Instability of Collisionless systems
Stellar system is described by the Collisionless Boltzmann and Poison equation.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} - \nabla\Phi \cdot \frac{\partial f}{\partial \underline{v}} = 0 \quad (1)$$

$$\nabla^2 \Phi(\underline{x}, t) = 4\pi G \int f(\underline{x}, \underline{v}, t) d^3\underline{v} \quad (2)$$

In equilibrium, there is a time independent solution

$$\underline{v} \cdot \frac{\partial f_0}{\partial \underline{x}} - \nabla\Phi_0 \cdot \frac{\partial f_0}{\partial \underline{v}} = 0 \quad (3)$$

$$\nabla^2 \Phi_0(\underline{x}) = 4\pi G \int f_0(\underline{x}, \underline{v}) d^3\underline{v} \quad (4)$$

Suppose a stellar system is in equilibrium. Introduce a minor perturbation to this system:

$$f(\underline{x}, \underline{v}, t) = f_0(\underline{x}, \underline{v}) + \varepsilon f_1(\underline{x}, \underline{v}, t) \quad (5)$$

$$\Phi(\underline{x}, t) = \Phi_0(\underline{x}) + \varepsilon \Phi_1(\underline{x}, t) \quad (6)$$

It can be possible for f_1, Φ_1 to be large, as long as ε is small.

1. Extract a version of equation in background. (i.e. it is possible to subtract all parts of the background equation from the combined expression)
2. Remove all terms that are second order or higher in perturbations: ε is small, so ε^2 is negligible.
3. May be able to remove some differentials of background variables, e.g. $\partial/\partial t$ - background is time independent.

From equations (1-4), to first order:

$$\frac{\partial f_1}{\partial t} + \underline{v} \cdot \frac{\partial f_1}{\partial \underline{x}} - \nabla \cdot \Phi_0 \frac{\partial f_1}{\partial \underline{v}} - \nabla\Phi_1 \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$\nabla^2 \Phi_1(\underline{x}, t) = 4\pi G \int f_1(\underline{x}, \underline{v}, t) d^3\underline{v} \quad (7)$$

The equilibrium is *unstable* if the perturbation grows over time. This always happens in a rotating disk with zero random velocities.

Compare a perfect fluid (no random velocities) with the equations for perturbation. The equations for a prefect fluid are:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad \text{(Continuity)}$$

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla}\Phi \quad \text{(Momentum)}$$

$$\nabla^2 \Phi = 4\pi G \rho \quad \text{(Poisson equation)}$$

$$p = p(\rho(\underline{x}, t)) \quad \text{(Equation of State)}$$

Taking the continuity equation:

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla}\rho + \rho \underline{\nabla} \cdot \underline{v} = 0$$

$$\rho = \rho_0 + \rho_1, \; \underline{v} = \underline{v}_0 + \underline{v}_1$$

$$\frac{\partial\rho_0}{\partial t}+\frac{\partial\rho_1}{\partial t}+\left(\underline{v}_0+\underline{v}_1\right)\cdot\left(\underline{\nabla}\rho_0+\underline{\nabla}\rho_1\right)+\left(\rho_0+\rho_1\right)\left(\underline{\nabla}\cdot\underline{v}_0+\underline{\nabla}\cdot\underline{v}_1\right)=0$$

$$\boxed{\frac{\partial\rho_0}{\partial t}}+\frac{\partial\rho_1}{\partial t}+\boxed{\underline{v}_0\cdot\underline{\nabla}\,\rho_0}+\underline{v}_0\cdot\underline{\nabla}\,\rho_1+\cancel{\underline{v}_1\cdot\underline{\nabla}\,\rho_0}+\xcancel{\underline{v}_1\cdot\underline{\nabla}\rho_1}$$

$$+\boxed{\rho_0\underline{\nabla}\cdot\underline{v}_0}+\rho_0\underline{\nabla}\cdot\underline{v}_1+\cancel{\rho_1\underline{\nabla}\cdot\underline{v}_0}+\xcancel{\rho_1\underline{\nabla}\cdot\underline{v}_1}=0$$

