{M_1} = \frac{a_1}{a_2}$$

$$\frac{M_3}{M_1} = \frac{a_1}{a_3}$$

etc…

A definition of mass in terms of how a body accelerates → inertial mass.
This mass scale is tied to a standard kilogram (Pt/Ir alloy kept at Sévres, France)
The unit of Force, the Newton, is defined as the force which imparts an acceleration of $1ms^{-2}$ to a mass of 1kg.
So $1N=1kgms^{-2}$.

Newton III

Forces come in pairs.
All forces are the result of some mutual interaction.
"If I push on the table, the table pushes back on me". Recall the "Isolated system".

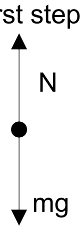
To be a real force there must be another object subject to an equal and opposite force.
Newton III: "If a body A exerts a force on body B (action) then B exerts a force on A (reaction) such that action $\underline{F}_{AB}$ = reaction $\underline{F}_{BA}$."

$$\underline{F}_{AB}=-\underline{F}_{BA}$$

2.2 Applications of Newton's Laws

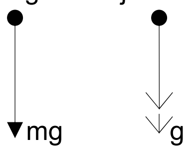
Free-Body Diagrams

First step to solving a dynamics problem, e.g. a book at rest on a table.



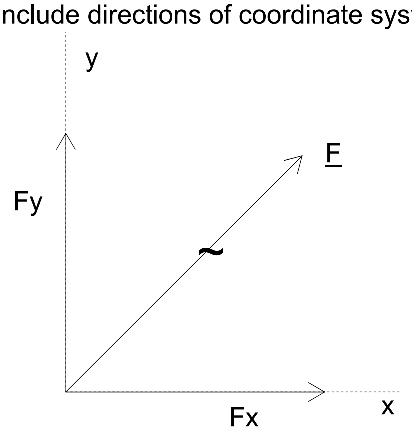
Static equilibrium.

e.g. an object falling



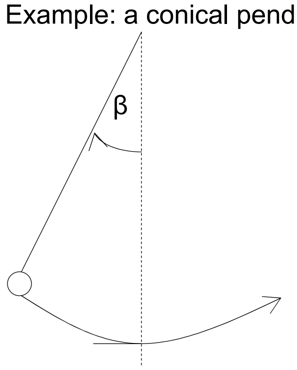
Rules for drawing these diagrams:

1. All forces act on a point that represents the object (Later: centre of mass)
2. Only draw forces on diagrams. Draw any acceleration next to the force diagram.
3. Do not draw any other objects on the same diagram. Draw separate diagrams for each object. No ropes, surfaces, …
4. Include directions of coordinate systems

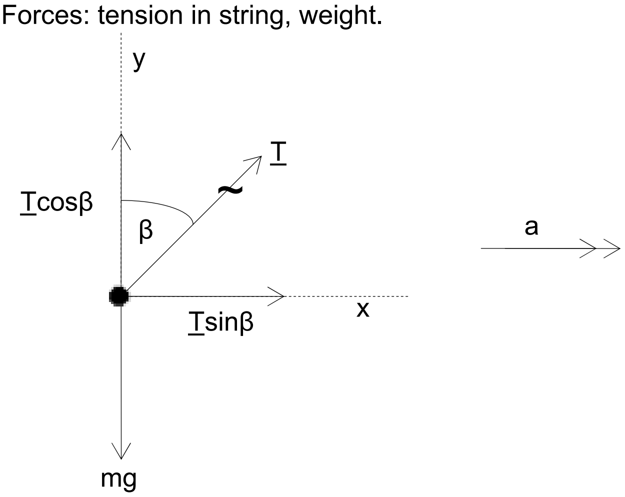


Usual practice to put a wiggly line through the force that has been resolved into components.

5. Usual to label an object's weight as mg.
   Example: a conical pendulum:

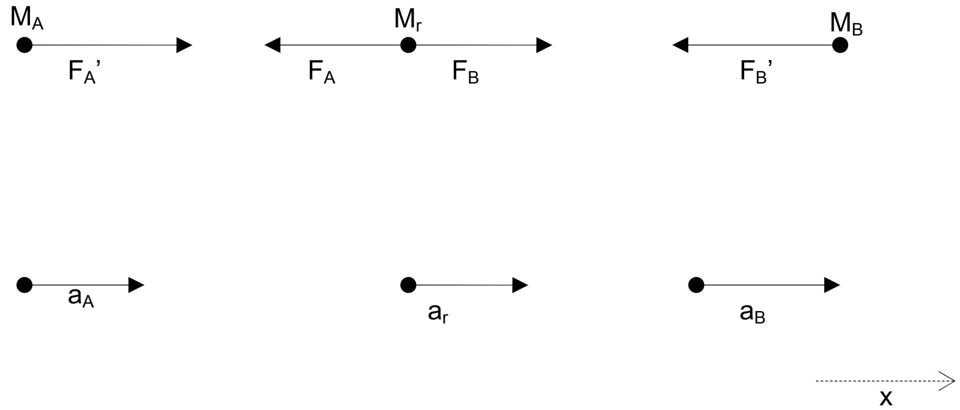


Forces: tension in string, weight.



The Astronaut's Tug of War

Two astronauts are at rest in free space. They pull on either end of a rope of mass M_r.
(A) pulls with a force F_a and has a mass of M_a
(B) pulls with a force F_b and has a mass of M_a
Find the motion of these two astronauts.
FB diagrams:



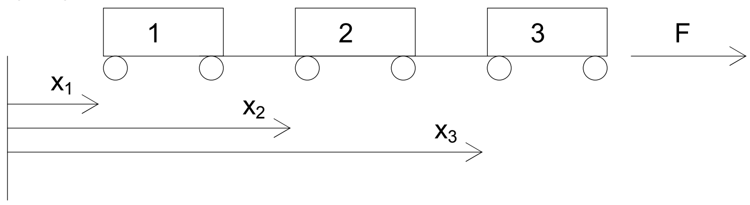
Newton's 3rd law
→ $F_A'=F_A$ and $F_B'=F_B$
EOM (Equation of Motion) for the rope:
$F_B-F_A=M_r a_r$
Ignore the mass of the rope
→ $F_A=F_B$
Either applies the same force to the rope.

EOM for (A) $a_A = \frac{F_A}{M_A}$

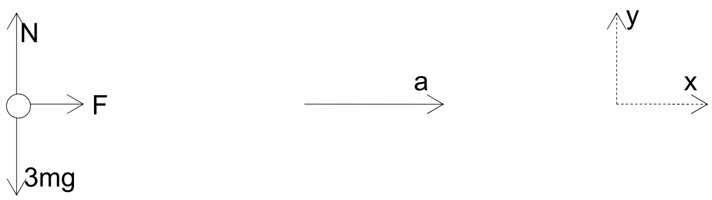
EOM for (B) $a_B = -\frac{F_B}{M_B} = -\frac{F_B}{M_B}$

Note the negative sign.

The Train



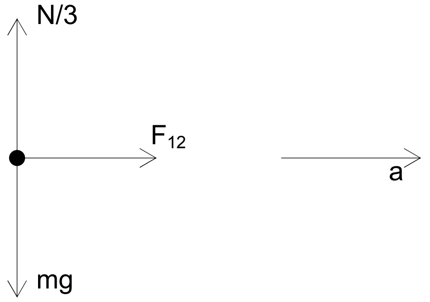
Three trucks each of mass m are pulled by an engine providing force F.
Constraints: x_2-x_1=const. etc… → $a_1=a_2=a_3=a$
FB diagram for the whole system:



Static equilibrium in y direction N=3mg
x-direction EOM

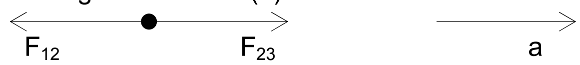
F-3ma or $a = \frac{F}{3m}$

FB diagram for truck (1)



$$F_{12} = ma = \frac{F}{3}$$

FB diagram for truck (2)

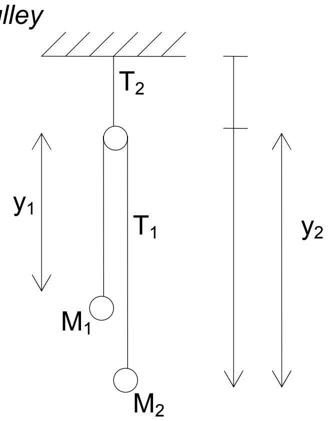


$$F_{23} - F_{12} = ma$$

$$F_{23} = \frac{F}{3} + \frac{F}{3} - \frac{2F}{3}$$

So net force on truck 2 $F_{23} - F_{12} = \frac{F}{3}$

Pulley



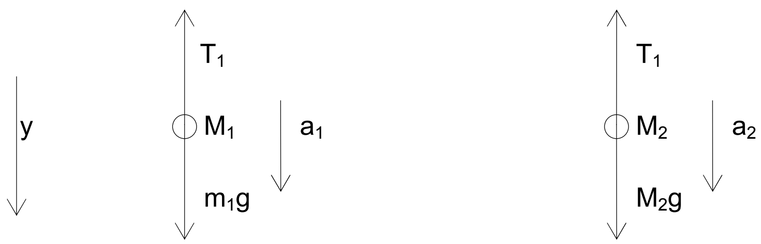
y_1+y_2=const.
Equation of constraint:

$$\dot{y} \equiv \frac{dy}{dt}$$

$$\ddot{y} \equiv \frac{d^2y}{dt^2}$$

Equation of constant → $\ddot{y}_1 = -\ddot{y}_2$

FB Diagram:



EOM: $M_1 g - T_1 = M_1 a_1$ (1) $M_2 g - T_1 = M_2 a_2 = -M_2 a_1$ (2)

(1)-(2)

$$(M_1 - M_2)g = (M_1 + M_2)a_1$$

$$a_1 = \frac{(M_1 - M_2)}{(M_1 + M_2)}g$$

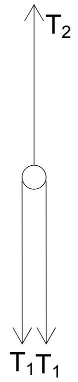
If $M_1=M_2$, a=0 (OK)
If $M_2=0$, a_1=g (OK)

(1)+(2)

$$(M_1 + M_2)g - 2T_1 = (M_1 - M_2)a_1$$

$$T_1 = \frac{1}{2}\left[M_1(g - a_1) + M_2(g + a_1)\right]$$

FB diagram for pulley:

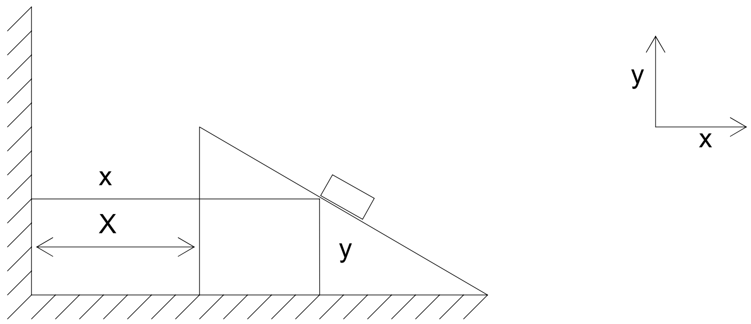


$T_2=2T_1$
If $a_1=0$ ($M_1=M_2$) then:

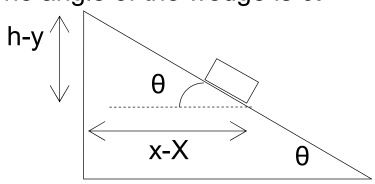
$$T_1 = \frac{1}{2}(M_1 + M_2)g$$

$$T_2 = (M_1 + M_2)g$$

Seems OK.



All surfaces are frictionless
The angle of the wedge is θ.



$x - X = (h - y)\cot\theta$ (1) equation of constraint.

$\ddot{x} - \ddot{X} = -\ddot{y}\cot\theta$ (2)

Note that working in coordinate system fixed to the wedge would be incorrect → a non-inertial frame of reference.

<u>Problems involving plane polar coordinates</u>

Recall:

$$\frac{d\underline{e}_r}{dt} = \dot{\underline{e}}_r = \omega \underline{e}_\theta = \dot{\theta}\dot{\underline{e}}_\theta$$

$$\frac{d\underline{e}_\theta}{dt} = \dot{\underline{e}}_\theta = \omega \underline{e}_r = \dot{\theta}\dot{\underline{e}}_r$$

Then with $\underline{r} = r\underline{e}_r$

$$\underline{v} = \frac{d}{dt}\left(r\underline{e}_r\right) = \dot{r}\underline{e}_r + r\dot{\underline{e}}_r$$

$$\boxed{\underline{v} = \dot{r}\underline{e}_r + r\dot{\underline{e}}_r}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\theta}\underline{e}_\theta + \ddot{r}\underline{e}_\theta + r\ddot{\theta}\underline{e}_\theta$$

$$\boxed{\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e}_\theta}$$

e.g. block on a strong (horizontal plane)

Ignore gravity

Circular path radius R

Constant angular velocity $\omega \equiv \dot{\theta}$

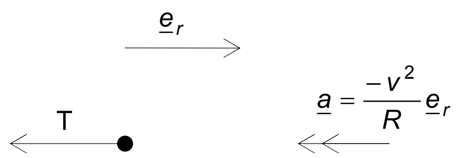
So $\ddot{r} = \dot{r} = \ddot{\theta} = 0$

$$\underline{a} = -r\dot{\theta}^2\underline{e}_r$$

$$\boxed{\dot{\theta} = \frac{v}{r} = \omega}$$

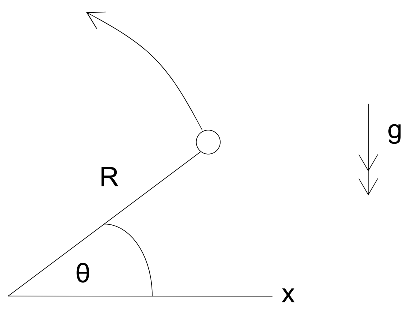
$$\underline{a} = \frac{-v^2\underline{e}_r}{R}$$

FB diagram

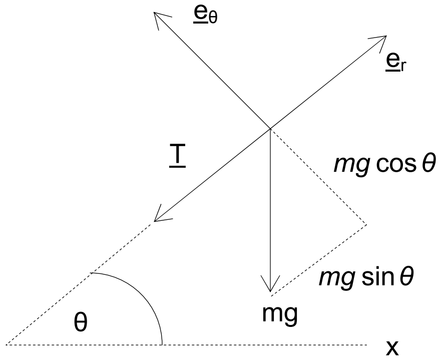


$$T = \frac{mv^2}{R}$$

Block on a string in the vertical plane with gravity:



Tension in strong T(t)>0
Find T
FB diagram:



Constraints? $\dot{r} = \ddot{r} = 0, r = R$

$\underline{a} = -r\dot{\theta}^2\underline{e}_r + r\ddot{\theta}\underline{e}_\theta$

Radial EOM:

$-T - mg\sin\theta = -mr\theta^2$

$T - mR\dot{\theta}^2 = -mg\sin\theta$

Tangential direction $(\underline{e}_\theta)$

$-mg\cos\theta = mr\ddot{\theta}$

$\ddot{\theta} = \frac{-g\cos\theta}{R}$

$T > 0$

$\therefore mR\dot{\theta}^2 > mg\sin\theta$

Max value of sinθ=1

So $\dot{\theta} > \sqrt{\frac{g}{R}}$ or string will go slack.

