Jeans Equations

Spatial density ν of stars

$$\nu = \int f d^3 \underline{v}$$

Mean velocity v of stars

$$\overline{v}_i = \frac{1}{\nu} \int f v_i d\underline{v}$$

From before,

$$\int \frac{\partial f}{\partial t} d^3 v + \int \underline{v} \cdot \underline{\nabla} f \, d^3 v - \int \underline{\nabla} \Phi \cdot \underline{\nabla}_v f \, d^3 v = 0$$

$$\frac{\partial}{\partial t} \int f \, d^3 v + \int v_i \frac{\partial f}{\partial x_i} d^3 v - \int \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} d^3 v = 0$$

where $i = 1,2,3$ (implied summation). Providing that the potential doesn't depend on velocity, and noting that $d^3 v \equiv dv_i$,

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i} \int v_i f \, d^3 v - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3 v = 0$$

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu \overline{v}_i)}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} [f]_{-\infty}^{\infty} = 0$$

$[f]_{-\infty}^{\infty} = 0$ providing $f \to 0$ as $v_i \to \pm\infty$. We are left with the *continuity equation*

$$\frac{\partial \nu}{\partial t} + \frac{\partial \nu \overline{v}_i}{\partial x_i} = 0 \quad (1)$$

Starting again from

$$\frac{\partial f}{\partial t} + \underline{v} . \underline{\nabla} f - \underline{\nabla} \Phi \cdot \underline{\nabla}_v f = 0 ,$$

$$\int v_j \frac{\partial f}{\partial t} d^3 v + \int v_j v_i \frac{\partial f}{\partial x_i} d^3 v - \int v_j \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} d^3 v = 0$$

Focus on the last term of this equation.

$$\int v_j \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} d^3 v = \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3 v$$

Using integration by parts, and $[f]_{-\infty}^{\infty} = 0$, it can be shown that

$$\int v_j \frac{\partial f}{\partial v_i} d^3 v = \int \frac{\partial v_j}{\partial v_i} f d^3 v = \int \delta_{ij} f d^3 v = \delta_{ij} \int f d^3 v = \delta_{ij} \nu$$

The Kroenecker delta appears as the integral will give 1 if $i = j$, and 0 otherwise. The first two terms of the above equation are the same as the above except with an extra factor of v_j, so that we get the *momentum transfer equation*

$$\frac{\partial \nu \overline{v}_j}{\partial t} + \frac{\partial \nu \overline{v_i v_j}}{\partial x_i} + \delta_{ij} \nu \frac{\partial \Phi}{\partial x_j} = 0 \quad (2)$$

as $\overline{v_j} = \frac{1}{\nu} \int f v_j d^3 v$ and $\overline{v_i v_j} = \frac{1}{\nu} \int f v_i v_j d^3 v$.

The third Jeans equation can be obtained from the momentum transfer equation, subtracting v_j times the continuity equation.

$$\frac{\partial \nu \bar{v}_j}{\partial t} + \frac{\partial \nu \overline{v_i v_j}}{\partial x_i} + \delta_{ij} \nu \frac{\partial \Phi}{\partial x_j} - v_j \left(\frac{\partial \nu}{\partial t} + \frac{\partial \nu \bar{v}_i}{\partial x_i} \right) = 0$$

$$\nu \frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j \frac{\partial (\nu \bar{v}_i)}{\partial x_i} + \frac{\partial \left(\nu \overline{v_i v_j} \right)}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j}$$

Setting $\sigma_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$,

$$\nu \frac{\partial v_j}{\partial t} + \nu v \frac{\partial v_i}{\partial x_i} + \nu \frac{\partial \phi}{\partial x_i} = -\frac{\partial \nu \sigma_{ij}{}^2}{\partial x_i} \quad (3)$$

where

$$\sigma_{ij}{}^2 = \sqrt{(v_i - \bar{v}_i)(v_j - \bar{v}_j)}$$

Distribution function in spherical systems

If f describes all the mass in the system,

$$\nabla^2 \Phi = 4\pi G \rho = 4\pi G \int f d^3 \underline{v}$$

which gives

$$\frac{1}{r^2} \frac{d}{dr} \left(v^2 \frac{d\Phi}{dr} \right) = 4\pi G \int f(E, \underline{L}) d^3 \underline{v}$$

Define a relative potential

$$\Psi = -\Phi + \Phi_0$$

where Φ_0 is a background potential, and define relative energy

$$\varepsilon = -E + \Phi_0 = \Psi - \frac{1}{2} v^2$$

with $f = 0$ for $\varepsilon \le 0$ and $\Psi \to \Phi_0$ for $x \to \infty$.

$$\nabla^2 \Psi = -4\pi G \rho$$

Isotropic systems

Here, $f = f(\varepsilon)$. The velocity ellipsoid is isotropic,

$$\bar{v}_r{}^2 = \bar{v}_\theta{}^2 = \bar{v}_\phi{}^2$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi}{dr} \right) = -4\pi G \int f(\varepsilon) d^3 \underline{v}$$

Given f, this is a linear equation for solving for Ψ (and hence ρ).