Those with boxes around sum up to zero by the continuity equation in the background, while those with crosses through are removed as they are second order in the perturbation. Those with a single line through are removed by the Jeans ‘swindle’, which states that with constant ρ_0 → $\underline{\nabla}\rho_0=0$, and constant $\underline{v}_0$ → $\underline{\nabla}\cdot\underline{v}_0=0$. So we are left with

$$\frac{\partial\rho_1}{\partial t}+\underline{v}_0\cdot\underline{\nabla}\,\rho_1+\rho_0\underline{\nabla}\cdot\underline{v}_1=0$$

Taking the momentum transfer;

$$\frac{\partial\underline{v}}{\partial t}+\left(\underline{v}\cdot\underline{\nabla}\right)\underline{v}=-\frac{1}{\rho}\underline{\nabla}p-\underline{\nabla}\Phi$$

$$\underline{v}=\underline{v}_0+\underline{v}_1,\ p=p_0+p_1,\ \Phi=\Phi_0+\Phi_1$$

$$\frac{\partial\underline{v}_0}{\partial t}+\frac{\partial\underline{v}_1}{\partial t}+\left(\underline{v}_0\cdot\underline{\nabla}\right)\left(\underline{v}_0+\underline{v}_1\right)+\left(\underline{v}_1\cdot\underline{\nabla}\right)\left(\underline{v}_0+\underline{v}_1\right)=-\left(\rho_0+\rho_1\right)^{-1}\left(\underline{\nabla}p_0+\underline{\nabla}p_1\right)-\underline{\nabla}\Phi_0-\underline{\nabla}\Phi_1$$

$$\frac{\partial\underline{v}_0}{\partial t}+\frac{\partial\underline{v}_1}{\partial t}+\left(\underline{v}_0\cdot\underline{\nabla}\right)\underline{v}_0+\left(\underline{v}_0\cdot\underline{\nabla}\right)\underline{v}_1+\left(\underline{v}_1\cdot\underline{\nabla}\right)\underline{v}_0$$

$$+\left(\underline{v}_1\cdot\underline{\nabla}\right)\underline{v}_1=-\frac{\left(1+\frac{\rho_1}{\rho_0}\right)^{-1}}{\rho_0}\left(\underline{\nabla}p_0+\underline{\nabla}p_1\right)-\underline{\nabla}\Phi_0-\underline{\nabla}\Phi_1$$

Through a Taylor expansion, $\left(1+\frac{\rho_1}{\rho_0}\right)^{-1}\approx\left(1-\frac{\rho_1}{\rho_0}\right)$, hence

$$\boxed{\frac{\partial\underline{v}_0}{\partial t}}+\frac{\partial\underline{v}_1}{\partial t}+\boxed{\left(\underline{v}_0\cdot\underline{\nabla}\right)\underline{v}_0}+\left(\underline{v}_0\cdot\underline{\nabla}\right)\underline{v}_1+\cancel{\left(\underline{v}_1\cdot\underline{\nabla}\right)\underline{v}_0}$$

$$+\xcancel{\left(\underline{v}_1\cdot\underline{\nabla}\right)\underline{v}_1}=-\frac{1}{\rho_0}\left(\boxed{\underline{\nabla}p_0}+\underline{\nabla}p_1-\cancel{\frac{\underline{\nabla}p_0}{\rho_0}\rho_1}-\xcancel{\frac{\underline{\nabla}p_1}{\rho_0}\rho_1}\right)\boxed{-\underline{\nabla}\Phi_0}-\underline{\nabla}\Phi_1$$

which leaves

$$\frac{\partial\underline{v}_1}{\partial t}+\left(\underline{v}_0\cdot\underline{\nabla}\right)\underline{v}_1=-\frac{\underline{\nabla}p_1}{\rho_0}-\underline{\nabla}\Phi_1$$

Taking the Poisson equation,

$$\nabla^2\Phi=4\pi G\rho$$

$$\boxed{\nabla^2\Phi_0}+\nabla^2\Phi_1=\boxed{4\pi G\rho_0}+4\pi G\rho_1$$

$$\nabla^2\Phi_1=4\pi G\rho_1$$

The Equation of State can be expanded,

$$p = p(\rho)$$
$$p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_{p_0} d\rho$$
$$p - p_0 = p_1$$
$$p_1 = \left(\frac{\partial p}{\partial \rho}\right)_{p_0} \rho_1 = v_s^{\,2}\rho_1$$

where v_s is the sound speed.