2.3 Momentum

We express the second law in the form:

$\underline{F} = m\underline{a}$ (1)

This is not the original form of the law.

$\underline{F} = \frac{d\underline{P}}{dt}$ (2)

where $\underline{P} = m\underline{v}$

If you have a particle with constant mass

$\underline{F} = \frac{d}{dt}(m\underline{v}) = m\underline{a}$

→ (1) and (2) are the same.

For complex systems the 2nd form (2) is more useful – can be used in situations where the mass is not constant.
Consider a system of interacting particles.
e.g. sun & planets.
e.g. an object comprised of atoms.

Suppose N particles with masses $M_1, M_2, \ldots, M_n$.
Position of the j^{th} particle by $\underline{r}_j$.
Force on the j^{th} particle by $\underline{F}_j$

$\underline{F}_j$ can be split into internal forces and a force due to interactions with objects outside the system.

$$\underline{F}_j = \underline{F}_j^{(e)} + \sum_{k=1}^{n} \underline{F}_{jk}$$

$\underline{F}_{jk}$ is the force on particle j due to k.

For each particle we have

$$\underline{F}_j = \frac{d}{dt}\underline{P}_j$$

We have n such equations. If we add these:

$$\underline{F}_1 + \underline{F}_2 + \ldots + \underline{F}_n = \frac{d}{dt}\left(\underline{P}_1 + \underline{P}_2 + \ldots + \underline{P}_n\right)$$

or

$$\sum_{i=1}^{n} \underline{F}_1 = \frac{d}{dt}\sum_{i=1}^{n} \underline{P}_1$$

If we substitute in for $\underline{F}_1$ we get

$$\sum_{i=1}^{n} \underline{F}_i^{(e)} + \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{F}_{ij} = \frac{d}{dt}\sum_{i=1}^{n} \underline{P}_i$$

$\sum_{i=1}^{n}\sum_{j=1}^{n} \underline{F}_{ij} = 0$ as all forces are paired with each other due to Newton's 3rd $\underline{F}_{ij} = -\underline{F}_{ji}$

If we set

$\underline{F} = \sum_{i=1}^{n} \underline{F}_i^{(e)}$ Total external force

and

$\underline{F} = \sum_{i=1}^{n} \underline{P}_i$ Momentum of system

We get:

$$\underline{F} = \frac{d}{dt}\underline{P}$$

Looking for an equation in the form $\underline{F} = m\underline{\ddot{R}}$ where $\underline{F}$ is the total applied force and

$$M = \sum_{i=1}^{n} M_i$$

$$\underline{P} = \sum_{i=1}^{n} M_i \underline{\dot{r}}_i$$

$$M\underline{\ddot{R}} = \frac{d\underline{P}}{dt} = \sum_{i=1}^{n} M_i \underline{\ddot{r}}_i$$

True if we define

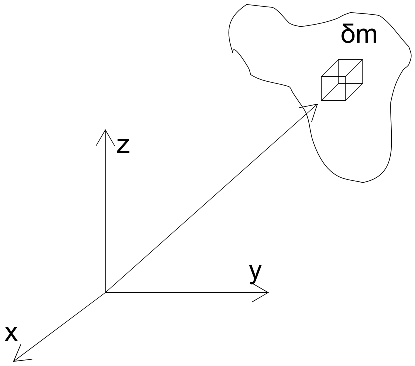
$$\underline{R} = \frac{1}{M}\sum_{i=1}^{n} M_i \underline{r}_i$$

Position vector of the centre of mass.

$$\underline{F} = M\underline{\ddot{R}}$$

The system behaves as though all its' mass is concentrated at one point.
All external forces act through this point. This tells you everything you need to know about what happens to the centre of mass but says nothing about the orientation in space of the object – see later.

$\underline{R} = \frac{1}{M}\sum_{i=1}^{n} M_i \underline{r}_i$ is OK for a collection of point particles – but what do we do about extended bodies?



Imagine the body divided up into N regions each of mass $\delta M = \frac{M}{N}$

$$\underline{R} = \frac{1}{M}\sum_{i=1}^{n} \delta m \underline{r}_i$$

By definition as N→∞

$$\frac{1}{M}\sum_{i=1}^{n} \delta m \underline{r}_i \to \frac{1}{M}\int \underline{r}_i\, dm$$

For a continuous body:

$$\underline{R} = \frac{1}{M}\int \underline{r}\, dm$$

If you know the density P(r)

$$\underline{R} = \frac{1}{M}\int P\underline{r}\, dv$$ (dv = volume element)

This is a volume integral.

Simple example:

Centre of mass of a long, thin beam.

Beam has mass M and length L
Density (mass per unit area length)=M/L
So:

$$\underline{R} = \left[\frac{1}{M}\int \frac{M}{L} x\, dx\right]\underline{i}$$

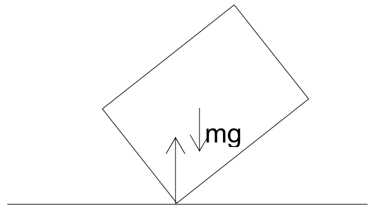
$$\underline{R} = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L \underline{i}$$

$$\underline{R} = \frac{L}{2}\underline{i}$$

2D problems involve density/unit area 2D integrals.
3D problems involve density/unit volume 3D integrals.
More complicated maths.

Falling box

Centre of mass in geometric centre of box. What forces act on the system?



Neither force has a horizontal component.
Acceleration of Com must be vertical.
As the base tumbles the COM falls straight down. COM motion but we say nothing at this stage about changes of orientation.

Conservation of Momentum.

Newton II $\underline{F} = \frac{d\underline{P}}{dt}$

Suppose we have an isolated system

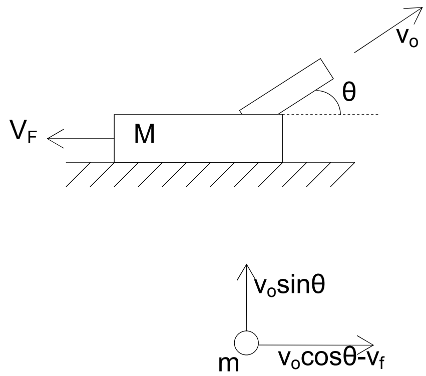
$\underline{F} = \underline{0}, \frac{d\underline{P}}{dt_N} = 0$

Net momentum $\underline{P} = \sum_{i=1}^{N} \underline{P}_i$ is a constant.

The total momentum is <u>conserved</u>.
Note that this conservation law is applicable in quantum mechanics and relativity as well.

Example

Gun recoil.



Muzzle velocity v_o. Two components to the velocity of the cannon ball.

Conservation of momentum in x-direction.

$0 = \left(m\left(v_o \cos\theta - v_F\right) - Mf_F\right)\underline{i}$

$V_F = \frac{mv_o \cos\theta}{m+M}$

In the y-direction there is a normal force acting on the gun. This is an external force so we cannot use conservation of momentum in this direction.

Impulse

$\underline{F} = \frac{d\underline{P}}{dt}$

$\Rightarrow \int_0^t \underline{F}dt = \underline{P}(t) - P(0)$

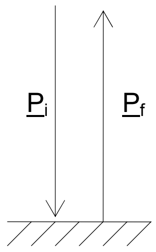
$\int_0^t \underline{F}dt$ is known as the impulse.

Example: rubber ball.

Mass = 200g

Velocity ~ $8ms^{-1}$

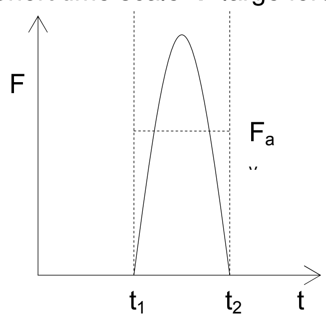
Ball in contact with the floor for ~ 1ms = 10^{-3}s



$$\Delta\underline{P} = \underline{P}_f - \underline{P}_i = 3.2kgms^{-1}$$

$$\underline{F}_{av} = \frac{\Delta P}{\Delta t} = 3200N$$

Short time scale → large forces.



$$\Delta t = t_f - t_i$$

$$\mathrm{Im}\,pulse = \int \underline{F}dt$$

Momentum and Mass flow problems

Often we have to deal with problems that involve a flow of mass.

→ Snowball rolling downhill

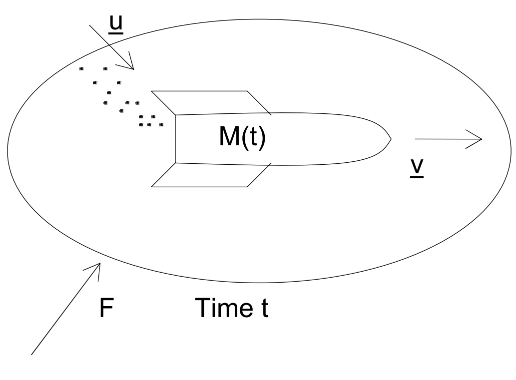
→ Flow of sand

→ Rocket propulsion

Such problems are best viewed from the point of view of momentum transfer.

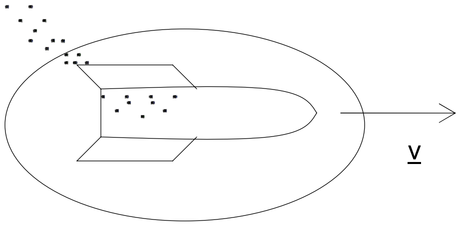
It is important to clearly define the dynamical system.

Example:



What value of F will keep the rocket moving at uniform $\underline{v}$?

time t+Δt



Mass of rocket M(t)+ΔM

ΔM is the mass added to the rocket in time Δt.

$$\underline{P}(t) = M(t)\underline{v} + \Delta M\underline{u}$$

$$P(t+\Delta t) = M(t) + \Delta M\underline{v}$$

$$\Delta\underline{P} = (v-u)\Delta m$$

Rate of change of momentum

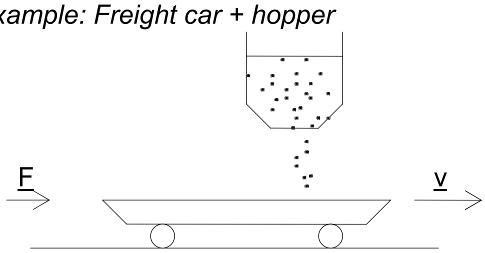
$$\frac{\Delta\underline{P}}{\Delta t} = (v-u)\frac{\Delta M}{\Delta t}$$

$$\Delta t \to 0$$

$$\frac{d\underline{P}}{dt} = (v-u)\frac{dm}{dt} = \underline{F}$$

$$\underline{v} = \underline{u} \Rightarrow \underline{F} = 0$$

Example: Freight car + hopper



Sand falls from a hopper onto a car moving with velocity v. What F is required?
The change of momentum is due to the change in mass.

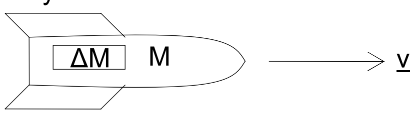
$$\frac{d\underline{P}}{dt} = \underline{v}\frac{dm}{dt} = F$$

If you drain the sand through a hole in the car then no change of momentum takes place → no force needed.

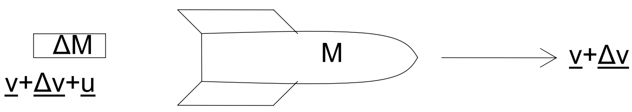
Problems with Rockets

Consider a rocket ejecting fuel at an exhaust velocity u.

System at time t:



Time t+Δt



$\underline{P}(t) = (M + \Delta M)\underline{v}$
$\underline{P}(t + \Delta t) = M(\underline{v} + \underline{\Delta v}) + \Delta M(\underline{v} + \underline{\Delta v} + \underline{u})$
$\Delta\underline{P} = M\underline{\Delta v} + \Delta M\underline{u}$
Note that we ignore ΔmΔv.

$$\frac{d\underline{P}}{dt} = \frac{Md\underline{v}}{dt} + \underline{u}\frac{dM}{dt}$$

$$\frac{d\underline{P}}{dt} = \frac{Md\underline{v}}{dt} - \underline{u}\frac{dm}{dt}$$

$$\left[\frac{dM}{dt} = -\underline{u}\frac{dm}{dt}\right]$$

$$\boxed{\underline{F} = \frac{Md\underline{v}}{dt} - \underline{u}\frac{dm}{dt}}$$

$\underline{F}$ is the external force on the system.

Suppose the rocket is in free space…
$\underline{F} = 0$

$$\frac{Md\underline{v}}{dt} = \underline{u}\frac{dm}{dt}$$

$$\frac{d\underline{v}}{dt} = \underline{u}\frac{1}{M}\frac{dm}{dt}$$

Suppose $\underline{u}$ is constant. Then:

$$\int_{t_0}^{t_f}\frac{d\underline{v}}{dt}\,dt = \int_{t_0}^{t_f}\underline{u}\frac{1}{M}\frac{dM}{dt}\,dt$$

$$\boxed{v_f - v_i = \underline{u}\ln\left(\frac{M_f}{M_0}\right)}$$

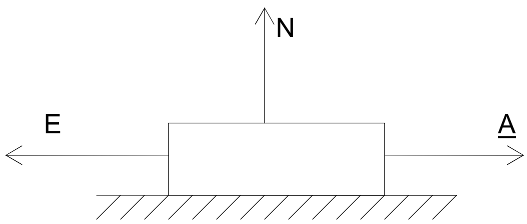
Fixed velocity of rocket doesn't depend on the time the fuel was burnt for.

