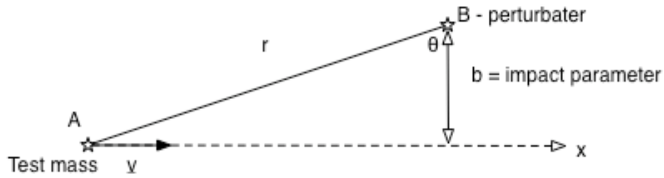
Time-dependent stellar systems



Assume a star with some velocity with respect to other 'field' stars. Assume a close encounter with another star, with impact parameter b. Consider force perpendicular to stellar motion.

$$F_p = \frac{Gm^2}{r^2}\cos\theta = \frac{Gm^2 b}{r^3} = m v_p$$

$$m\frac{dv_p}{dt} = \frac{Gm^2 b}{\left(b^2 + x^2\right)^{3/2}} = \frac{Gm^2}{b^2\left(1 + x^2/b^2\right)^{3/2}}$$

$$\frac{dv_p}{dt} \approx \frac{Gm}{b^2}\left|1 + \left(\frac{vt}{b}\right)^2\right|^{-3/2}$$

Integrate

$$\delta v_p \approx \frac{Gm}{bv}\int_{-\infty}^{\infty}\left(1 + s^2\right)^{-3/2} ds = \frac{2Gm}{bc}$$

Total number of encounters with impact parameter between b and $b + db$

$$\delta n = \frac{N}{\pi R^2} 2\pi b db = \frac{2N}{R^2} b db$$

where R is the radius of the galaxy, N is the total number of stars, and stars are uniformly distributed.

The change in velocity due to δn (encounters due to a single crossing of the galaxy)

$$\delta v_p{}^2 = \left(\frac{2GM}{bc}\right)^2 \frac{2N}{R^2} b db \,.$$

Integrate over all impact parameters b

$$\Delta v_p{}^2 = \int_{b_{\min}}^{R} dv_p{}^2 \approx 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda$$

where we're integrating from a minimum impact parameter $b_{\min}$ (the theory can't handle 0) to R the galactic scale. $\lambda = R / b_{\min}$. Assume $b_{\min} = \frac{Gm}{v^2}$.

Assume

$$v^2 \approx E_p \approx \frac{GNm}{R}$$

$$\frac{\delta v_p{}^2}{v^2} = \frac{8\ln\Lambda}{N}$$

The number of crossings required to randomize the velocity $\left(v^2 = v_p{}^2\right)$ is

$$n_{relax} = \frac{N}{8\ln\Lambda}$$

The relaxation time is

$$t_{relax} = n_{relax} \times t_{cross} = n_{relax}\frac{R}{v} = \frac{NR}{8v\ln\Lambda}$$

with

$$\Lambda = \frac{R}{b_{\min}} \approx \frac{Rv^2}{Gm} \approx N$$

For galaxies, $N = 10^{11}$, $n \approx 100$. Relaxation is unimportant.
For globular clusters, $N = 10^5$, $t_{cross} = 10^5 yr$. Relaxation is important over cluster lifetime $\left(10^{10} yr\right)$.
Cores of globular clusters, $N = 10^4$, $t_{cross} = 10^3 yr$. Relaxation dominant.

Overall, we expect non-relaxed velocity distributions (based on the 2-body estimate). However, stellar velocities are approximately relaxed. Why? Violent relaxation (probably). D. Lynden-Bell (1967). He said that the energy of each star,

$$E = \frac{1}{2}v^2 + \Phi,$$

is constant in a fixed potential. However, the potential is not constant – it is a function of both position and time, $\Phi = \Phi(\underline{r},t)$. This means that the energy of the star won't be constant either. An example would be the passage of a spiral arm through a 'stellar neighbourhood'.

$$\frac{dE}{dt} = \frac{1}{2}\frac{d\left(v^2\right)}{dt} + \frac{d\Phi}{dt}$$

where $\frac{d}{dt}$ is a derivative following the motion of the star. We can expand the last term as

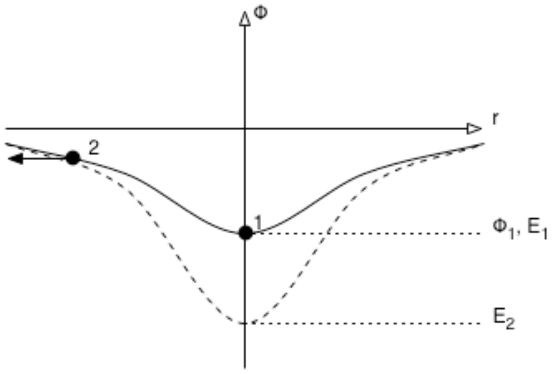
$$\frac{dE}{dt} = \frac{1}{2}\frac{d\left(\underline{v}\cdot\underline{v}\right)}{dt} + \left.\frac{\partial\Phi}{\partial t}\right|_{\underline{v}} + \underline{v}\cdot\underline{\nabla}\Phi$$

$$= \underline{v}\cdot\frac{d\underline{v}}{dt} + \left.\frac{\partial\Phi}{\partial t}\right|_{\underline{v}} - \underline{v}\cdot\dot{\underline{v}} = \left.\frac{\partial\Phi}{\partial t}\right|_{\underline{v}}$$

