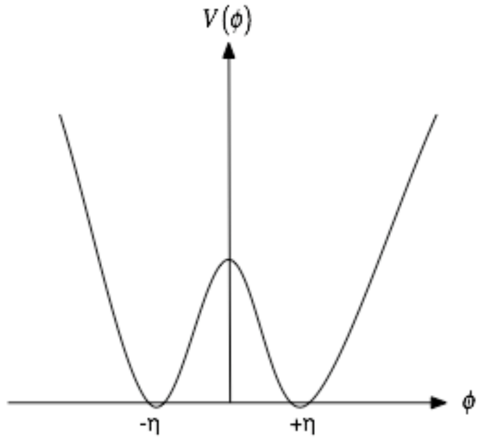
One is forced to make a choice of vacuum, and the resulting phenomena is known as spontaneous symmetry breaking (SSB).

1. Discrete Goldstone Model

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

where $\phi \in \mathbb{R}$ and $V(\phi) = V(-\phi)$, i.e. Z_2-symmetric. Choose $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2$.



→ there exists two minima $\phi = \eta, -\eta$ which are related by Z_2 symmetry.

What we do is choose one of the vacua, and expand around it. $\phi = \eta + \psi$.

$$\rightarrow V(\phi) = \frac{\lambda}{4}\left(2\eta\psi + \psi^2\right)^2 = \lambda\eta^2\psi^2 + \lambda\eta\psi^3 + \frac{\lambda}{4}\psi^4$$

Compare this to $L = \frac{1}{2}\partial_\mu \psi \partial^\mu \psi - \frac{1}{2}m_\psi{}^2\psi^2 + L_{\text{int}}$:-

$$\rightarrow \frac{1}{2}m_\psi{}^2 = \lambda\eta^2$$

$$L_{\text{int}} = -\lambda\eta\psi^3 - \frac{\lambda}{4}\psi^4$$

NB: if we had chosen $\phi = -\eta + \psi$ then $m_\psi{}^2$ would be the same, but the interaction Lagrangian would have changed slightly.

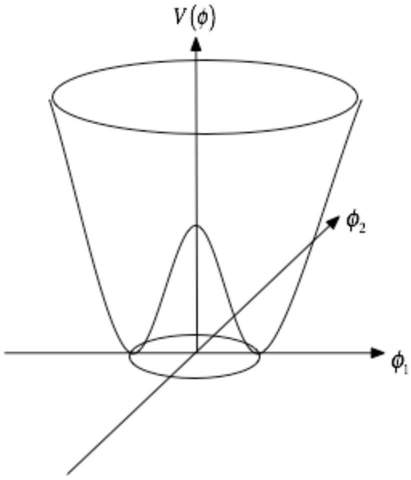
We say that the symmetry is spontaneously broken by the vacuum and the resulting particle has mass $\sqrt{2\lambda\eta^2}$.

2. U(1) Goldstone Model

$$L = \partial_\mu \Phi \partial^\mu \bar{\Phi} - \frac{\lambda}{4}\left(|\Phi|^2 - \eta^2\right)^2$$

$$\Phi \in \mathbb{C};\ \Phi = \phi_1 + i\phi_2$$

$\Phi \rightarrow e^{i\alpha}\Phi$ is a global symmetry of the Lagrangian, so it is $U(1)$ symmetric.



This is sometimes called the "Mexican Hat Potential".
There exists vacua along the circle (S^1) in the complex plane (ϕ_1, ϕ_2) given by $|\phi| = \eta$, all of which are related by the action of the symmetry group $\Phi \to e^{i\alpha}\Phi$.

As before, to find the theory about the vacuum, choose one point in the vacuum and expand around it.

$$\Phi = \eta + \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$|\Phi|^2 = \eta^2 + \sqrt{2}\eta\psi_1 + \frac{1}{2}(\psi_1{}^2 + \psi_2{}^2)$$

$$L = \frac{1}{2}\partial_\mu\psi_1\partial^\mu\psi_1 + \frac{1}{2}\partial_\mu\psi_2\partial^\mu\psi_2 - \frac{1}{2}\lambda\eta^2\psi_1{}^2 + O(3)$$

There are two particles ψ_1 and ψ_2, but one is massless since there is no quadratic term for ψ_2.