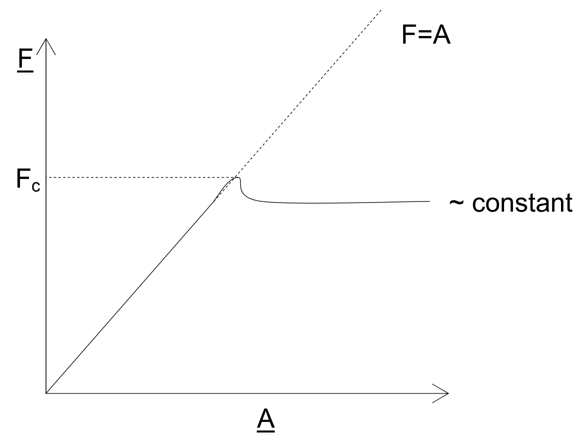
$E = -mg$?

2.4 Friction and Viscosity

What is friction?
– Force which opposes motion
– Static (body is stationary) & Dynamic (body is moving)

The edge of matter consists of ragged edges, rather than smooth ones. As such, a very small fraction of the geometric areas of the objects are in contact.

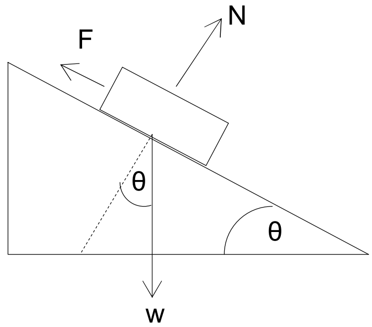




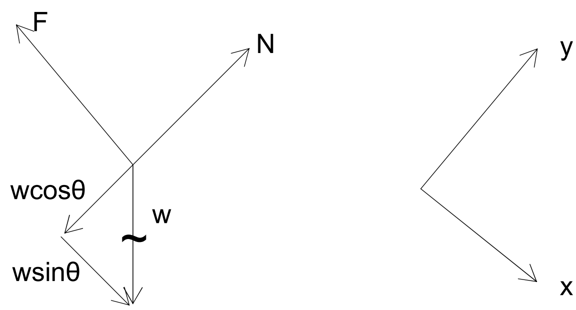
$F_c = \mu N$
N = normal force between the surfaces.
μ = coefficient of static friction.

When A>μN, object starts to move. Thereafter (A>μN) the frictional force is roughly constant.
$F = \mu_d N$
when μ_d is slightly less than μ .

Example Block and a wedge with friction.



FB Diagram:



Coefficient of friction μ block mass m
At what value of θ does the block start to move?
EOM:
$M\ddot{x} = w\sin\theta - F$
$M\ddot{y} = N - w\cos\theta$
When the block begins to slide:
$F = \mu N, \ddot{x} = 0$
Just at the limit when motion starts.
So:

$w \sin\theta_m = \mu N$

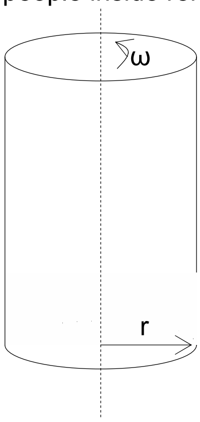
$w \cos\theta_m = N$

$\therefore \tan\theta_m = \mu$

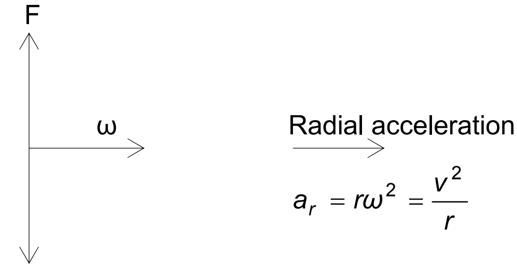
For wooden surfaces, μ~0.2-0.5

Example 2: The spinning terror

Large vertical drum that spins around its central axis → the floor is removed and the people inside remain fixed to the side.



What value of ω is needed?



w=mg

$$W = mR\omega^2$$

$$F \le \mu N = \mu m R\omega^2$$

$$F = mg$$

$$g \le \mu R\omega^2$$

$$\omega^2 \ge \frac{g}{\mu R}$$

So the smallest value of ω which will work is:

$$\omega = \sqrt{\frac{g}{\mu R}}$$

Cloth on wood gives $\mu \ge 0.3$

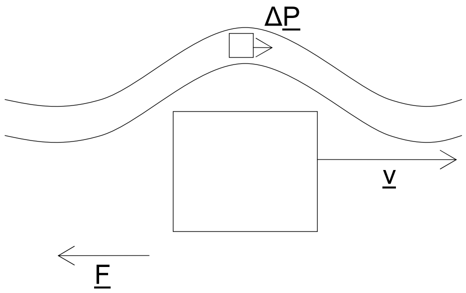
Take R=2m

$$\omega = \sqrt{\frac{9.81}{0.3x2}} = 4.0 rad.s^{-1}$$

$$Period = \frac{2\pi}{\omega} = 1.6 \sec onds$$

Viscosity

A body moving through a fluid is retarded by the force of velocity exerted on it by the fluid.



By dragging elements of the fluid along with the body momentum is transferred to the fluid → force.
Newton's III → resistive force on body
The viscous force is linear in v i.e. $\underline{F}_v = -C\underline{v}$. C can be calculated for simple shapes.
In PC1101 we will take C as a given constant.

Example

Free motion in a fluid.
A body moves with velocity v in a fluid. There are no other forces.
Solve the equation of motion

$$F = m\underline{a} = m\frac{d\underline{v}}{dt}$$

Here:

$$\underline{F} = \underline{F}_v = -c\underline{v}$$

$$-c\underline{v} = m\frac{d\underline{v}}{dt}$$

$$\frac{dv}{dt} + \frac{cv}{m} = 0$$

$$\frac{1}{V}dv = \frac{-c}{m}dt$$

$$[\ln v]_{v_o}^{v} = \frac{-c}{m}(t - t_o)$$

$$\ln\left(\frac{v}{v_o}\right) = \frac{-c}{m}t$$

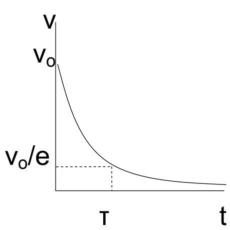
$$[t_o = 0]$$

$$\boxed{v = v_o e^{\frac{-ct}{m}}}$$

Because the exponential must have directionless argument → $\left[\frac{m}{c}\right] = T$

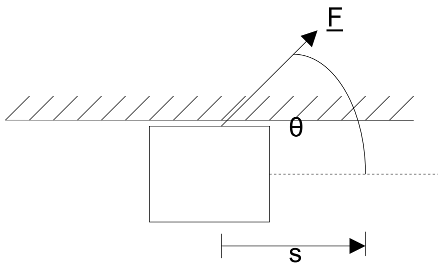
$$\left[\frac{m}{c}\right] = \tau$$

τ is the time taken for velocity to fall from $v_o \to \frac{v_o}{e} \approx 0.37v_o$

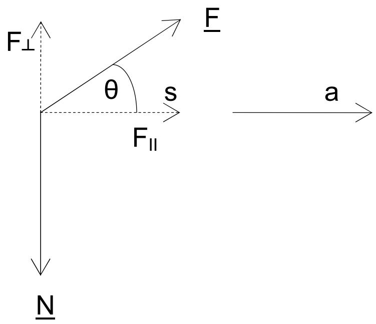


3. Conservation of Energy

Work done is dependant on the magnitude of the force and the distance moved.



Block against a wall – no friction.

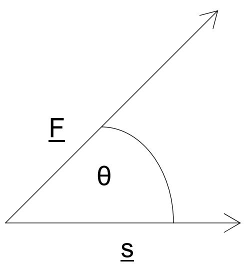


$F_\perp$ does no work.

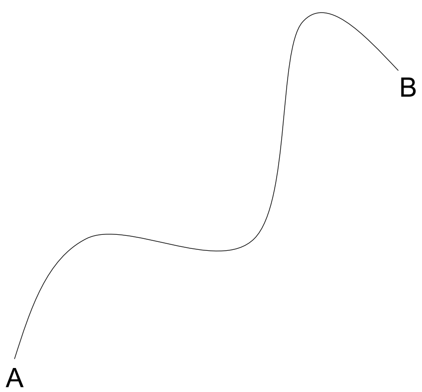
F_{II} (Component of force along the axis of displacement) does the work.

$W = F_{II} s = Fs\cos\theta = \underline{F}.\underline{s}$

Units 1 Joule = 1Nm



$W = \underline{F}.\underline{s} = Fs\cos\theta$



If the object is moved from A to B how do we determine the work done?

Path is broken down into a series of small displacements.

Total work done:

$$W = \sum_{i=1}^{N} \underline{F}_i \, \underline{ds}_i$$

So with N→∞ and $\underline{ds}$→0

$W = \int_A^B \underline{F}\,\underline{ds}$

(Path integral of line integral)

In the previous example the block accelerates and the final velocity will depend on the work done.

$v_f^2 = v_1^2 + 2as$

$as = \frac{v_f^2 - v_i^2}{2}$

$F_{II} = ma$

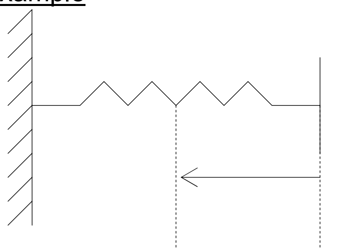
$\therefore W = F_{II} s = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

Useful in problems when you are trying to determine a speed $|\underline{v}|$. We define this as the kinetic energy.

$\boxed{E_k = \frac{1}{2}mv^2}$

$\boxed{W = E_{kf} - E_{ki}}$

<u>Example</u>



x=0 is the equilibrium position of the spring.
What is the work done in moving from x=0 to x=x'?
f=-kx Hooke's law
k = spring constant, x = displacement from equilibrium.

$W = \int \underline{F}.\underline{ds}$

$W = \int_{x=0}^{x=x'} kxdx$

$W = \frac{1}{2}kx'^2$

or more generally:

$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$

in going from x_1 to x_2.

3.1 Work-Energy Theorem

We have shown that in the case of uniform acceleration that $W = k_2 - k_1$.

Consider straight line motion where F may depend on position.

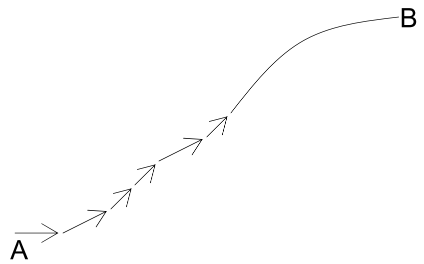
$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$W = \int_{x_1}^{x_2} Fdx = \int_{x_1}^{x_2} Mv\frac{dv}{dx}dx$

Using $F = ma = mv\frac{dv}{dx}$

So:

$$W = \int_{v_1}^{v_2} mvdv = \frac{1}{2}mv_2^{\ 2} - \frac{1}{2}mv_1^{\ 2}$$
$$v_1 = v(x_1)$$
$$v_2 = v(x_2)$$

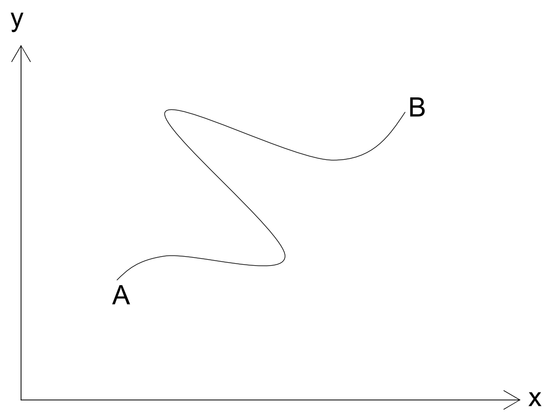


Curve can be considered as a series of small linear steps.
For each small step we can use:
$\Delta W_i = \underline{F}_i \Delta s_i = \Delta K_i$ (Change in KE at the i[th] step)
$$W_{AB} = \sum_{i=1}^{N} \Delta W_i = E_{KB} - E_{KA}$$
or more formally:
$$\boxed{\int_A^B \underline{F}\underline{ds} = E_{KB} - E_{KA}}$$
(Line integral → see Vectors, Fields and Matrices)
In any situation the work done is the change in kinetic energy.

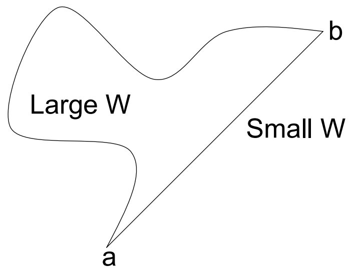
<u>Power</u>

The power is the rate at which work is done.
$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dw}{dt}$$
Units of power: 1 watt = 1Js^{-1}
$$P = \frac{dw}{dt} = \frac{\underline{F}\underline{ds}}{dt}$$

$\boxed{P = \underline{F}\underline{v}}$ - Instantaneous power

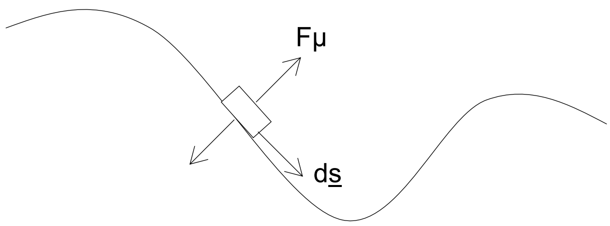


In general it is difficult to see how to apply the work / energy theorem → to evaluate W you need to know the path the system takes from a → b. Now always! If F is <u>conservative</u> then W depends only on a and b, but not the details of the path.



If you have friction then the force is non-conservative. The larger path involves more work.
The Work-Energy theorem is also useful in cases of conserved motion.

Rollercoaster:



$Fd\underline{s} = 0$
WD by forces of constraint = 0
Example: work done by uniform force.
e.g. m$\underline{g}$.
Particle moves from $\underline{r}_a$ to $\underline{r}_b$ along some path. The force is constant.