using $\dot{\underline{v}} = -\underline{\nabla}\Phi$, i.e. the acceleration is caused by the change in potential. Some stars will gain energy as Φ changes, while others will loose energy. It depends on the situation of the star. Overall, populations of stars with different original E become mixed. The energy per unit mass is shared. This leads to relaxation on time $\propto \dot{\Phi}^{-1}$. The exact timescale is

$$\tau_{VR} = \frac{1}{2}\left\langle\frac{\dot{\Phi}^2}{\Phi^2}\right\rangle^{-1/2} \sim \tau_{dyn} = \frac{1}{\sqrt{G\rho}},$$

i.e. the RMS average over many perturbations. It is approximately equal to the free-fall, or dynamical, time. It is around $10^8 yrs$ for the milky way.



A star in situation (1) will loose energy, as the potential it is in is deepening. The star at (2), which has an initial velocity away from the center of the potential, will not loose energy at the start, and once the potential has deepened it has an increased potential energy, which can be converted to kinetic energy when it stars falling into the potential well. When this star reaches the centre after the most compact configuration of the potential has been reached, then it will have – and can hold on to – sufficient energy to reach the other side of the potential. It can reach the original level in the potential with more KE than it had at the start.
2-body relaxation relaxes in energy, with energy pumped from high mass to low mass objects. Violent relaxation only relaxes with energy per unit mass.

Phase space density
We have a 7 dimensional phase space, $t, v_x, v_y, v_z, x, y, z$. The number of stars with position $\underline{x}$ and velocity $\underline{v}$ is $dn = f(\underline{x}, \underline{v}, t)\, d^3\underline{x}\, d^3\underline{v}$. The differentials include any necessary Jacobians for the coordinate systems, e.g. $d^3\underline{x} = J d\xi_1 d\xi_2 d\xi_3$ $f(\underline{x}, \underline{v}, t)$ is the *distribution function* or *phase space density*. The total number of stars is conserved,

$$N = \int\int f(\underline{x}, \underline{v}, t)\, d^3\underline{x}\, d^3\underline{v}\,.$$

The differential with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z} + \frac{dv_x}{dt}\frac{\partial f}{\partial v_x} + \frac{dv_y}{dt}\frac{\partial f}{\partial v_y} + \frac{dv_z}{dt}\frac{\partial f}{\partial v_z}$$

$$= \frac{\partial f}{\partial t} + v_x\frac{\partial f}{\partial x} + v_y\frac{\partial f}{\partial y} + v_z\frac{\partial f}{\partial z} + \dot{v}_x\frac{\partial f}{\partial v_x} + \dot{v}_y\frac{\partial f}{\partial v_y} + \dot{v}_z\frac{\partial f}{\partial v_z}$$

$$= \frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \underline{\dot{v}}\cdot\underline{\nabla}_v f$$

(Note: this differs slightly from the version given in the lectures, hopefully correctly…) The last part is a form of the Boltzmann equation. In this form, it is valid in all orthogonal coordinate systems. Note that $\underline{\nabla}_v$ denotes the grad operator with velocity rather than position. Again using $\underline{\dot{v}} = -\underline{\nabla}\Phi$,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f - \underline{\nabla}\Phi\cdot\underline{\nabla}_v f$$

In cylindrical coordinates, this is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \left(\frac{v_\phi^{\,2}}{R} - \frac{\partial\Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R}\left(v_R v_\phi + \frac{\partial\Phi}{\partial\phi}\right)\frac{\partial f}{\partial v_R} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z}$$

Collisionless Boltzmann equation
If stars do not make sudden jumps in f,

$$\frac{df}{dt} = \underline{v} \cdot \underline{\nabla} f - \underline{\nabla}\phi \frac{\partial f}{\partial \underline{v}} + \frac{\partial f}{\partial t} = 0$$

If collisions are important,

$$\frac{df}{dt} = \left(\frac{df}{dt}\right)_{collisions}$$

If the motion of a star is determined by the mean potential, the system is collisionless. If not, sudden velocity changes occur due to chance encounters with individual stars.

In the absence of collisions, the phase space density is constant. f is said to be incompressible.

Jeans Theorem
(A) *Any steady state solution of the Collisionless Boltzmann equation is fully described as a function of the integrals of motion, and* (B) *any function of the integrals of motion will yield a steady-state solution of the Collisionless Boltzmann equation.*

Strong Jeans Theorem
The distribution function of a steady-state galaxy with regular stellar orbits can be described as a function of only three independent isolating integrals.

Jeans Theorems
Integral of motion is time independent along an orbit.

$$\frac{d}{dt} I\left(\underline{x}(t), \underline{v}(t)\right) = 0$$

$$\frac{df}{dt} = \underline{\nabla} \underline{I} \cdot \frac{d\underline{x}}{dt} + \underline{\nabla}_{\underline{v}} \underline{I} \cdot \frac{d\underline{v}}{dt} = 0$$

or

$$\underline{v} \cdot \nabla \underline{I} - \nabla \Phi \cdot \nabla_{\underline{v}} \underline{I} = 0$$