The choice above is not unique. More generally,

$$\Phi = \eta e^{i\alpha} + \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \text{ for any } \alpha.$$

$$\to L = \frac{1}{2}\partial_\mu\psi_1\partial^\mu\psi_1 + \frac{1}{2}\partial_\mu\psi_2\partial^\mu\psi_2 + \frac{1}{2}\mu_{ij}\psi_i\psi_j + L_{\text{int}}$$

where $\mu_{ij} = \mu_{ij}(\alpha)$ is the mass matrix. In order to find the particle masses, diagonalise μ_{ij} and the eigenvalues would be 0, $\lambda\eta^2$.
→ there are two particles, one with mass $\lambda\eta^2$ and the other which is massless.

<u>3. O(N)</u>
<u>General case:</u>

$$L = \frac{1}{2}\partial_\mu\underline{\phi}\partial^\mu\underline{\phi} - V(\underline{\phi})$$

Potential is invariant under G, i.e. $V(g\underline{\phi}) = V(\underline{\phi})$
Assume that min $V(\underline{\phi}) = 0$ and define the vacuum manifold

$$M_0 = \left\{ \phi : V\left(\underline{\phi}\right) = 0 \right\}$$

Now assume that G acts transitively on M_0, that is, the action of G generates the whole of M_0 from any given point.

→ given any $\underline{\phi}_1, \underline{\phi}_2 \in M_0$, $\exists \gamma \in G$ such that $\underline{\phi}_1 = \gamma \underline{\phi}_2$.

Define the stability group of a point $\underline{a} \in M_0$ to be

$$H_{\underline{a}} = \left\{ h \in G : h\underline{a} = \underline{a} \right\}$$

Although H varies with $\underline{a}$, it does so in a simple way due to the transitivity property.

$$\gamma \underline{a}_1 = \underline{a}_2 \rightarrow H_{\underline{a}_2} = \gamma H_{\underline{\alpha}_1} \gamma^{-1}$$

$$\rightarrow H_{\underline{a}_1} \cong H_{\underline{a}_2}$$

i.e. they are isomorphic.

Given $\underline{a} \in M_0$, $M_0 = \left\{ \gamma \underline{a} : \gamma \in G \right\}$.

$$\gamma_1 \underline{a} = \gamma_2 \underline{a} \rightarrow \gamma_2 \gamma_1^{-1} \in H$$

This means that γ_1 and γ_2 are in the same coset of H in G.

$$M = \frac{G}{H}$$

Expansion around $\phi = \underline{a} + \underline{\psi}$, where $V\left(\underline{a}\right) = 0; \frac{\partial V}{\partial \phi}\left(\underline{a}\right) = 0$.

$$L = \frac{1}{2}\partial_\mu \underline{\phi} \partial^\mu \underline{\phi} - \frac{1}{2}\mu_{ij}\psi_i \psi_j + L_{\text{int}}\left(\underline{\psi}\right)$$

where $\mu_{ij} = \dfrac{\partial^2 V}{\partial \phi_i \partial \phi_j}\left(\underline{a}\right)$

Masses of the particles are the eigenvalues of μ_{ij}.

e.g. $SO(N) \rightarrow SO(N-1)$; 1 massive, $N-1$ massless.

<u>N=3, G=SO(3)</u>

$$L = \frac{1}{2}\partial_\mu \underline{\phi} \partial^\mu \underline{\phi} - \frac{\lambda}{4}\left(\left|\phi\right|^2 - \eta^2\right)^2$$

$$\underline{\phi} = \left(\phi_1, \phi_2, \phi_3\right)$$

$SU(3)$ symmetry.