Example, assume

$$f(\varepsilon) = \frac{\rho_1}{\left(2\pi\sigma^2\right)^{3/2}} e^{-\varepsilon/\sigma^2}$$

Integrate over all velocities,

$$\rho = \rho_0 e^{-\Psi/\sigma^2}$$

where we have used

$$\varepsilon = \psi - \frac{1}{2} v^2 .$$

Use Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)=-4\pi G\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right)=-\frac{4\pi G}{\sigma^2}v^2\rho$$

with solution

$$\rho(r)=\frac{\sigma^2}{2\pi G r^2}$$

This describes a singular isothermal sphere with

$$F(v)=N\exp\left(-\frac{1}{2}\frac{v^2}{\sigma^2}\right)$$

independent of r.

Remember that

- Kinematic LSR is based on average motions of local stars, $\langle\Theta\rangle$.
- Dynamical LSR is a perfect circular orbit at solar circle Θ_c.

We want to find the difference between these, $\langle\Theta\rangle-\Theta_c$ using the velocity dispersion tensor σ_{ij}.

Velocity Ellpsoid

Cylindrical polar coordinates:

$$\underline{v}=(R,\theta,z)$$

$$\underline{\dot{r}}=\underline{v}=(\Pi,\Theta,Z)=(\dot{R},\dot{\theta},\dot{z})$$

Collisionless Boltzmann equation,

$$\frac{\partial f}{\partial t}+\underline{v}\cdot\underline{\nabla}f+\underline{\dot{v}}\cdot\underline{\nabla}_v f=0$$

Expanding this,

$$\frac{\partial f}{\partial t}+\Pi\frac{\partial f}{\partial R}+\frac{\Theta}{R}\frac{\partial f}{\partial\theta}+Z\frac{\partial f}{\partial z}+\dot{\Pi}\frac{\partial f}{\partial\Pi}+\dot{\Theta}\frac{\partial f}{\partial\Theta}+\dot{Z}\frac{\partial f}{\partial Z}=0$$

For a test star, abandoning radial force assumption,

$$\dot{R}-R\dot{\theta}^2=-\frac{\partial\Phi}{\partial R}$$

$$R\ddot{\theta}+2\dot{R}\dot{\theta}=-\frac{1}{R}\frac{\partial\Phi}{\partial\theta}$$

$$\ddot{z}=-\frac{\partial\Phi}{\partial z}$$

In terms of the velocity components,

$$\dot{\Pi}=R\left(\frac{\Theta}{R}\right)^2-\frac{\partial\Phi}{\partial R}$$

$$\dot{\Theta}=-2\Pi\frac{\Theta}{R}-\frac{1}{R}\frac{\partial\Phi}{\partial\theta}$$

$$\dot{Z}=-\frac{\partial\Phi}{\partial z}$$

Assumptions:

- Steady state → $\frac{\partial f}{\partial t} = 0$
- Axial symmetry → $\frac{\partial \Phi}{\partial \theta} = \frac{\partial f}{\partial \theta} = 0$
- Centrifugal balance → $\frac{\partial \Phi}{\partial R} = \frac{\Theta_c^{\ 2}}{R}$
- Constant acceleration in the z direction → $K_Z = +\frac{\partial \Phi}{\partial z}$

Substituting these into the Boltzmann equation,

$$\Pi \frac{\partial f}{\partial R} + Z\frac{\partial f}{\partial z} + \frac{1}{R}\left(\Theta^2 - \Theta_c^{\ 2}\right)\frac{\partial f}{\partial \Pi} - \frac{\Pi\Theta}{R}\frac{\partial f}{\partial \Theta} - K_z\frac{\partial f}{\partial z} = 0$$

The continuity equation for a steady state is

$$\underline{\nabla} \cdot \left(\rho \underline{\bar{v}}\right) = 0 = \frac{\partial\left(\rho\bar{\Pi}\right)}{\partial R} + \frac{\partial\left(\rho\bar{Z}\right)}{\partial z} + \rho\frac{\bar{\Pi}}{R} = 0$$

It gives relations between averaged quantities.