$\underline{F} = F_o \hat{\underline{n}}$

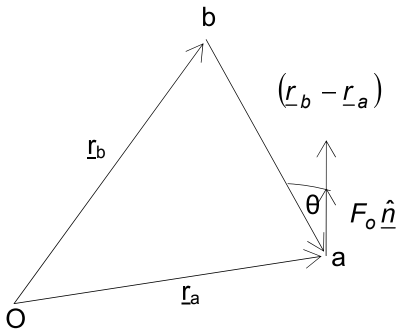
$W_{ba} = \int_{\underline{r}_a}^{\underline{r}_b} \underline{F}\underline{dr} = \int_{\underline{r}_a}^{\underline{r}_b} F_o \hat{\underline{n}} dr$

$\underline{dr} \equiv \underline{i}dx + \underline{j}dy + \underline{k}dz$

$W_{ba} = F_o \hat{\underline{n}} \left[\underline{i}\int_{x_a}^{x_b} dx + \int_{y_a}^{y_b} dy\, \underline{j} + \int_{z_a}^{z_b} dz\underline{k} \right]$

$W_{ba} = F_o \hat{\underline{n}} \left[\underline{i}(x_b - x_a) + \underline{j}(y_b - y_a) + \underline{k}(z_b - z_a) \right] = F_o \hat{\underline{n}} \left[(r_b - r_a) \right]$

$W = F_o \cos\theta \left[\underline{r}_b - \underline{r}_a \right]$



In the case of m$\underline{g}$
Work done $= W_{ba} = mg\Delta h$
(Δh = change in height)

Example: Escape velocity
A mass m is shot vertically from Earth with initial speed v.
→ What is the maximum altitude it reaches?
→ What speed does it need to escape Earth's gravitational field?

$$F = \frac{-FM_E m}{r^2}$$

$$k(r) - k(R_E) = W = \int_{R_E}^{r} F(r)dr = -GM_E m \int_{R_E}^{r} \frac{dr}{r^2}$$

or

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_o^2 = -GM_E m\left(\frac{1}{r} - \frac{1}{R_E}\right)$$

Find the largest value of $r=r_{max}$ by setting v=0

$$V_o^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right)$$

Note that $g = \frac{GM_E}{R_E^2}$

$$V_o^2 = 2gR_E^2\left[\frac{1}{R_E} - \frac{1}{r_{max}}\right]$$

$$V_o^2 = 2gR_E\left[1 - \frac{R_E}{r_{max}}\right]$$

For escape velocity let $r_{max} \rightarrow \infty$

$$V_o^2 = 2gR_E$$

$$V_o = \sqrt{2gR_E}$$

$$V_o = \sqrt{2x9.8x6.4x10^6} = 1.1x10^4 ms^{-1}$$

Potential Energy

For a conservative force the work done in going from a to b depends only on the end points not the detail of the path.

$$\int_{\underline{r}_a}^{\underline{r}_b} \underline{F}\underline{dr} = f^n(\underline{r}_b) - f^n(\underline{r}_a)$$

or we can define u

$$\int_{\underline{r}_a}^{\underline{r}_b} \underline{F}\underline{dr} = -u(\underline{r}_b) + u(\underline{r}_a)$$

Note sign convention.

$W_{ba} = k_b - k_a = -u_b + u_a$ where $u(\underline{r}_b) = u_b$

$k_b + u_b = k_a + u_a$ if force is conservative

Total mechanical energy = K+U

where K is the kinetic energy, and U is the potential energy.

This is conserved during the motion.

Example: Potential energy for a uniform field

$$\underline{F} = -mg\underline{k}$$

$$U_b - U_a = -\int_{z_a}^{z_b}(-mg)dz = mg(z_b - z_a)$$

So we can adopt

$$U = mgh(+const.)$$

Sometimes you need to get F from U.

Consider this in 1D

$$U_b - U_a = -\int_{x_a}^{x_b}(F(x))dx$$

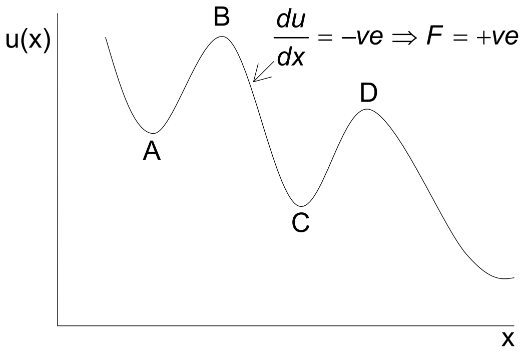
Suppose that $x_a=x$ and $x_b=x+\Delta x$

$$\Delta u = u(x+\Delta x) - u(x) = -\int_{x}^{x+\Delta x} f(x)dx$$

If Δx is small then

$F(x) \approx const.$
$\Delta u \approx -f(x)[(x + \Delta x) - x]$
$\Delta u \approx -f(x)\Delta x$
or
$\boxed{f(x) = -\frac{du}{dx}}$

Stability



The particle will be stable at A and C.

$-\frac{du}{dx} = 0, F = 0$

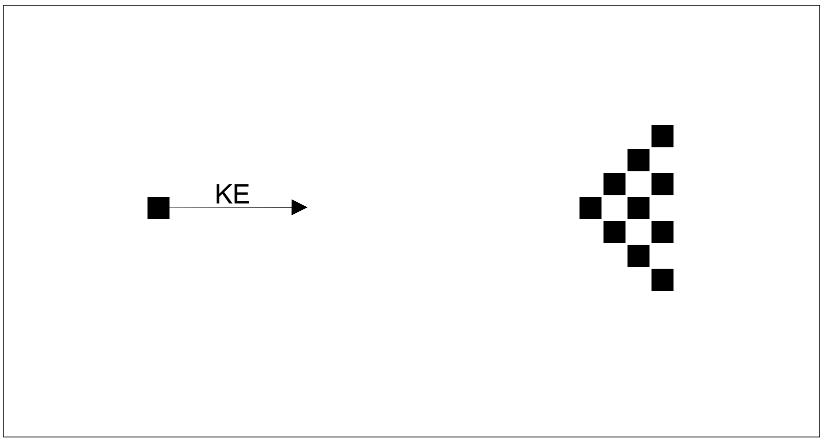
At B and D, F=0 also. However, these are not considered to be positions of stability (there are always variations in a system).
The system is only really stable at A and C → minima in u(x).

3.2 – Conservation of energy in collisions between particles

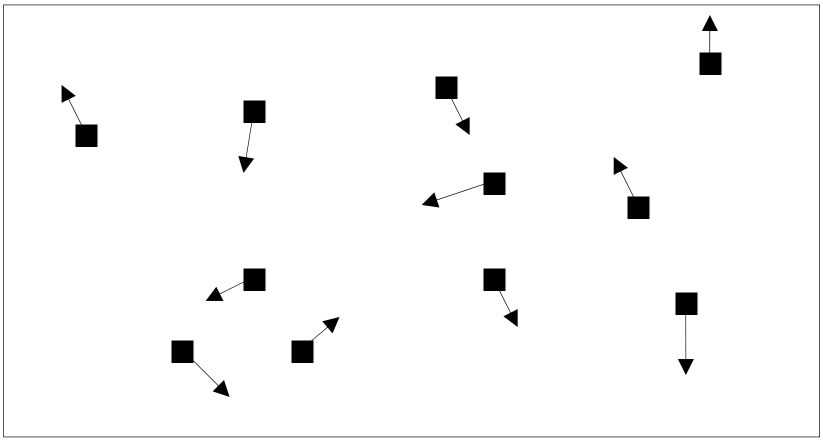
The general law of conservation of energy

The basic forces of nature (e.g. gravity, electric, magnetic) are <u>conservative</u>.
So, how do non-conservative forces arise?
In the case of friction, mechanical energy is lost as a block slides along a surface (to heat, sound).
In a series of detailed experiments, James Joule showed that heat is a form of energy.
Mechanical energy → can be converted into heat.
The reverse can happen, but has never been observed.

Consider a collision between a fast helium atom and a group of stationary atoms:



Before: KE only



After: "heat"

KE conserved, but randomly shared with all atoms
At microscopic level, every atomic collision is elastic.
In practice, we never see reverse process where all individual KE's are returned to one particle. ("Arrow of time")

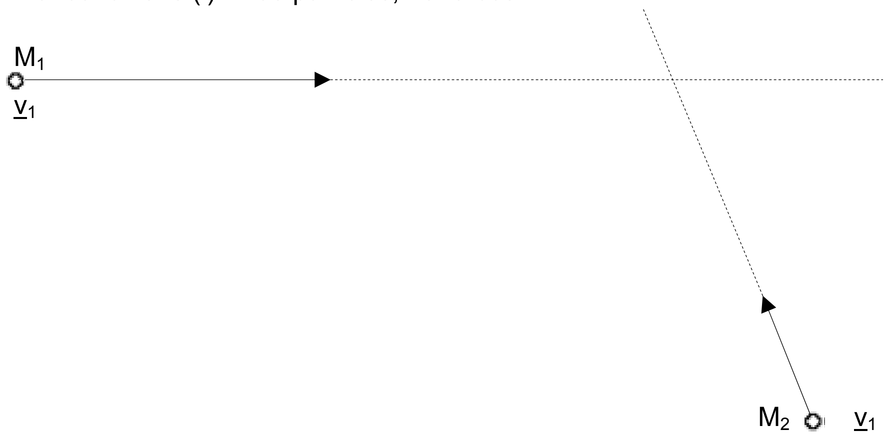
Heat = energy contained in random movements of atoms.
Ordered system → disordered system (easy)
Disordered system → ordered system (Unlikely)

If all forms of energy are taken into account (Kinetic, Potential, Heat, Sound, Light & radiation [Mass $E=mc^2$]), the total energy of a closed system is <u>conserved</u>.

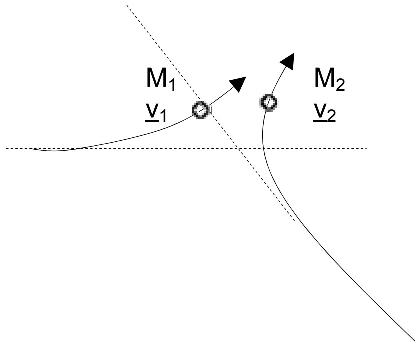
Conservation laws and particle collisions
Much of our knowledge of atoms, nuclei and elementary particles comes from scattering experiments. (e.g. Rutherford was led to the nuclear model of the atom through alpha particle scattering)

Particle collisions: 3 stages

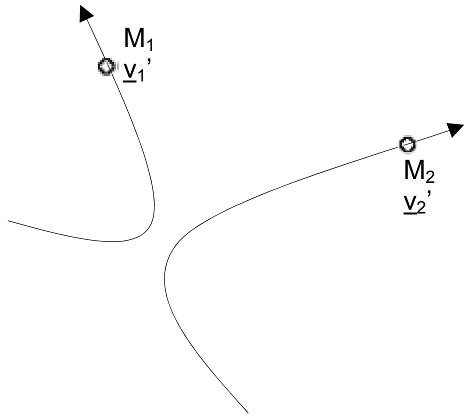
a) Long before collision
Initial conditions (i). Free particles, no forces.



b) Interactive stage
Forces act to change energies and momentum.



c) Long time after
Final conditions (f)



We have $\underline{P}_i = \underline{P}_f$ Momentum conservation.
(3 scalar equations, but 6 unknowns – $\underline{v}_1'$, $\underline{v}_2'$)
→ Need to consider the energy equation.
A collision is elastic if the total kinetic energy (KE) is unchanged.
If entirely conservative forces → elastic collision.

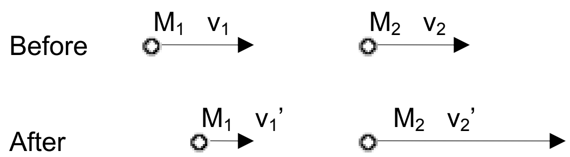
If some part of the interaction force is none conservative the collision is inelastic and KE is converted to/from other forms of energy. In general:

$k_i = k_f + Q$

where Q is the energy converted into another form.

Q>0, inelastic (e.g. car crash)
Q=0, elastic
Q<0, inelastic (e.g. chemical reactions or nuclear collisions “reactions”)

Elastic collisions in 1D



Conservation of momentum
$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$
Conservation of energy:

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2$$

Example: Elastic collision of two balls

Conservation of momentum:

$3mv - mv = 3mv_1 + mv_2$

$2v = 3v_1 + v_2$

Conservation of energy:

$$\tfrac{3}{2}mv^2 + \tfrac{1}{2}mv^2 = \tfrac{3}{2}mv_1^{\,2} + \tfrac{1}{2}mv_2^{\,2}$$

$$4v^2 = 3v_1^{\,2} + v_2^{\,2}$$

$$4v^2 = 3v_1^{\,2} + (2v - 3v)^2$$

$$4v^2 = 3v_1^{\,2} + 4v^2 + 9v_1^{\,2} - 12vv_1$$

$$v_1^{\,2} - vv_1 = 0$$

$$v_1(v_1 - v) = 0$$

Therefore two solutions – v_1=v (Initial condition) and v=0 (Collision has occurred)

Therefore, v_2=2v

Collision in 3D

These are usually easier in centre-of-mass (C.O.M.) frame of reference.

Recall $\underline{R} = \dfrac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}$

R is the position vector of the C.O.M.

C.O.M. velocity relative to the lab frame,

$$V_{cm} \equiv \dot{\underline{R}} = \frac{m_1\underline{v}_1 + m_2\underline{v}_2}{m_1 + m_2}$$

Velocities in C.O.M frame

$$\underline{V}_{1c} = \underline{V}_1 - \underline{V}_{cm} = \frac{m_2}{m_1 + m_2}(\underline{V}_1 - \underline{V}_2)$$

$$\underline{V}_{2c} = \underline{V}_2 - \underline{V}_{cm} = \frac{-m_1}{m_1 + m_2}(\underline{V}_1 - \underline{V}_2)$$

Momentum in COM frame:

$$\underline{P}_1 = M_1\underline{V}_{1C} = \frac{M_1M_2}{M_1 + M_2}(\underline{V}_1 - \underline{V}_1)$$

$$\underline{P}_2 = M_2\underline{V}_{2C} = \frac{-M_1M_2}{M_1 + M_2}(\underline{V}_1 - \underline{V}_1)$$

For a 2-body system, this is called the reduced mass.

$$\mu = \frac{M_1M_2}{M_1 + M_2}$$

$$\underline{P}_{1C} = \mu\underline{V}_r$$

$$\underline{P}_{2C} = \mu\underline{V}_r$$

$$(\underline{V}_r = \underline{V}_1 - \underline{V}_2)$$

The reduced mass is the natural unit of mass in the COM frame of reference.