Minima occurs when $\left|\phi\right| = \eta$

$$M_0 = \left\{ \underline{\phi} : \left|\underline{\phi}\right| = \eta \right\} \cong S^2$$

Any point on the vacuum manifold can be obtained from any other by a rotation.

$$\underline{a} = \left(0, 0, \eta\right)$$

→ $H_{\underline{a}}$ is the group of rotations about the 2$^{\text{nd}}$ axis *(i.e. keep z fixed; can rotate around x, y freely)*. In general, it will be a $SO(2)$ subgroup of $SO(3)$.

$$G = SO(3),\ H = SO(2) \rightarrow \frac{G}{H} = \frac{SO(3)}{SO(2)} \cong S^2.$$

$$\underline{\phi} = \underline{a} + \underline{\psi}$$

$$L = \frac{1}{2}\partial_\mu \underline{\phi}\partial^\mu \underline{\phi} - \frac{\lambda}{4}\left(2\underline{a}\cdot\underline{\psi} + \underline{\psi}^2\right)^2$$
$$= \frac{1}{2}\partial_\mu \underline{\phi}\partial^\mu \underline{\phi} - \lambda\left(\underline{\alpha}\cdot\underline{\psi}\right)^2 + L_{\text{int}}$$
$$\psi = \psi_1 \underline{a} + \psi_2 \underline{b} + \psi_3 \underline{c}$$

where $(\underline{a}, \underline{b}, \underline{c})$ form an orthogonal triad $|\underline{a}| = |\underline{b}| = |\underline{c}| = \eta$

$$\rightarrow L = \frac{1}{2}\partial_\mu \underline{\phi}\partial^\mu \underline{\phi} - \lambda\eta^2\psi_i^2 + L_{\text{int}}$$

→ 1 massive particle, 2 massless.

3.3 Symmetry Restoration at High Temperature and the finite temperature effective potential

(Finite temperature = non-zero temperature)
The concept of Grand Unification suggests that the theory at very high temperatures is described by a simple Lie Group G and that there have been a number of Phase Transitions which lead to the symmetry being broken to the standard model.

$G \to H \to K \to \ldots \to SU(3)\times SU(2)\times U(1) \to SU(3)\times U(1)$

The next-to last of these is the Weinberg-Salam model + QCD. The last is QCD + EM.

3.3.1 Statistical Mechanics

Thermodynamic Potential:

$$\Omega = -T\log Z = E - TS - \mu N$$

where S is the entropy, μ is the chemical potential. $E - TS$. $\log Z$ is the grand partition function.

$$\log Z = \pm V_S \int d^3k \frac{1}{(2\pi)^3}\log\left[1 \pm \exp\left\{\frac{-(E(k)-\mu)}{T}\right\}\right]$$

where V_S is the volume, the positive case is for fermions and the negative case is for bosons.

Free Energy $(\mu = 0)$: $F = E - TS = \Omega(\mu = 0)$.

$$\rightarrow \frac{F}{V_S} = \mp\int \frac{d^3k}{(2\pi)^3}\log\left[1 \pm \exp\left\{-\frac{E(k)}{T}\right\}\right]$$

3.3.2 Effective Potential

The Lagrangians (e.g. Goldstone Model) studied so far are for a single field corresponding to possibly a number of particles related by a symmetry. But we like to couple these particles to a thermal heat bath of particles with temperature T.

Basic Concept: There exists an effective potential which encompasses the Thermal corrections.

It has been shown that the computation of these thermal corrections is the same as computing the free energy. Therefore:

$$V_{eff}\left(\underline{\phi}, T\right) = V\left(\underline{\phi}\right) + \sum_n f_n(\phi, T)$$

where $f_n(\phi,T)=\frac{F}{V}$ for the n$^{\text{th}}$ particle.

$$f_n(\phi,T)=\mp T\int\frac{d^3k}{(2\pi)^3}\log\left[1\pm\exp\left[-\frac{E(k)}{T}\right]\right]$$

and $E(k)=\sqrt{k^2+m_n^{\,2}(\phi)}$.

For Bosons,

$$f_n=-\frac{\pi^2}{90}T^4+\frac{m_n^{\,2}(\phi)}{24}T^2+O\left(m_n^{\,4}\right)$$

For Fermions,

$$f_n=-\frac{7\pi^2}{720}T^4+\frac{m_n^{\,2}(\phi)}{48}T^2+O\left(m_n^{\,4}\right)$$

Hence,

$$V_{eff}(\phi,T)=V(\phi)+\frac{1}{24}m^2(\phi)T^2-\frac{\pi^2}{90}NT^4$$

where $N=N_B+\frac{7}{8}N_f$, where N_B is the number of bosonic degrees of freedom and N_f is the number of dof for fermions.

$$m^2(\phi)=\sum_B m_n^{\,2}(\phi)+\frac{1}{2}\sum_f m_n^{\,2}(\phi)$$

3.3.3 Effective Potential of Goldstone Model with U(1) Symmetry

$$L=\partial_\mu\Phi\partial^\mu\bar{\Phi}-V(\underline{\Phi})$$

with $V(\Phi)=\frac{\lambda}{4}\left(|\Phi|^2-\eta^2\right)^2$.

