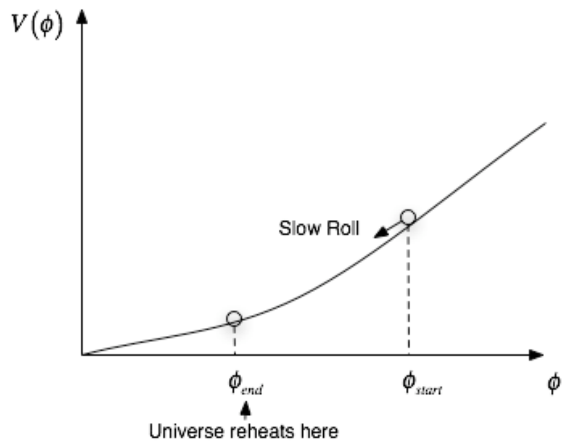
NB: de Sitter space is when the Hubble parameter H is constant.
("Hubble parameter = constant universe")

→ there are fluctuations in the scalar and tensor parts of the metric due to quantum mechanics. The scalar field fluctuates by $\delta\phi = \frac{H}{2\pi}$, and the metric / tensor part of gravity fluctuates by $\delta h = \frac{H}{2\pi}$.

Consider the inflationary potential.



No QM – the whole universe reheats at the same time.
With QM – different regions reheat at different times due to the uncertainty in ϕ.

This means that they experience slightly more or slightly less inflation. Therefore density fluctuations are created. The density contrast

$$\frac{\delta\rho}{\rho} \propto \frac{\delta t}{t_{\exp}},$$

where δt is the uncertainty in the time of reheating $\delta t = \frac{\delta\phi}{\dot{\phi}}$, and $t_{\exp}$ is the expansion timescale $t_{\exp} = H^{-1}$. From these,

$$\frac{\delta\rho}{\rho} \propto \frac{H}{\dot{\phi}}\delta\phi \propto \frac{H^2}{\dot{\phi}}$$

since $\delta\phi = \frac{H}{2\pi}$.

Similarly, gravitational waves are imprinted with amplitude proportional to H.

Define the scalar power spectrum (density waves)

$$P_s(k) = \left(\frac{\delta\rho}{\rho}\right)^2 = \frac{H^4}{\dot{\phi}^2},$$

and the tensor power spectrum (gravitational waves)

$$P_T(k) = (\delta h)^2 = \left(\frac{H}{m_{pl}}\right)^2.$$

[NB: In general, $P(k) \propto |\delta|^2$]

During slow roll,

$$3H\dot{\phi} = -\frac{dV}{d\phi}$$

$$H^2 = \frac{8\pi G}{3}V$$

$$\rightarrow \left[P_s(k)\right]^{1/2} \propto \frac{V^{3/2}}{m_{pl}{}^3 \frac{dV}{d\phi}}$$

$$\left[P_T(k)\right]^{1/2} \propto \frac{V^{1/2}}{m_{pl}{}^2}$$

Where these expressions are evaluated at ϕ such that $k = aH$, that is, when the comoving wave number k crosses the horizon.

$$\rightarrow \frac{dk}{k} = \frac{d\dot{a}}{\dot{a}} \approx Hdt = \frac{Hd\phi}{\dot{\phi}} = -\frac{8\pi GV}{\frac{dV}{d\phi}}d\phi$$

where the approximation is for de Sitter space.

$$\rightarrow d(\log k) = -8\pi G\frac{V}{V'}d\phi$$

$$\text{and } k = k_{60}\exp\left[-8\pi G\int_{t_{60}}^{\phi}\frac{V}{\frac{dV}{d\phi}}d\phi\right]$$

NB: k_{60} is the comoving wave number which left the horizon 60 e-foldings from the end of inflation and returned at the present day.

$$k_{60} = H_0$$

and ϕ_{60} can be computed from $60 = 8\pi G\int_{\phi_{end}}^{\phi_{60}}\frac{V}{\frac{dV}{d\phi}}d\phi$.

Now define the spectral indices n_S and n_T to be

$$n_S = 1 + \frac{d(\log P_S(k))}{d(\log k)}$$

$$n_T = \frac{d(\log P_T(k))}{d(\log k)}$$

i.e. $P_S(k) \propto k^{n_S - 1}$, $P_T(k) \propto k^{n_T}$. These are typically evaluated at ϕ_{60} .