Notice that in the COM frame the total momentum is zero.

Usually problems involving the relative coordinates are easier in the COM frame

4. Special Relativity

4.1 Introduction

Newtonian mechanics deals with the motion of particles under the influence of forces.

$\underline{F} = m\underline{a}$

The inertial mass m is a constant.

Even if $\underline{F}$ is not known Newtonian mechanics tells us that

$\underline{P} = \sum_i m_i v_i$

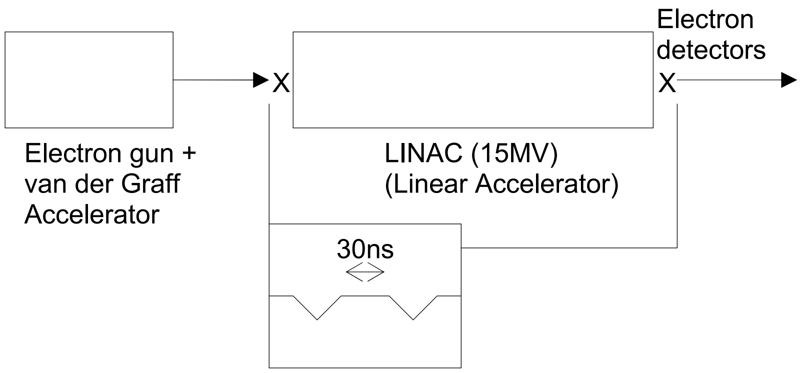
is a conserved quantity (no external forces)

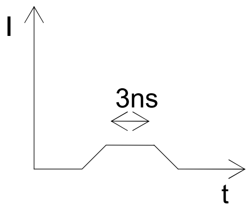
$\int_1^2 \underline{F} d\underline{r} = \frac{1}{2} M(V_2^{\ 2} - V_1^{\ 2})$

W.D. Change in KE

➔ Conservation of energy.

An experiment where this scheme fails:

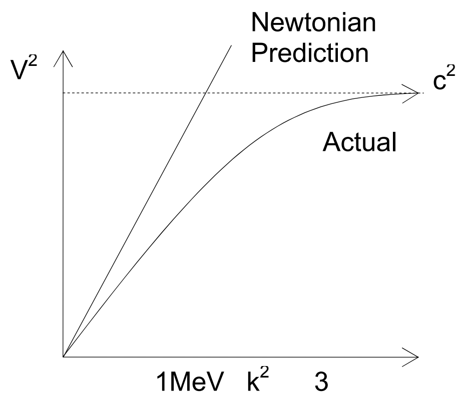




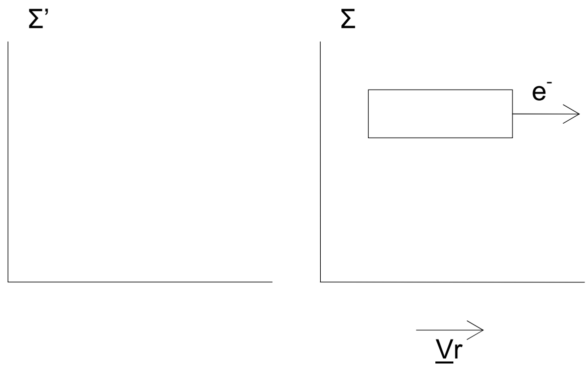
We can control the kinetic energy and measure the electrons velocity.

– W=eV
– W= ½ $m_e v^2 k$
– v^2=2k/M (Newtonian scheme)

W = Work Done
e = electron charge
V = voltage



The electrons never go faster than the speed of light! They have a maximum speed $c=2.99\times10^{8}ms^{-1}$ (The velocity of light).



If we view the experiment from another frame of reference $\sum'$ and $\sum$ moves with respect to $\sum'$ at a velocity $\underline{v}_r$ what limiting velocity do we see for the electron now?

What is v_{e^-}'(measured in $\sum'$)?

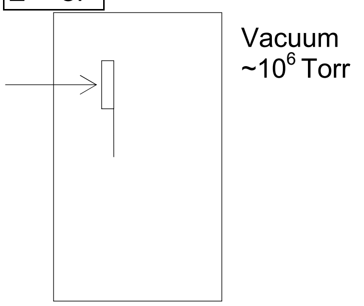
→ Either we are in a 'special' frame of reference, or there is something wrong with the way that we combine velocities.

→ Something wrong with the way we view space and time!

Photons:

What is so special about light?

- Experiments show that no matter what the energies of the photon (light of any frequency) has the same velocity in vacuum.
- Light is made up of photons – little packets of energy.
  $\boxed{E = hv}$ Energy of each packet or photon.
  h=Planck's constant. ≈6.63×10^{-34}Js.
- Light also has momentum (P)
  $\boxed{E = CP}$

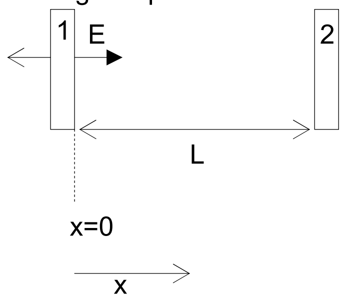


Measure momentum change.

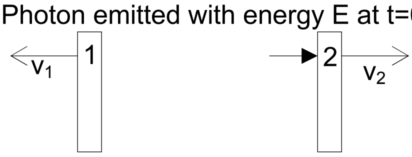
where $P = \frac{h}{\lambda}$ ← Wavelength

Also $E = hv$ ← frequency
Can be checked by measurements of radiation pressure.

Photons carry momentum and energy. Do they have mass?
A thought experiment:



Two objects initially at rest.
Photon emitted with energy E at t=0.



$t = \frac{L}{c}$

$$v_1 = -\frac{E}{m_1'c}$$

Mass of (1) $m_1 \rightarrow m_1'$ due to emission of photon.

$$x_1(t) = \frac{E}{m_1'c}t$$

$$x_2(t) = L + \frac{E}{m_2'c}\left(t - \frac{L}{c}\right)$$

After absorption.
Assume that the centre of mass is fixed.

$$M\bar{x} = m_1 0 + m_2 L$$

$$M\bar{x}' = -\frac{E}{c}t + m_2'L + \frac{E}{C}t - \frac{E}{c^2}L$$

$M=m_1+m_2$.

get $\Delta M = m_2' - m_2 = \frac{E}{c^2}$

pr $M_\gamma = \frac{E}{c^2}$

Consider the second block
M_γ is added.
E (of γ ray) is also added.

$\boxed{E = Mc^2}$ (Valid for all objects)
c^2 is huge – 9.0x1016m^2s^{-2}.
Small changes in mass correspond to huge changes of energy.
Normally the kinetic energies of everyday objects corresponds to very small changes in mass.
Develop a new dynamics for any object photons / electrons / anything else.
For photons E=cp (Experiment)

and $M = \frac{E}{c^2}$

So $M = \frac{p}{c}$ (1) for photons.

In Newtonian mechanics $M = \frac{p}{v}$ (2) Keep this.
(1) looks like a particular case of (2). v→c.
Newtonian mechanics we work with energy differences.
$dE = Fdx$ (Work done) This is still valid when working with the high velocities of photons.

$dE = \frac{dp}{dt}dx$

$dE = vdp$

Using (2) $E = \frac{c^2 p}{v}$ (3) (Photons and massive bodies)

$EdE = c^2 pdp$

Integrate both sides

$E^2 = c^2 p^2 + E_o^2$ (4) (E_o^2 is a constant of integration)

The constant of integration is ignored in Newtonian mechanics, but may have some role in new dynamics.

Substitute (3) into (4)

→ $E(v) = \frac{E_o}{\left(1 - v^2/c^2\right)^{1/2}}$ - only works if v<c. Applies to massive bodies only.

To convert this into mass, divide by c^2.

$m(v) = \frac{M_o}{\left(1 - v^2/c^2\right)^{1/2}}$

M_o is seen as the rest mass of an object. As v approaches c, the mass goes to infinity.
The inertial mass depends on velocity.
As the velocity gets very large it gets more difficult to move the electron.

What happens at very low velocity? (v<<c)

$E(v) = E_o\left(1 - v^2/c^2\right)^{-1/2}$

$e(v) \approx E_o\left(1 + \frac{1}{2}v^2/c^2\right)$

(Binomial expansion)

The kinetic energy:

$K = E(v) - E_o$

$K = \frac{1}{2}\frac{E_o}{c^2}v^2$

$\frac{E_o}{c^2}$ must be m_o (rest mass)

$k = \frac{1}{2}m_o v^2$ for low v.

The kinetic energy of the electron at high V≈c

$$K = m_o c^2\left[\frac{1}{\left(1 - v^2/c^2\right)^{1/2}} - 1\right]$$

Often you will see

$\gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}}$

We now have a new dynamics

$$\boxed{\begin{array}{l} \underline{P} = m(v)\underline{v} = m_o\gamma\underline{v} \\ \underline{F} = \dfrac{d\underline{p}}{dt} \\ E = m(v)c^2 = m_o\gamma c^2 \end{array}}$$

$$E^2 = C^2P^2 + (M_Oc^2)^2 = (M_0\gamma C^2)^2$$

Photons and massive particles Particles with mass

Example:

One of the reactions that generates energy in the sun.

$P + {}^2_1D \rightarrow {}^3_2He + \gamma$

Rest masses:

P: $1.672\text{x}10^{-27}$kg

D: $3.343\text{x}10^{-27}$kg

^{3}He: $5.0058\text{x}10^{-27}$kg

P+D=$5.0156\text{x}10^{-27}$kg

The excess of mass is taken away by the gamma ray.

$\Delta M = 9.8x10^{-30}\,kg$

Convert to an energy

$E = mc^2 = 8.8x10^{-13}\,J = 5.5MeV$

$\left(1eV \equiv 1.602x10^{-19}\,J\right)$

Back to basics:

Light propagates as a wave.

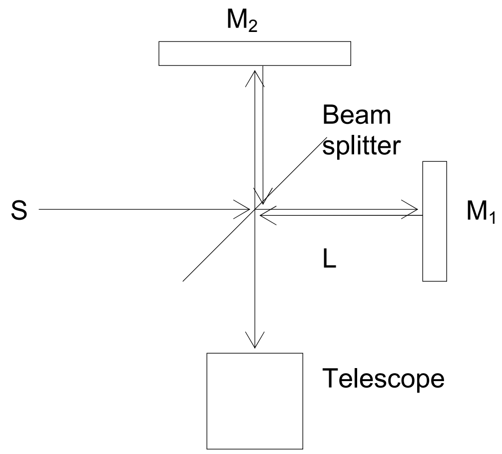
Waves are vibrations in some medium.

Dense media produces faster waves.

Light is very fast … so what is the medium?

The prevailing theory was that an ether exists in which light travels in.

(Mickleson-Morley experiment)



Can be shown:

$$t_1 = \frac{2L/c}{1 - v^2/c^2} \text{ BS} \rightarrow \text{M}_1 \rightarrow \text{BS}$$

$$t_2 = \frac{2L/c}{\left(1 - v^2/c^2\right)^{1/2}} \text{ BS} \rightarrow \text{M}_2 \rightarrow \text{BS}$$

There is no fringe shift observed as the apparatus is moved through 90^o.

The speed of light is not affected by the motion of the Earth.

This was the start of the end of the ether.

Galilean Relativity

That there are special frames of reference "inertial" frames. In these frames the laws of physics take the same form.

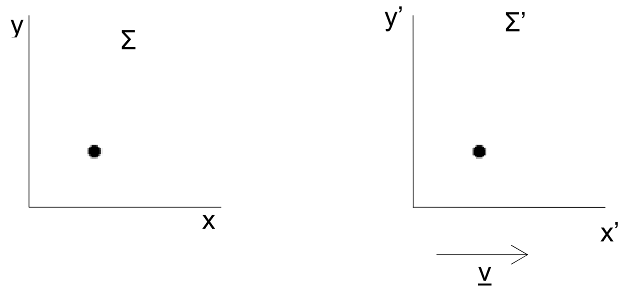
$\underline{F} = m\underline{a}$

→ may use different coordinates, but physics equations remain the same.

A rock dropped from the top of a mast of a moving ship still falls at the base of the mast.

→ you can't tell if the ship is moving by looking at things on the ship.

Transformations between frames of reference.



$x' = x - vt$

$y' = y$

$z' = z$

$t' = t$

^ Galilean transformations.

$u'_x = u_x - v$

$a'_x = a_x$

Suppose we have two particles

Suppose a force $F_{12} = f(x_2 - x_1)$

In frame $\sum$

$f(x_2 - x_1) = M_2 a_2$ EOM for (2)

In frame $\sum'$

$f(x_2 - x_1) = f(x'_2 + vt - (x'_1 + vt) = f(x'_2 - x'_2)$

And $a'_2 = a_2$

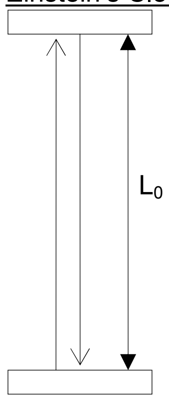
$f(x_2' - x_1') = m_2 a_2'$ in $\sum'$.

The idea of the Ether violates Galilean relativity.

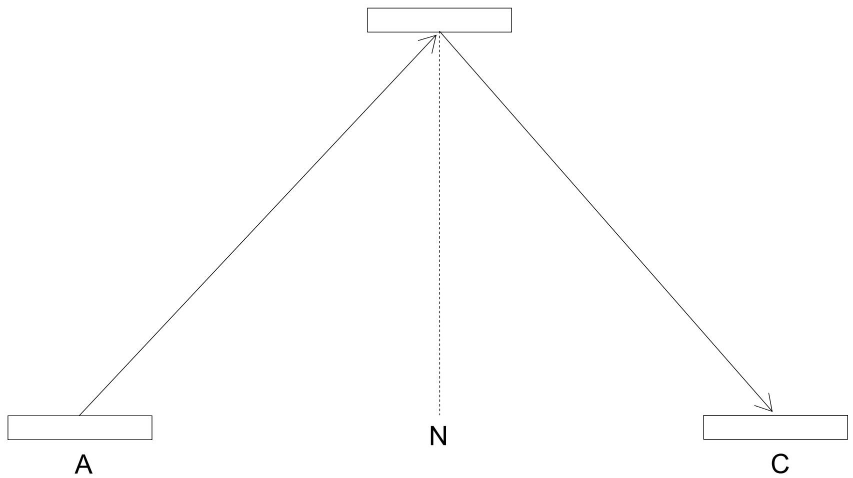
Einstein's Special Theory of Relativity

1) All inertial frames are equivalent with respect to the laws of physics
2) The speed of light in empty space is always c.