Consider a homogeneous medium of density ρ_0 and pressure p_0 and $v_0 = 0$. Assume $\Phi_0 = 0$ (Jeans's swindle). With $h_1 = v_s^{\,2}\dfrac{\rho_1}{\rho_0}$,

$$\frac{\partial \rho_1}{\partial t} + \underline{\nabla}\cdot\left(p_0 \underline{v_1}\right) = 0 \text{ (A)}$$
$$\frac{\partial \underline{v}_1}{\partial t} = -\underline{\nabla} h_1 - \underline{\nabla}\Phi_1 = -\frac{\nabla \rho_1}{\rho_0} - \underline{\nabla}\Phi_1 \text{ (B)}$$
$$\nabla^2 \Phi_1 = 4\pi G \rho_1 \text{ (C)}$$
$$p_1 = v_s^{\,2}\rho_1 \text{ (D)}$$

We want to reduce these four equations down to 1. Do this by eliminating v_1, Φ_1, h_1.

$$\frac{\partial \underline{v}_1}{\partial t} = -\frac{v_s^{\,2}}{\rho_0}\underline{\nabla}\rho_1 - \underline{\nabla}\Phi_1$$
$$\frac{\partial}{\partial t}\left(\underline{\nabla}\cdot \underline{v}_1\right) = -\frac{v_s^{\,2}}{\rho_0}\nabla^2\rho_1 - \nabla^2\Phi_1$$
$$= -\frac{v_s^{\,2}}{\rho_0}\nabla^2\rho_1 - 4\pi G \rho_1$$
$$-\frac{1}{\rho_0}\frac{\partial^2 \rho_1}{\partial t^2} = -\frac{v_s^{\,2}}{\rho_0}\nabla^2\rho_1 + 4\pi G \rho_1$$

as $\underline{\nabla}\cdot \underline{v}_1 = -\dfrac{1}{\rho_0}\dfrac{\partial \rho_1}{\partial t}$. This leaves us with a wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^{\,2}\nabla^2\rho_1 - 4\pi G \rho_0 \rho_1 = 0 \text{ (8)}$$

Solutions are of the form

$$\rho_1(\underline{x}, t) = Ce^{i(\underline{k}\cdot\underline{x} - \omega t)}$$

with $k = |\underline{k}|$ the wave number, and

$$\omega^2 = v_s^{\,2}k^2 - 4\pi G \rho_0 .$$

For small ρ or large k (small wavelength $\lambda = 2\pi / k$) solution is a sound wave. For very large λ, $\omega^2 < 0$ and $\rho_1 \propto e^{\pm\gamma t}$ exponentially growing / decaying solutions. System is unstable if

$$k^2 < k_J^{\,2} = \frac{4\pi G \rho_0}{v_s^{\,2}}$$

Perturbations are unstable if the wavelength exceeds the Jeans length λ_J.

$$\lambda^2 > \lambda_J{}^2 = \frac{\pi v_s{}^2}{G\rho_0}$$

For a stellar system the result is very similar. Assume

$$f_0(\underline{v}) = \frac{p_0}{\left(2\pi\sigma^2\right)^{3/2}} \exp\left(\frac{1}{2}\frac{v^2}{\sigma^2}\right)$$

$$\left(\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{v^2}{\sigma^2}\right) dv = \sqrt{2\pi\sigma^2}\right)$$

Instability occurs when

$$k^2 < k_J{}^2 = \frac{4\pi G\rho_0}{\sigma^2}$$

This is the same result, but with the velocity dispersion in place of the sound velocity.

Stellar systems are stable against small scale perturbations.

Application to real stellar systems

Virial theorem: $2K + W = 0$ (internal kinetic energy is minus half the potential energy).

For a system with random velocities v^2,

$$W = Mv^2 = \frac{GM^2}{r_g}$$

where r_g is an equivalent gravitational radius. A system with a velocity dispersion v^2 has typical size $r_g = \frac{GM}{v^2}$. Approximately, $\lambda_J \approx r_g$.

Instabilities tend to occur on size scales comparable to the size of the system.

Radial systems

A system is stable if there are no accessible configurations with lower energy.

One can show:

A spherical system with $f_0 = f_0(E)$ *and* $df_0 / dE < 0$ *is stable to all non-radial and radial perturbations*.

Isochrone and Plummer models are both stable → Globular clusters are stable.

If $f_0 = f_0(E, L)$ there may be bar-like instabilities.