Flat, uniformly rotating systems
Infinite fluid sheet with zero thickness, constant surface density Σ, and constant angular momentum Ω_z.

In a rotating coordinate frame,

$$\frac{\partial \Sigma}{\partial t} + \underline{\nabla} \cdot \left(\Sigma \underline{v} \right) = 0 \quad (9)$$

$$\frac{\partial \underline{v}}{\partial t} + \left(\underline{v} \cdot \underline{\nabla} \right) \underline{v} = -\frac{\nabla p}{\Sigma} - \underline{\nabla} \Phi - 2\underline{\Omega} \times \underline{v} + \underline{\Omega}^2 \left(x\underline{\hat{e}}_x + y\underline{\hat{e}}_y \right) \quad (10)$$

$$\nabla^2 \Phi = 4\pi G \Sigma \delta(z) \quad (11)$$

(Assume that centrifugal force is balanced by gravity, and $v_0 = 0$)

$$\frac{\partial \Sigma_0}{\partial t} + \frac{\partial \Sigma_1}{\partial t} + \underline{\nabla} \cdot \left(\left[\Sigma_0 + \Sigma_1 \right] \left[\underline{v}_0 + \underline{v}_1 \right] \right) = 0$$

$$\boxed{\frac{\partial \Sigma_0}{\partial t}} + \frac{\partial \Sigma_1}{\partial t} + \underline{\nabla} \cdot \left(\boxed{\Sigma_0 \underline{v}_0} + \Sigma_0 \underline{v}_1 + \underbrace{\cancel{\Sigma_1 \underline{v}_0}}_{\underline{v}_0 = 0} + \cancel{\Sigma_1 \underline{v}_1} \right) = 0$$

$$\rightarrow \frac{\partial \Sigma_1}{\partial t} + \underline{\nabla} \cdot \left(\Sigma_0 \underline{v}_1 \right) = 0$$

$$\frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \underline{\nabla} \cdot \underline{v}_1 + \cancel{\underline{v}_1 \cdot \underline{\nabla} \Sigma_0} = 0$$

$$\rightarrow \frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \underline{\nabla} \cdot \underline{v}_1 = 0 \quad (12)$$

$$\boxed{\frac{\partial \underline{v}_0}{\partial t}} + \frac{\partial \underline{v}_1}{\partial t} + \boxed{\left(\underline{v}_0 \cdot \underline{\nabla} \right) \underline{v}_0} + \cancel{\left(\underline{v}_0 \cdot \underline{\nabla} \right) \underline{v}_1} + \cancel{\left(\underline{v}_1 \cdot \underline{\nabla} \right) \underline{v}_0} + \cancel{\left(\underline{v}_1 \cdot \underline{\nabla} \right) \underline{v}_1}$$

$$= -\frac{\left(\boxed{\underline{\nabla} p_0} + \underline{\nabla} p_1 \right)}{\Sigma_0} \left(1 - \frac{\Sigma_1}{\Sigma_0} \right) - \boxed{\underline{\nabla} \Phi_0} - \underline{\nabla} \Phi_1 - \boxed{2\Omega \times \underline{v}_0}$$

$$-2\Omega \times \underline{v}_1 + \boxed{\Omega^2 \left(x\underline{\hat{e}}_x + y\underline{\hat{e}}_y \right)}$$

$$\frac{\partial \underline{v}_1}{\partial t} = -\frac{\underline{\nabla} p_1}{\Sigma_0} + \underbrace{\cancel{\frac{\underline{\nabla} p_0}{\Sigma_0^2} \Sigma_1}}_{\textit{const background pressure}} - \underline{\nabla} \Phi_1 - 2\underline{\Omega} \times \underline{v}_1$$

$$\rightarrow \frac{\partial \underline{v}_1}{\partial t} = -\frac{\underline{\nabla} p_1}{\Sigma_0} - \underline{\nabla} \Phi_1 - 2\underline{\Omega} \times \underline{v}_1 = \frac{{v_s}^2 \Sigma_1}{\Sigma_0} - \nabla \Phi_1 - 2\underline{\Omega} \times \underline{v}_1 \quad (13)$$