Einstein's Clock



Clock ticks with period $\Delta t = \frac{2L_0}{c}$



The clock is set in motion with velocity v.
Clock has period Δt in lab

$$AN = NC = V\frac{\Delta t}{2}$$

$$BN = L_0$$

so $AB + BC = 2\left[L_0{}^2 + \left(V\frac{\Delta t}{2}\right)^2\right]^{1/2}$

$$c\Delta t = 2\left[L_0{}^2 + \left(V\frac{\Delta t}{2}\right)^2\right]^{1/2}$$

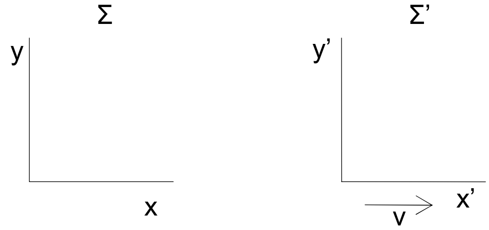
$$\Delta t = \frac{2L_0}{\left(c^2 - v^2\right)^{1/2}}$$

$$\Delta t = \frac{2L_0}{\left(1 - v^2/c^2\right)^{1/2}} = \gamma\Delta t'$$

$\gamma \geq 1$

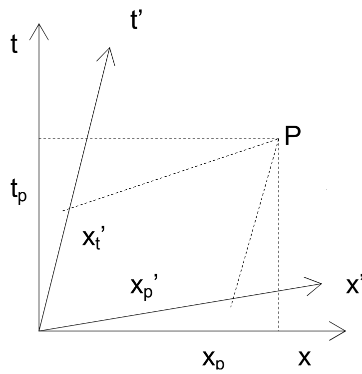
Called time dilation.

4.2 The Lorentz Transformations



Galilean Transformations:
x'=x-vt
x=x'+vt
y=y'
z-z'
t=t'

We want to replace these with a new set of transformation rules which embody the 2nd postulate.
We should be prepared for t≠t'.



Points to note:
1. x' axis is drawn to represent events at t'=0
2. t' axis is drawn to represent the "world line" of the point x'=0
3. None of this represents any real tilting of the x-axis.
   → Geometric representation of the transformation from Σ to Σ'.
4. Linear transformation.
5. x'=0, t'=0 and x=0, t=0 represent the same event.

Point (4):
x=ax'+bt' (1a)
x'=ax-bt (1b)
General linear transformations.

The motion of Σ as measured in Σ':
x=0
0=ax'+bt'

$$\frac{x'}{t'} = -\frac{b}{a} = -v$$

or $\frac{b}{a} = v$

Consider a beam of light moving in the x-direction in both Σ and Σ'.
Assume that flash of light starts at t=0, x=0.

$$\left.\begin{aligned} x &= ct \\ x' &= ct' \end{aligned}\right\} 2nd.postulate$$

Substitute into equation (1).

$$ct = (ac + b)t'$$
$$ct' = (ac - b)t$$
$$\Rightarrow c^2 = a^2\left(c - \frac{b}{a}\right)\left(c + \frac{b}{a}\right)$$
$$c^2 = a^2\left(c^2 - \left(\frac{b}{a}\right)^2\right)$$
$$a = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma(v)$$
$$NB : \frac{b}{a} = v$$

So:

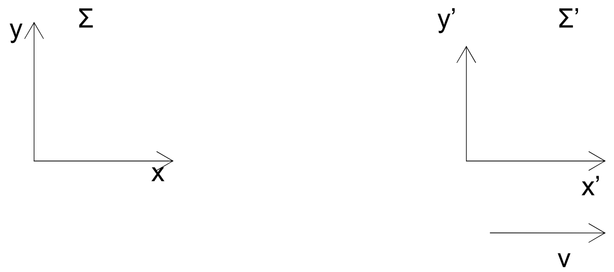
$$x = \gamma(x' + vt')$$
$$x' = \gamma(x - vt)$$

For low velocities, γ tends to 1, and you get the Galilean transformations.
Can also obtain the equations:

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

The full set of equations $(x,t) \leftrightarrow (x',t')$ are known as the Lorentz transformations.

Summery:



Axes coincide at t=t'=0

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ x &= \gamma(x + vt) \\ \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ t &= \gamma\left(t' + \frac{vx}{c^2}\right) \end{aligned}}$$

<u>Example</u>:
Suppose that Σ' has velocity v with respect to Σ. v=0.6c
Clocks are set so that t=t'=0 when x=x'=0.
Two events occur.

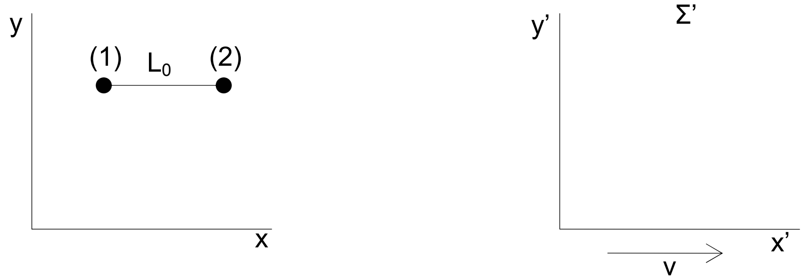
1) x_1=10m, t_1=2x10^{-7}s
2) x_2=50m, t_2=3x10^{-7}s

What is the distance between these two events as measured in Σ'?

$$(x_2' - x_1') = \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right]$$
$$\gamma = \frac{1}{\sqrt{1 - 9/25}} = \frac{5}{4}$$
$$x_2' - x_1' = \frac{5}{4}\left[(50 - 10) - \frac{3}{5}(3 - 2)x10^{-7}c\right]$$
$$x_2' - x_1' = 27.5m$$

Lorentz Contraction:

What is the length of an object?



$L_0=x_2-x_1$ at any time t in Σ.

Need two events at the same t'.

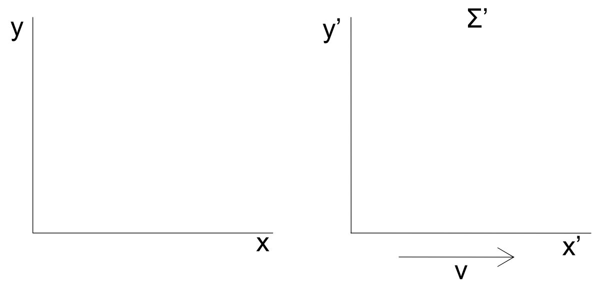
$$x_1 = \gamma(x_1' + vt_1')$$
$$x_2 = \gamma(x_2' + vt_2')$$
$$t_1' = t_2' = t'$$
$$x_2 - x_1 = \gamma(x_2' - x_1')$$
$$L_0 = \gamma L'$$

The object is shorter in the moving frame by a factor $\gamma \geq 1$.

Called Lorentz contraction (only along x-direction)

4.3 Transformation of Velocities

$$x = \gamma(x' + vt)$$
$$y = y'$$
$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$



$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

By definition

$$u_x' = \frac{dx'}{dt'}$$

(x component of velocity)

$$u_y' = \frac{dy'}{dt'}$$

$$u_x = \frac{dx}{dt}, u_y = \frac{dy}{dt}$$

$$\frac{dx}{dt'} = \gamma\left(\frac{dx'}{dt'} + v\right) = \gamma(u_x' + v)$$

$$\frac{dy}{dt} = u_y'$$

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{v}{c^2}u_x'\right)$$

$$u_x = \frac{dx}{dt'}\frac{dt'}{dt} = \frac{u_x' + v}{\left(1 + \frac{vu_x'}{c^2}\right)} \quad (1)$$

$$u_y = \frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} = \frac{u_y'}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)} \quad (2)$$

Does (1) give the correct behaviour at low v/c?

$$\frac{v}{c} << 1$$

$$u_x \approx u_x' + v$$

OK.

Also $u_y \approx u_y'$

Example 1:

Suppose $v = u_x' = \frac{c}{2}$

$$u_x = \frac{0.5c + 0.5c}{1 + 0.5^2} = \frac{4}{5}c$$

Example 2:

Suppose $u_x' = c$

Light propagating along x' direction.

$$u_x = \left[\frac{c + v}{1 + v/c}\right] = \left[\frac{1 + v/c}{1 + v/c}\right]c = c$$

In addition to (1) and (2) we can obtain

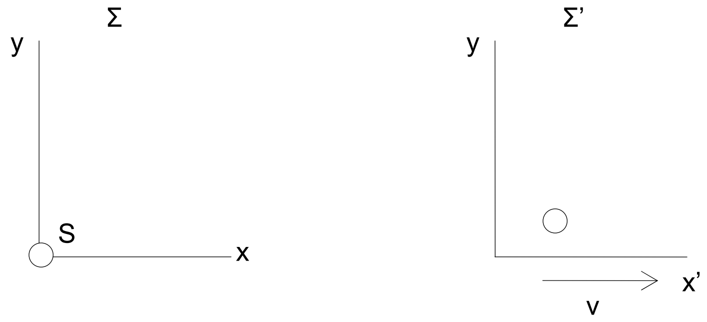
$$u_x' = \frac{u_x - v}{1 - vu_x/c^2} \quad (3)$$

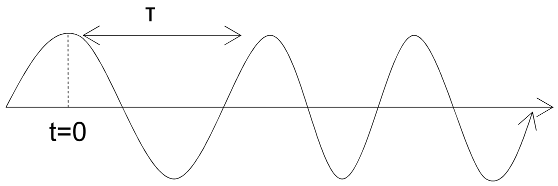
$$u_y' = \frac{u_y}{\gamma(1 - vu_x/c^2)} \quad (4)$$

(1) through (4) → remember.

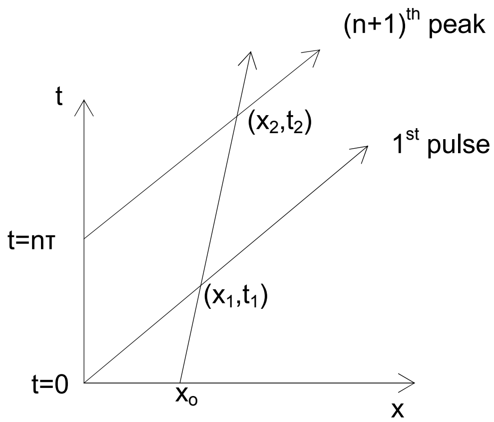
4.4 The Doppler Effect

Here we will consider EM radiation from a source in frame Σ (origin) observed in frame Σ' (origin)





in Σ.
Consider a train of n peaks
nth peak at time (in Σ) $(n-1)\tau$
Space time diagram in Σ:



$$x_1 = ct_1 = x_o + vt_1$$

and

$$x_2 = c(t_2 - n\tau) = x_o + vt_2$$

$$t_2 - t_1 = \frac{nc\tau}{c-v}$$

and

$$x_2 - x_1 = \frac{vnc\tau}{c-v}$$

From the Lorentz transformations:

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - v(x_2 - x_1)/c^2\right]$$

Substitute from above:

$$t_2' - t_1' = \gamma\left(\frac{cn\tau}{c-v} - \frac{v}{c^2}\frac{vcn\tau}{c-v}\right)$$

So this time difference is n periods in Σ’.

$$\tau' = \frac{\gamma c\tau}{c-v}\left[1 - \frac{v^2}{c^2}\right]$$

Let $\beta = \dfrac{v}{c}$

$$\tau' = \frac{\gamma\left(1-\beta^2\right)}{\left(1-\beta\right)}\tau$$

$$\tau' = \gamma\left(1+\beta\right)\tau$$

But

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

So

$$\tau' = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}\tau$$

or in terms of frequency v

$$v' = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} v$$

If v is positive → β is positive.

$v' < v$

observed frequency is lower.

If source is moving towards observer:

$v' > v$

Observed frequency higher.

4.5 Relativistic Mechanics

$$m(v) = \gamma M_o$$

$$\underline{P} = \gamma M_o \underline{v}$$

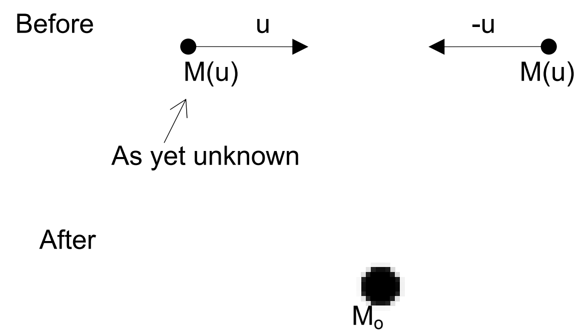
$$\underline{F} = \gamma M_o c^2 = Mc^2$$

$$E^2 = \left(M_o c^2\right)^2 + c^2 p^2$$

$$\gamma(v) = \frac{1}{\sqrt{1-\beta^2}}, \beta = v/c$$

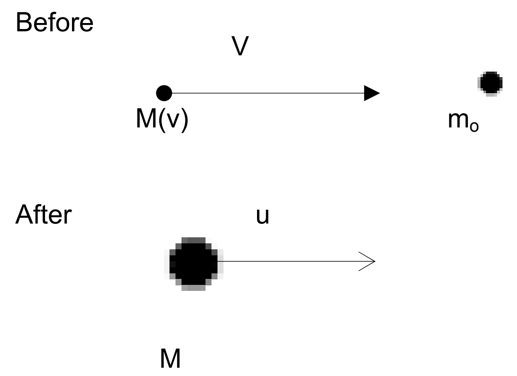
Consider the following inelastic collision:

Σ' (Zero momentum frame)



Stationary composite particle in Σ'

Σ lab frame: stationary target



Recall $u_x = \dfrac{u_x' + v}{1 + \dfrac{v u_x'}{c^2}}$

Here $u_x = V, u_x' = v = u$

$$V = \frac{2u}{1 + \dfrac{u^2}{c^2}} \; *$$

V = velocity of projectile as seen in Σ.
Suppose we have the following conservation laws:
Momentum conservation: $m(v)v = Mu$
Mass conservation: $m(v) + m_o = M$
Eliminate M $\dfrac{m(v)}{m_o} = \dfrac{u}{v - u}$
Using * (relates v and u)
→ $u^2 - 2\left(\dfrac{c^2}{v}\right)u + c^2 = 0$

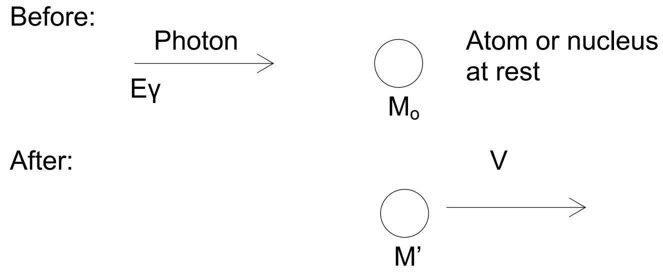
Solution: $\dfrac{c^2}{v} \pm \left[\left(\dfrac{c^2}{v}\right)^2 - c^2\right]^{1/2}$

$u \to \dfrac{v}{2}$ as V<<c → (-)

$$\left[\frac{m(v)}{m_o} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma(v)\right]$$

We have linked the mechanics that was suggested by the thought experiment with the Lorentz transformations.