Take moments of the Boltzman equation: note that (for example)

$$\langle \Pi Z \rangle = \frac{\int f \Pi Z d^3 \underline{v}}{\int f\, d^3 \underline{v}} = \frac{1}{\rho}\int f \Pi Z\, d\Pi\, d\Theta\, dZ$$

Multiply the Boltzmann equation by Π and integrate;

$$\frac{\partial\left(\rho\langle\Pi^2\rangle\right)}{\partial R} + \frac{\partial\left(\rho\langle\Pi Z\rangle\right)}{\partial z} + \frac{\rho}{R}\langle\Pi^2\rangle + \frac{\rho}{R}\left(\Theta_c^{\ 2} - \langle\bar{\Theta}^2\rangle\right) = 0$$

Multiply the Boltmann equation by Θ and integrate;

$$\frac{\partial\left(\rho\langle\Pi\Theta\rangle\right)}{\partial R} + \frac{\partial\left(\rho\langle Z\Theta\rangle\right)}{\partial t} + \frac{Z\rho}{R}\langle\Pi\Theta\rangle = 0$$

and the same for Z;

$$\frac{\partial\left(\rho\langle\Pi Z\rangle\right)}{\partial R} + \frac{\partial\left(\rho\langle Z^2\rangle\right)}{\partial z} + \frac{\rho}{R}\langle\Theta Z\rangle = \rho K_z$$

These equations represent conservation of average momentum in each coordinate.
Let

$$\Pi = \langle\Pi\rangle + \Pi'$$
$$\Theta = \langle\Theta\rangle + \Theta'$$

noting that this does not assume $\langle\Theta\rangle = \Theta_c$. Define the velocity dispersion tensor

$$\underline{\underline{\sigma}} = \begin{bmatrix} \langle\Pi'^2\rangle & \langle\Pi'\Theta'\rangle & \langle\Pi' Z'\rangle \\ \langle\Theta'\Pi'\rangle & \langle\Theta'^2\rangle & \langle\Theta' Z'\rangle \\ \langle Z'\Pi'\rangle & \langle Z'\Theta'\rangle & \langle Z'^2\rangle \end{bmatrix}$$

We can write $\sigma_{RR} = \sigma_u^{\ 2}$, $\sigma_{\theta\theta} = \sigma_v^{\ 2}$ and $\sigma_{zz} = \sigma_w^{\ 2}$ as shorthand.

Making the additional assumption is that there is no radial streaming $\left(\langle\Pi\rangle = 0\right)$, the radial equation becomes

$$\frac{\partial(\rho\sigma_{RR})}{\partial R}+\frac{\partial(\rho\sigma_{RZ})}{\partial z}+\frac{\rho}{R}\sigma_{RR}+\frac{\rho}{R}\left[\Theta_c-\left(\langle\Theta\rangle^2+\sigma_{\theta\theta}\right)\right]=0$$

Neglect σ_{RZ} : this is zero in the galactic plane ($z=0$) by symmetry.

Assume $\langle\Theta\rangle\sim\Theta_c$, so

$$\Theta_c{}^2-\langle\Theta\rangle^2=\left(\langle\Theta\rangle+\Theta_c\right)\left(\Theta_c-\langle\Theta\rangle\right)\approx 2\Theta_c\left(\Theta_c-\langle\Theta\rangle\right)$$

The drift of the KLSR relative to the DLSR is then

$$\langle\Theta\rangle-\Theta_c\approx\frac{\sigma_{RR}}{2\Theta_c}\left[\frac{\partial\left(\ln(\rho\sigma_{RR})\right)}{\partial(\ln R)}+\left(1-\frac{\sigma_{\theta\theta}}{\sigma_{RR}}\right)\right]\propto\sigma_{RR}$$

$\langle\Theta\rangle$ is a property of the large-scale gravitational potential. ρ, $\underline{\underline{\sigma}}$, Θ_c are functions of f in a particular star sample.

If the term in [ ] is independent of stars chosen, this gives $\langle\Theta\rangle-\Theta_c\propto\sigma_{RR}$. This is known as the Stromberg drift.

For a given group of stars, $\sigma_{RR},\sigma_{\theta\theta}$ hardly vary.

$$\langle\Theta\rangle-\Theta_c\propto\frac{\partial(L\rho)}{\partial(LR)}<0$$

So the KLSR will lag behind the DLSR. This is because ρ falls with R on average, so we observe more stars with guiding centres inside the solar circle than outside. Those from inside tend to be seen at the slowest point of their orbits, so we see more stars with $\Theta<\Theta_c$ than with $\Theta>\Theta_c$.