Equation of state:

$$p = p(\Sigma)$$

$$p = p_0 + \left(\frac{\partial p}{\partial \Sigma} \right)_{p_0} d\Sigma$$

$$p - p_0 = p_1 = \left(\frac{\partial p}{\partial \Sigma} \right)_{p_0} \left(\Sigma - \Sigma_0 \right) = \left(\frac{\partial p}{\partial \Sigma} \right)_{p_0} \Sigma_1 = {v_s}^2 \Sigma_1$$

Poisson equation:

$$\nabla^2\Phi = 4\pi G\Sigma\delta(z)$$

$$\nabla^2\Phi_1 = 4\pi G\Sigma_1\delta(z) \quad (14)$$

The solution we're after is:

$$\Phi_1 = \Phi_a e^{i(kx-\omega t)-|kz|}$$

$$\frac{\partial\Phi_1}{\partial x} = ik\Phi_1 \qquad \frac{\partial^2\Phi_1}{\partial x^2} = -k^2\Phi_1$$

$$\frac{\partial\Phi_1}{\partial z} = \pm k\Phi_1 \qquad \frac{\partial^2\Phi_1}{\partial z^2} = +k^2\Phi_1$$

For all $z \neq 0$,

$$\nabla^2\Phi = k^2\Phi_1 - k^2\Phi_1 = 0 = 4\pi G\Sigma_1\delta(z)$$

If $z = 0$,

$$\nabla^2\Phi = -k^2\Phi_1 = 4\pi G\Sigma_1$$

This means that we must still relate Σ_1 to Φ_1, or Σ_a to Φ_a (where $\Sigma_1 = \Sigma_a e^{i(kx-\omega t)}$ - note that this naturally has no z dependence). For $z = 0$,

$$-k^2\Phi_a e^{i(kx-\omega t)} = 4\pi G\Sigma_a e^{i(kx-\omega t)}$$

$$\rightarrow -k^2\Phi_a = 4\pi G\Sigma_a$$

(Don't use this – use the later one)

NB: Velocity $\underline{v}_1 = (x, y, t) = \left(v_{0z}\hat{\underline{e}}_z + v_{0y}\hat{\underline{e}}_y\right)e^{i(kx-\omega t)}$.

Doing an integral over z, such that we're looking at positions just above and below z ($\delta z = \xi$)

$$\lim_{\xi\to 0}\int_{-\xi}^{\xi}\frac{\partial^2\Phi}{\partial z^2}dz = \lim_{\xi\to 0}\left.\frac{\partial\Phi_1}{\partial z}\right|_{-\xi}^{\xi} = 4\pi G\Sigma_1\int_{-\xi}^{\xi}\delta(z)dz = 4\pi G\Sigma_1$$

But

$$\frac{\partial\Phi_1}{\partial z} = \Phi_a e^{i(kx-\omega t)-|kz|}\left(\pm|k|\right)$$

$$\left.\frac{\partial\Phi_1}{\partial z}\right|_{-\xi}^{\xi} = k\Phi_a e^{i(kx-\omega t)-k\xi} - (-k)\Phi_a e^{i(kx-\omega t)-|k(-\xi)|}\left(\pm|k|\right)$$

$$= \Phi_a e^{i(kx-\omega t)}\left[ke^{-k\xi} + ke^{-k\xi}\right] = 4\pi G\Sigma_a e^{i(kx-\omega t)-|kz|}$$

$$\rightarrow 2\Phi_a = 4\pi G\Sigma_a$$

(We now have a $|kz|$ in the Σ_1 exponential – this is a "calculus fiddle".)

Poisson's equation requires:

$$\Phi_1(x, y, t) = -\frac{2\pi G}{|k|}e^{i(kx-\omega t)-|kz|}$$

Substitute

$$-i\omega\Sigma_a = -ik\Sigma_0 v_{ax}$$

$$-i\omega v_{ax} = -\frac{{v_s}^2 ik\Sigma_a}{\Sigma_0} + \frac{2\pi Gi\Sigma_a k}{|k|} + 2\Omega v_{ay}$$

$$-i\omega v_0 = -2\Omega v_{ax}$$

with as solution the dispersion relation

$$\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0 |k| + k^2 v_s^{\,2} \quad (15)$$

The gravity term will provide instability, while the Ω^2 term provides support – so overall the disk is stabilized.