Example: Absorption of photons



Find V.
Conservation of Energy: $E = M_o c^2 + E_\gamma = M' c^2$

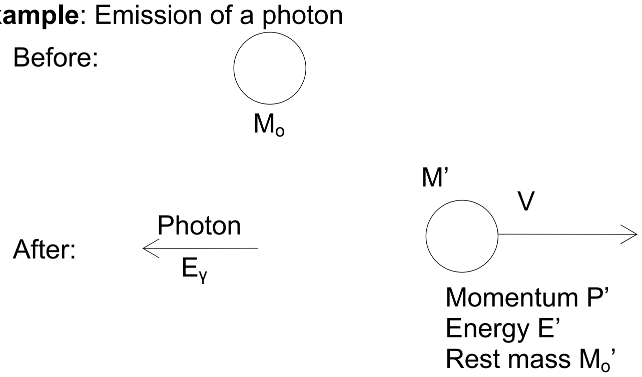
Conservation of momentum: $P = \frac{E_\gamma}{c}$ $\left(E^2 = \left(M_o c^2\right)^2 + c^2 p^2\right)$

$$M' = M_o + \frac{E_\gamma}{c^2}$$

$$\beta = \frac{v}{c} = \frac{E_\gamma}{\left(M_o c^2 + E_\gamma\right)}$$

Recoil velocity.

Example: Emission of a photon



Energy and momentum conservation:

$$E = M_o c^2 = M' c^2 + E_\gamma = E' + E_v$$

$$P = 0 = M' V - \frac{E_\gamma}{c} = P' - \frac{E_\gamma}{c}$$

So:

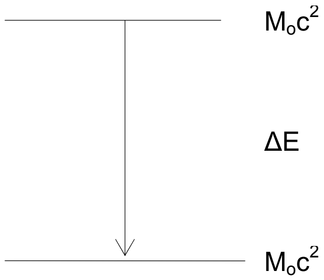
$$E' = M_o c^2 - E_\gamma$$

$$cP' = E_\gamma$$

$$\left(M_o' c^2\right)^2 = (E')^2 - (cP')^2$$

$$\left(M_o' c^2\right)^2 = \left(M_o c^2 - E_\gamma\right)^2 - E_\gamma^{\,2}$$

$$\left(M_o' c^2\right)^2 = \left(M_o c^2\right)^2 - 2M_o c^2 E_\gamma$$



Energy levels within the atom / nucleus.

$$M_o' c^2 = M_o c^2 - \Delta E$$

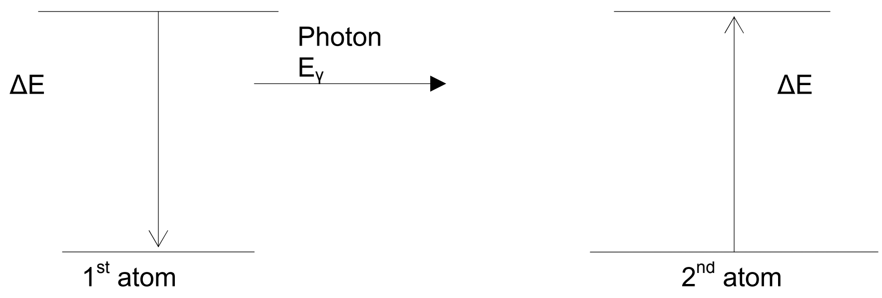
Square this expression:

$$\left(M_o' c^2\right)^2 = \left(M_o c^2\right)^2 - 2M_o c^2 \Delta E + \Delta E^2\ 9$$

Combine with *

$$E_\gamma = \Delta E\left[1 - \frac{\Delta E}{2M_o c^2}\right]$$

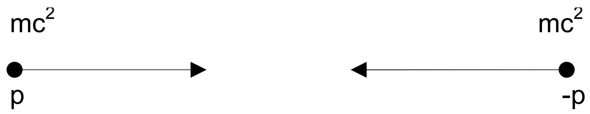
The photon does not have exactly the same energy difference of the levels – it is slightly less. $\Delta E << M_o c^2$. This is significant because



γ is not absorbed by the second atom – light gets out.

Example: Creation of a new particle

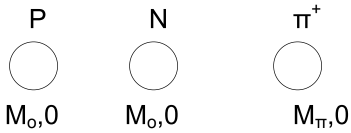
Two protons in zero momentum frame



$$E = 2mc^2$$

$$\left[P + P \to P + N + \pi^+\right]$$

Collision creates a pion.



Zero momentum

$$E = 2M_o c^2 + M_\pi c^2$$

$$\Rightarrow \frac{M}{M_o} = 1 + \frac{M_\pi}{2M_o}$$

$$\frac{M}{M_o} = \gamma = 1.074$$

$$\Rightarrow \frac{v}{c} = 0.37$$

(In zero momentum frame)

$$\beta = \frac{v}{c}$$

$$\beta_1 = \frac{v_1}{c}$$

$$\beta_1 = \frac{\beta + \beta}{1 + \beta^2}$$

$$\beta_1 = 0.65$$

$$\Rightarrow \gamma_1 = 1.31$$

Kinetic energy:

$$E_K = mc^2 - m_o c^2 = (\gamma_1 - 1) M_o c^2$$

$$E_k = 290 MeV$$

5. Angular Momentum and Rotation of Rigid Bodies

5.1 Introduction

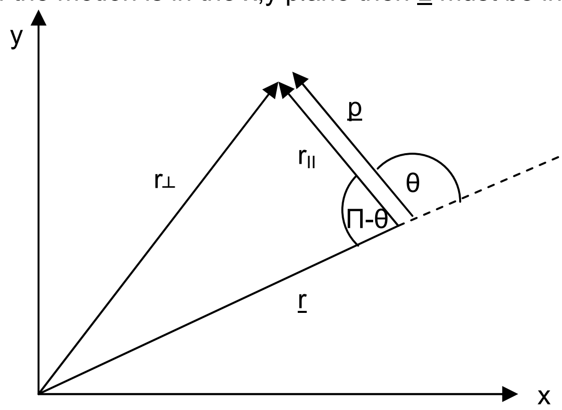
5.1.1 Angular Momentum

Angular momentum is defined for a particle as

$\underline{L} = \underline{r} x \underline{p}$

$\underline{L}$: Angular momentum. Units of kgm^{2}s^{-1} (No special names)
$\underline{r}$: Position
$\underline{P}$: momentum

If the motion is in the x,y plane then $\underline{L}$ must be in the z-direction.



$\underline{r}$ decomposed into $\underline{r}_\perp$ and $\underline{r}_{||}$.
But $\sin(\pi - \theta) = \sin\theta$

$|\underline{r} x \underline{p}| = r_\perp p = |L|$

Note that $\underline{L}$ gives a sense of the rotation.
$\underline{L}$ is along positive z axis for anticlockwise
$\underline{L}$ is along negative z axis for clockwise.
(Rotation in x,y plane)

$$\underline{L} = \underline{r} x \underline{P} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ r_x & r_y & 0 \\ P_x & P_y & 0 \end{vmatrix} = (r_x P_y - r_y P_x)\underline{k}$$

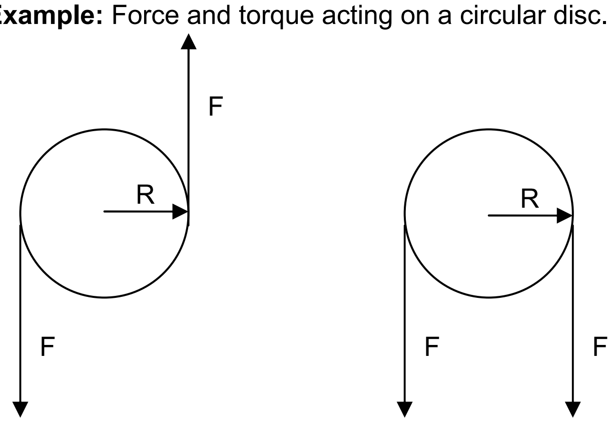
Note that $\underline{L}$ depends on the choice of origin.

5.1.2 Torque

$\underline{\tau} = \underline{r} x \underline{F}$

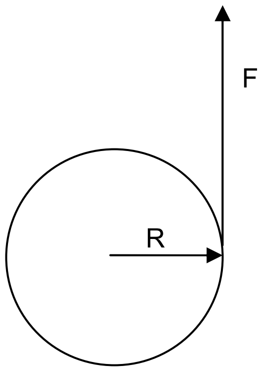
$\underline{r}$: Position
$\underline{F}$: Force acting on the particle.

Example: Force and torque acting on a circular disc.



$\tau = 2RF$
Net force = 0

$\tau = 0$
Net force = 2F



$\tau = RF$
Net force = f

Torque is important because it determines the rate of change of Angular Momentum.

$$\frac{d\underline{L}}{dt} = \frac{d}{dt}(\underline{r}x\underline{P}) = \dot{\underline{r}}x\underline{P} + \underline{r}x\dot{\underline{P}} = \dot{\underline{r}}x\underline{P} + \underline{r}x\underline{F}$$

The second term in this is the torque τ
Term 1 is 0 as both vectors are in the same direction.

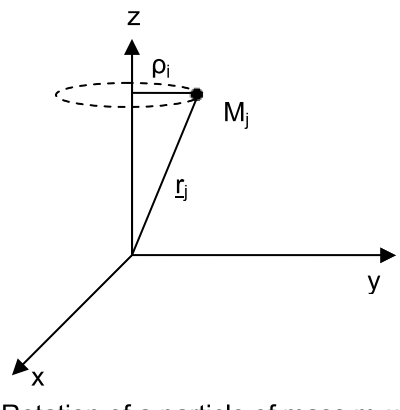
$\underline{P} = m\underline{v} \equiv m\dot{\underline{r}}$

$\dot{\underline{r}}x\dot{\underline{r}} = 0$

$$\boxed{\frac{d\underline{L}}{dt} = \underline{\tau}} \quad \left(\frac{d\underline{P}}{dt} = \underline{F}\right)$$

5.2 Rigid Body Rotation about a fixed axis

Often applications occur when the rotation is about a "fixed axis", i.e. one which doesn't rotate in space.
If the body is rigid then every particle within the system remains at a fixed distance from that axis.



Rotation of a particle of mass m_j with position $\underline{r}_j$ about the z-axis.
Circular motion about z. Radius is ρ_j

$|v_j| = \omega\rho_j$

$\underline{L}(j) = \underline{r}_j.m\underline{v}_j$

Only interested in L_z.

$L_z(j) = m_j v_j.d_z$

(d_z is the distance to the z-axis)

$L_z(j) = m_j v_j \rho_j = m_j P_j^2 \omega$

Total AM about z is

$$L_z = \sum_j L_z(j) = \sum_j M_j \rho_j^{\,2} \omega$$

$$L_z = I\omega$$

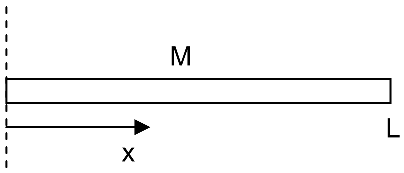
I is the moment of inertia.

$$I = \sum_j M_j \rho_j^{\,2}$$

For a continuous body

$$\sum_j M_j \rho_j^{\,2} = \int \rho^2 dm$$

and $I = \int \rho^2 dm = \int \left(x^2 + y^2\right) dm$



Uniform stick pivoted about x=0

$$I = \int_0^L x^2 dm = \frac{1}{3} ML^2$$

Dynamics of a pure rotation.

Postulate that internal torques sum to zero.

→ Angular momentum is conserved.

$$L_z = I\omega, \tau = \frac{d\underline{L}}{dt}$$

$$\Rightarrow \tau_z = \frac{d}{dt}(I\omega) = I\dot{\omega} = I\alpha$$

α is the angular acceleration.

$\tau_z = I\alpha$ (Recall $F = ma$)

Kinetic energy:

$$K = \sum_j \frac{1}{2} m_j v_j^{\,2} = \sum \frac{1}{2} M_j \rho_j^{\,2} \omega^2$$

$$K = \frac{1}{2} I\omega^2$$

(Similar to $k = \frac{1}{2} mv^2$)

5.3 Translation and rotation

Often systems show both translational and rotational motion. e.g. a drum rolling down a hill.

$$L_z = I_o \omega + \left(\underline{R} x M \underline{V}\right)_z$$

I_o is the moment of inertia about an axis through the COM

$\underline{R}$ is the position of the centre of mass

M is the total mass.