$\Phi\in\mathbb{C}\rightarrow\Phi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$

$$L=\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1+\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2-\frac{\lambda}{4}\left(\frac{1}{2}\left(\phi_1^{\,2}+\phi_2^{\,2}\right)-\eta^2\right)^2$$

$$\rightarrow\ \frac{\partial^2V}{\partial\phi_i\partial\phi_j}=\frac{\lambda}{2}\left[\frac{1}{2}\left(\phi_1^{\,2}+\phi_2^{\,2}\right)-\eta^2\right]\delta_{ij}+\frac{\lambda}{2}\phi_i\phi_j$$

NB: $|\phi|^2=\frac{1}{2}\left(\phi_1^{\,2}+\phi_2^{\,2}\right)$

Eigenvalues give the masses:

$$m_1^{\,2}(\phi)=\frac{\lambda}{2}\left(3|\phi|^2-\eta^2\right)$$

$$m_2^{\,2}(\phi)=\frac{\lambda}{2}\left(|\phi|^2-\eta^2\right)$$

NB: If $|\phi|^2=\eta^2$, i.e. on the vacuum manifold, then $m_1^{\,2}=\lambda\eta^2$ and $m_2^{\,2}=0$.

$$\rightarrow\ V_{eff}(\phi,T)=\frac{\lambda}{4}\left(|\phi|^2-\eta^2\right)^2+\frac{\lambda}{24}\left(2|\phi|^2-\eta^2\right)T^2-\frac{\pi^2}{45}T^4$$

$$\frac{1}{2}m_{eff}^{\,2}=\text{coefficient of }|\phi|^2=\frac{\lambda}{12}\left(T^2-6\eta^2\right)=\frac{\lambda}{12}\left(T^2-T_c^{\,2}\right)$$

where $T_c = \sqrt{6}\eta$ is the critical temperature.
For $T > T_c$, $m_{eff}^2(T) > 0$ → single minima and full $U(1)$ symmetry.
For $T < T_c$, $m_{eff}^2(T) < 0$ → spontaneous symmetry breaking, and there exists degenerate vacua.
The symmetry is said to have been restored at high temperature and is broken at low temperature.

3.4 Phase Transitions
3.4.1 2nd Order Phase Transitions
Consider

$$V_{eff}(\phi,T) = \frac{1}{2}m_{eff}^2(T)(\phi)^2 + \frac{\lambda}{4}(\phi)^4$$
$$m_{eff}^2 = \frac{\lambda}{6}\left(T^2 - 6\eta^2\right)$$

Sketch the curve $f(x) = \frac{1}{2}m_{eff}^2(T)x^2 + \frac{\lambda}{4}x^4$, $x > 0$.

$f'(x) = m_{eff}^2 x + \lambda x^3$

$f'(x) = 0$ → $x = 0$ & $x = -\frac{m_{eff}^2}{\lambda} = \frac{1}{6}\left(T_c^2 - T^2\right)$.

$f''(x) = m_{eff}^2 + 3\lambda x^2$

$x = 0$, $f''(x) = m_{eff}^2$

$x = -\frac{m_{eff}^2}{\lambda}$, $f''(x) = -2m_{eff}^2$

If $m_{eff}^2 > 0$ $\left(T > T_c\right)$, then there exists a single minimum at $x = 0$.
If $m_{eff}^2 < 0$ $\left(T < T_c\right)$, then there exists a maxima at $x = 0$ and a minimum of $x = \frac{1}{\sqrt{6}}\left(T_c^2 - T^2\right)^{1/2}$.
The parameter grows continuously with T for $T < T_c$ towards $x = \eta$, which is characteristic of a second order phase transition. This is the only kind of phase transition we will discuss here.