Three cases:

1. If $\Omega = 0$, sheet is unstable for $|k| < k_J = \dfrac{2\pi G\Sigma_0}{v_s^{\,2}}$
2. If in addition $v_s = 0$, sheet is violently unstable at all Λ.
3. Rotation and pressure stabilize the sheet if
   $$\frac{v_s\Omega}{G\Sigma_0} \geq \frac{\pi}{2} = 1.57$$
   Equivalent result for razor-thin stellar sheet: stability requires
   $$\frac{\sigma\Omega}{G\Sigma_0} \geq 1.68$$
   Finite thickness in z reduces the required σ (velocity distribution).

Conclusions

1. A cold sheet is violently unstable
2. A minimum sound speed or velocity dispersion can stabilize the sheet
3. Fluid and stellar sheets behave very similarly.

Spiral Arms

Properties:

- Typically 2 arms, starting at opposite sides of the nucleus
- Grand design: global pattern
- Later spirals have irregular arms, local patterns
- Barred spiral arms: arms attach to opposite ends of the bar
- M51 spiral arm attached to a small companion galaxy
- Arms are clearest in young (blue) stars and interstellar gas
- Galaxies without gas show no spiral arms

Parameters:

m: The shape is invariant under rotation under $\dfrac{2\pi}{m}$ radians.
Can talk about $m-$fold symmetry, mostly $m = 2$ but sometimes $m = 3$.
m spiral arms (usually)

i: pitch angle i at radius r is the angle between the tangent to the arm and the tangent to the circle $|r| = const$. By definition $0 < i < 180°$.
Observations show i in the range $5 - 20°$.

Arms appear to be *trailing* the rotation, not leading.

Winding dilemma

Differential rotation destroys the arms. Take a linear, radial structure in disk galaxy: $\phi(R) = \phi_0$. Let the structure rotate with the stars. At time t,

$$\phi(R,t) = \phi_0 + \Omega(R)t$$

The pitch angle is

$$\cot i = R\frac{\partial \phi}{\partial R} = Rt\left|\frac{d\Omega}{dR}\right|$$

For small i and radial inter-arm spacing $\Delta R << R$,

$$\Delta R = \frac{2\pi R}{\cot i}$$

For our galaxy, $\Omega R = v_c = 220 km\, s^{-1}$, $R = 10 kpc$, $t = 10^{10} yrs$:

$$i = 0.25°,\ \Delta R = 0.28 kpc$$

Predicts a much tightly wound spiral structure (by around a factor of 10).

Solutions to the Winding Dilemma:

1. Arms have a very short life span but continuously reform
2. Temporary phenomenon due to sudden violent disturbance
3. Detonation of star formation: *forest fire model*
4. Density wave following an enhancement to the gravitational potential

Kinematic density wave

Stellar orbit with radial period T_r. Recall

$$T_\phi = \frac{2\pi}{\Delta\phi} T_r$$

During T_r, azimuthal angle increases by $\Delta\phi$. The radial frequency is $\omega_r = \dfrac{2\pi}{T_r}$, and the Azimuthal frequency is $\omega_a = \dfrac{\Delta\phi}{T_r}$. Assume a rotating coordinate frame with angular sped Ω_p ("pattern speed").

$$\Delta\phi_p = \Delta\phi - \Omega_p T_r$$

If $\Delta\phi_p$ can be written as $\Delta\phi_p = 2\pi n / m$, with n, m integers,

$$\Omega_p = \frac{\Delta\phi}{T_r} - \frac{2\pi}{T}\frac{n}{m} = \omega_0 - \omega_r \frac{n}{m} \approx \Omega_c - \frac{n\kappa}{m}$$

where we used the epicycle approximation: Ω_c is the circular angular speed and κ is the epicycle frequency.

The orbit closes after m radial oscillations. The pattern formed by these orbits is stationary in the rotating frame, or rotating at Ω_p for a stationary observed. Ω_p is the *pattern speed*.