Consider N particles .

$$\underline{L} = \sum_{j=1}^{N} \left(\underline{r}_j x M_j \dot{\underline{r}}_j\right)$$

COM (Centre Of Mass) $\underline{R} = \sum_{j=1}^{N} \frac{M_j \underline{r}_j}{M}$

$$\underline{r}_j = \underline{R} + \underline{r}_j'$$

$\underline{r}_j'$ is the position in COM frame.

$$\underline{L} = \sum_j \left(\underline{R} + \underline{r}_j'\right) x M_j \left(\underline{\dot{R}} + \underline{\dot{r}}_j'\right)$$

$$= \underline{R}x\sum_j M_j \underline{\dot{R}} + \sum_j M_j \underline{r}_j' x\underline{\dot{R}} + \underline{R}x\sum_j M_j \underline{\dot{r}} + \sum_j M_j \underline{r}_j' x\underline{\dot{r}}_j'$$

$\sum_j M_j \underline{r}_j' = 0$ as it is the COM in the COM frame.

$\sum_j M_j \underline{\dot{r}}_j' = 0$ as above.

$\underline{R}x\underline{\dot{R}}\sum_j M_j = \underline{R}x\underline{\dot{R}}M = \underline{R}xM\underline{V}$ (M is the total mass, V is the velocity of COM)

The final term is the angular momentum in the COM frame.

$$\underline{L} = \underline{R}xM\underline{V} + \sum_j M_j \underline{r}_j' x\underline{\dot{r}}_j'$$

$$\boxed{L_z = \left(RxM\underline{V}\right)_z + I_o\omega}$$

$\left(RxM\underline{V}\right)_z$ is the orbital term, and $I_o\omega$ the spin term.

Torque can also be divided into two components.

$$\underline{\tau} = \sum_j \left(\underline{r}_j x\underline{F}_j\right) = \sum_j \left(\underline{r}_j' + \underline{R}\right) x\underline{F}_j = \sum_j \left(\underline{r}_j' x\underline{F}_j\right) + \underline{R}x\underline{F}$$

when $\underline{F} = \sum_j \underline{f}_j$ net force.

Take z components

$\boxed{\tau_z = \tau_o + \left(\underline{R}x\underline{F}\right)_z}$ *

τ_o is the torque about z in COM.

$\left(\underline{R}x\underline{F}\right)_z$ is the torque <u>on</u> COM.

We can show that the angular acceleration (α) $\alpha \equiv \dot{\omega}$ depends on torque about the axis through the COM (τ_o)

$$L_z = \left(\underline{R}xM\underline{V}\right)_z + I_o\omega$$

$$\frac{dL_z}{dt} = I_o\alpha + \frac{d}{dt}\left(\underline{R}xM\underline{V}\right)_z$$

$$= I_o\alpha + \left(\underline{R}xM\underline{a}\right)_z$$

$$= I_o\alpha + \left(\underline{R}x\underline{F}\right)_z$$

From * this must be:

$$\tau_o + \left(\underline{R}x\underline{F}\right)_z = I_o\alpha + \left(\underline{R}x\underline{F}\right)_z$$

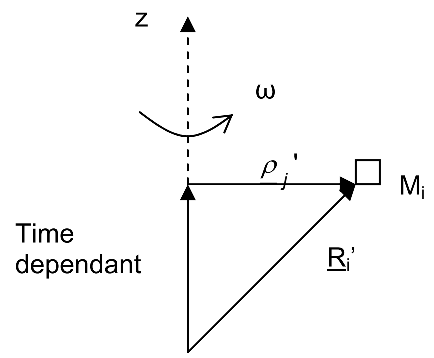
$$\boxed{\tau_o = I_o\alpha}$$

This equation is independent of the motion of the COM.
The COM may even be accelerating, and this equation will still hold.

Kinetic energy of system?

$$K = \frac{1}{2}\sum_j M_j v_j^2 = \frac{1}{2}\sum_j M_j \left(\underline{\dot{R}} + \underline{\dot{r}}_j'\right)^2$$

$$\left(\underline{A}^2 = \underline{A}.\underline{A} = |A|^2\right)$$



Fixed axis rotation

$\dot{\underline{r}}_j{}' = \dot{\underline{\rho}}_j{}'$

$$K = \frac{1}{2}\sum_j M_j\left(\dot{\underline{R}} + \dot{\underline{\rho}}_j\right)^2$$

$$= \frac{1}{2}\sum_j M_j \dot{\underline{\rho}}_j{}'^2 + \sum_j \dot{\underline{R}}.\dot{\underline{\rho}}_j{}' m_j + \frac{1}{2}\sum_j M_j v^2$$

$\sum_j \dot{\underline{R}}.\dot{\underline{\rho}}_j{}' m_j$ is 0 in COM.

$\dot{\underline{\rho}}_j{}' = \rho_j{}'\omega$

So $\boxed{K = \frac{1}{2}I_o\omega^2 + \frac{1}{2}MV^2}$

$\frac{1}{2}I_o\omega^2$ is the spin KE

$\frac{1}{2}MV^2$ is the orbital KE

Work-Energy Theorem

Recall $K_b - k_a = W_{ab}$

$$W_{ab} = \int_{\underline{r}_a}^{r_b} \underline{F}d\underline{r}$$

For the COM

$$\underline{F} = m\frac{d^2\underline{R}}{dt^2}$$

$$= M\frac{d\underline{V}}{dt}$$

$$\underline{F}.d\underline{R} = m\frac{d\underline{V}}{dt}.\underline{V}d$$

$$= d\left(\frac{1}{2}mv^2\right)$$

Integrate

$$\int_{\underline{R}_a}^{R_b} \underline{F}d\underline{R} = \frac{1}{2}mv_b{}^2 - \frac{1}{2}mv_a{}^2$$

What about rotation?

$$\tau_o = I_o\alpha = I_o\frac{d\omega}{dt}$$

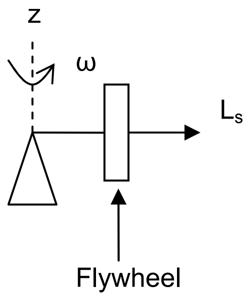
Multiply both sides by dθ=ωdt

$$\tau_o = I_o \frac{d\omega}{dt}\omega\omega dt$$
$$= d\left(\frac{1}{2}I_o\omega^2\right)$$

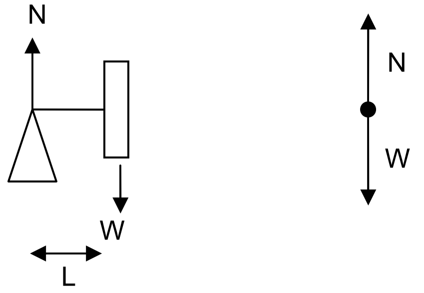
(Rotational kinetic energy)

$$\boxed{\int_{\theta_a}^{\theta_b}\tau_o d\theta = \frac{1}{2}I_o{\omega_b}^2 - \frac{1}{2}I_o{\omega_a}^2}$$

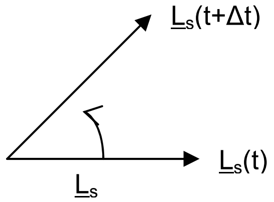
Uniform precession of the Gyroscope



Why doesn't it fall down?



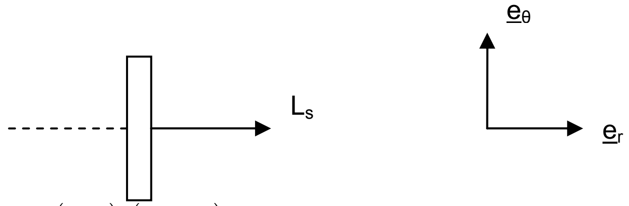
Why doesn't it rotate around the COM?
There is a net torque
$\tau = wL$



$\underline{L}_s = I_o\omega\underline{e}_r$

ω is the frequency of flywheel.
I_o is the moment of inertia of flywheel.

$$\frac{d\underline{L}_s}{dt} = \Omega L_s\underline{e}_\theta$$



$\underline{\tau} = (L\underline{e}_r)x(-w\underline{e}_z)$
$= wL\underline{e}_\theta$

For uniform procession:

$$\Omega L_s = wL$$

$$\Omega = \frac{wL}{I_o \omega_s}$$

Syllabus

1. **Defining Space, Time And Motion**
   Frames of reference, coordinate systems and vectors
   Velocity and acceleration
   Circular Motion
2. **Newton's Laws of Motion**
   Inertial frames and Newton I, Equilibrium conditions, forces and torque
   Newton II, Equations of Motion, Impulse
   Forces. Action at a distance.
   Momentum conservation and Newton III
   Applications of Newtonian Mechanics
3. **Conservation of Energy**
   Conservation principles in physics
   Kinetic energy and work
   Potential energy. Conservation forces.
   Not covered specifically, but throughout the course.
4. **Conservation of Linear and Angular Momenta**
   Conservation of linear momentum, internal forces for a collection of particles
   Centre of mass
   Angular momentum and Newton II
   Conservation of angular momentum.
5. **Special Relativity**
   Galilean transformations. The Postulates of special relativity
   Lorentz transformations. Lorentz contraction and time dilation
   Simultaneity. Transformation of velocities. The Doppler effect.
6. **Relativistic Momentum & Energy**
   Relativistic energy and momentum of massive particles.
   Energy equivalence of mass; pair production, nuclear energy.
   Massless particles. The Compton effect
7. **Classical Dynamics of Rigid Bodies**
   Equation of motion; kinetic energy, angular momentum and moments of inertia
   Finding moments of inertia. Gyroscopic motion.
   No questions on gyroscopic motion.
8. **Universal Gravitation**
   Kepler's Laws and planetary motion.
   Law of gravitation, field and potential.
   Binary systems and reduced mass.
   Inertial and gravitational mass.
   Section dropped!!!

Questions in exams will be more difficult than the exercises in the tutorials, but easier than problems.
Do as many questions as possible as practice!
Exam will consist of one general question covering various topics in the course, which must be answered, and three more specific questions, of which two must be answered.

<u>Equations:</u>

Definition of velocity and acceleration vectors:

$$\underline{V} = \frac{d\underline{R}}{dt}$$

$$a = \frac{d\underline{V}}{dt} = \frac{d^2\underline{R}}{dt^2}$$

Linear motion with uniform acceleration.

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$v = u + at$$

Velocity and Acceleration of Polar Coordinates

$$\underline{r} = e\underline{r}_r$$

$$\frac{d\underline{e}_r}{dt} = \dot{\theta}\underline{e}_r = \omega\underline{e}_r$$

$$\frac{d\underline{e}_\theta}{dt} = -\dot{\theta}\underline{e}_r = -\omega\underline{e}_r$$

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

$$\underline{a} = \left(\ddot{r} - r\omega^2\right)\underline{e}_r + \left(r\dot{\omega} + 2\dot{r}\omega\right)\underline{e}_\theta$$

Newton's laws:

$$\underline{P} = M\underline{V}$$

$$\underline{F} = \frac{d\underline{P}}{dt} = M\underline{a}$$

Work done, Potential and Kinetic Energy

$$W_{ba} = \int_a^b \underline{F}d\underline{r} = K_b - K_a$$

$$K = \frac{1}{2}MV^2$$

$$W_{ba} = -\left(I_b - U_a\right)$$

$$K_a + U_a = K_b + U_b$$

$$U_b - U_a = -\int_a^b \underline{F}d\underline{r}$$

$$F(r) = -\frac{dU(r)}{dr}$$

(Conservational forces only)

Special Potentials:

$$U_G = \frac{-Gm_1m_2}{r_{12}}$$

$$U_C = \frac{q_1q_2}{4\pi\varepsilon_o r_{12}}$$

Centre of Mass and 2-body collisions

$$\underline{R} = \frac{\sum_{j=1}^{N} m_1 \underline{r}_1}{\sum_{j=1}^{N} m_j}$$

$$\underline{V}_c = \frac{\sum_{j=1}^{N} m_j \underline{v}_j}{\sum_{j=1}^{N} m_j}$$

$$\underline{V}_r = \underline{v}_1 - \underline{v}_2$$

$$\underline{P}_{1c} = \mu \underline{V}_r$$

$$P_{2c} = -\mu \underline{V}_r$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Angular momentum and torque:

$$\underline{L} = \underline{r} x \underline{p}$$

$$\underline{\tau} = \underline{r} x \underline{f}$$

$$\frac{d\underline{L}}{dt} = \underline{\tau}$$

Pure rotation about a fixed axis:

$$I = \sum_{j=1}^{N} m_j \rho_j{}^2 \rightarrow \int \rho^2 dm$$

$$L = I\omega$$

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

$$K = \frac{1}{2} I\omega^2$$

$$WD = \int_{\theta_a}^{\theta_b} \tau d\theta = \frac{1}{2} I\omega_b{}^2 - \frac{1}{2} I\omega_a{}^2$$

Rotation and Translation

$$I_o = \sum_{j=1}^{N} m_j \rho'_j{}^2 \rightarrow \int \rho'^2\, dm$$

$$L_z = I_o\omega + \left(M\underline{R} x \underline{V}\right)_z$$

$$\tau_z = \tau_o \left(\underline{R} x \underline{F}\right)_z$$

$$\tau_o = I_o\alpha = I_o \frac{d\omega}{dt} = I_o \frac{d^2\theta}{dt^2}$$

$$K = \frac{1}{2} I_o\omega^2 + \frac{1}{2} MV^2$$

$$WD_{rot} == \int_{\theta_a}^{\theta_b} \tau_o d\theta = \frac{1}{2} I_o\omega_b{}^2 - \frac{1}{2} I_o\omega_a{}^2$$

Lorentz transformations

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$
$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma\left(t' + \frac{vx}{c^2}\right)$$

Intervals:

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$
$$\Delta x = \gamma(\Delta x' + v\Delta t')$$
$$\Delta y = \Delta y'$$
$$\Delta z = \Delta z'$$
$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x}{c^2}\right)$$

Lorentz contraction and time dilation as special cases

Relativistic transformation of velocity

$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$
$$u_y = \frac{u_y'/\gamma}{1 + \frac{vu_x'}{c^2}}$$
$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$
$$u_y' = \frac{u_y/\gamma}{1 - \frac{vu_x}{c^2}}$$

Relativistic mechanics

$$E = mc^2 = m_o \gamma c^2$$

$$K = (\gamma - 1) m_o c^2$$

$$\underline{P} = m\underline{V} = m_o \gamma \underline{V}$$

$$E^2 = c^2 p^2 + (m_o c^2)^2$$

$$\underline{F} = \frac{d\underline{P}}{dt}$$

All these formulae all need to be memorized!!!