Example: $V = \frac{1}{2}m^2\phi^2$; $\frac{dV}{d\phi} = m^2\phi$

$$\rightarrow \left[P_S(k)\right]^{1/2} \propto \frac{m}{m_{pl}}\left(\frac{\phi}{m_{pl}}\right)^2 ; \left[P_T(k)\right]^{1/2} \propto \frac{m}{m_{pl}}\left(\frac{\phi}{m_{pl}}\right)$$

$$d(\log k) = -4\pi G\phi d\phi$$

Working out ϕ_{60} ;

$$60 = 2\pi G\left(\phi_{60}{}^2 - \phi_{end}{}^2\right)$$

$$\rightarrow \phi_{60} \approx \left(10 - \frac{1}{6\pi}\right)m_{pl}$$

(approx $2\pi \approx 6$)

$$\phi_{60} \approx \sqrt{10}m_{pl}$$

Normalize to the observed fluctuations on the largest scales given by COBE, WMAP, etc.

$$\frac{\delta\rho}{\rho} \sim 10^{-5}$$

$$\rightarrow \left[P_S(k)\right]^{1/2} = \frac{10m}{m_{pl}} \sim 10^{-5}$$

$$\rightarrow m \approx 10^{13} GeV$$

NB: > 60 e-foldings required $m \le \left(\frac{\pi}{15}\right)^{1/2} m_{pl}$ → the potential is <u>very</u> flat, and the total number of e-foldings is around $N_{tot} \approx 10^{13}$ → $\frac{a_{end}}{a_{start}} = e^{10^{13}}$, which is very large.

So inflation happened for much longer than it needed to – the universe is ultra-flat.

Remember that this is just for this model, which’s fairly naïve.

$$d(\log P_s) = \frac{4d\phi}{\phi} ;\ d(\log P_T) = \frac{2d\phi}{\phi}$$

$$\rightarrow \left.\frac{d(\log P_S)}{d(\log k)}\right|_{k=k_{60}} \approx -0.03 ;\ \left.\frac{d(\log P_T)}{d(\log k)}\right|_{k=k_{60}} \approx -0.02$$

So $n_S \approx 0.97$ and $n_T \approx 0.02$. WMAP measures $n_S \approx 0.93 \pm 0.03$.

3. Phase Transitions in the Early Universe

3.1 Simple Field Theory Models

We have already discussed the case of a real scalar field $\phi = \phi(x,t) \in \mathbb{R}$ as inflatons. In Minkowski spacetime, the Lagrangian

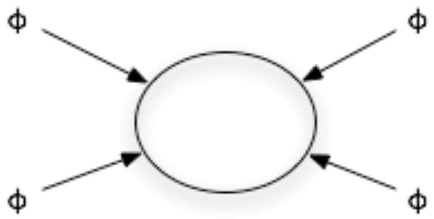
$$L = \partial_\mu \phi^\mu \phi - V(\phi)$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\nabla\phi|^2 - V(\phi)$$

EOM → $\Box\phi + \frac{dV}{d\phi} = 0$ where $\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$

If $V(\phi) = \frac{1}{2}m^2\phi^2$ this leads to the Klein-Gordon equations for massive (i.e. with mass) bosonic particles. Adding a 4-point interaction $\left(V_{\text{int}}(\phi) = \frac{\lambda}{4}\phi^4\right)$ leads to the simplest Quantum Field Theory (QFT).

e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + V_{\text{int}}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$.



The interaction term corresponds to scattering!
We will discuss mainly classical behaviour of these fields with a little bit of QM added in.

First note that $V(\phi) = V(-\phi)$, and therefore the Lagrangian is Z_2 symmetric, that is, it is symmetric under the action of the group $Z_2 = \{+1,-1\}$. This is a global symmetry since the element of Z_2 is independent of position.

We would like to discuss more complicated fields (complex scalars, vectors) and more complicated symmetry groups ($U(1)$, $SU(2)$, $SO(N)$).

(1) Complex Scalars

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \in \mathbb{C}$$

where $\overline{\Phi}$ is its complex conjugate.

$$L = \partial_\mu \Phi \partial^\mu \overline{\Phi} - V(\Phi)$$

$$\text{EOM: } \Box\Phi + \frac{dV}{d\overline{\Phi}} = 0$$

The kinetic term (the first part of the Lagrangian) is clearly symmetric under $\Phi \to \Phi' = e^{i\alpha}\Phi$ where α is a constant phase. If $V(\Phi) = V\left(e^{i\alpha}\Phi\right)\forall\alpha$, then the model is $U(1)$ symmetric.

(2) Complex Vectors

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \in \mathbb{C}^2$$
$$L = \partial_\mu \Phi^\dagger \partial^\mu \Phi - V(\Phi)$$

where $\Phi^\dagger = \overline{\Phi}^T$. If $U \in SU(2)$, then the kinetic term is symmetric under $\Phi \to \Phi^\dagger = U\Phi$.

$$\partial_\mu \Phi'^\dagger \partial^\mu \Phi' = \partial_\mu \left[(U\Phi)^\dagger\right] \partial^\mu (U\Phi) = \partial_\mu \Phi^\dagger U^\dagger U \partial^\mu \Phi = \partial_\mu \Phi^\dagger \partial^\mu \Phi$$

as $U^\dagger U = I$.

We require that $V(U\Phi) = V(\Phi)\forall U \in SU(2)$, e.g. $V(\Phi) = \frac{1}{2}m^2\Phi^\dagger\Phi + \frac{1}{4}\lambda\left(\Phi^\dagger\Phi\right)^2$.

(3) $SO(N)$

$$\underline{\Phi} = (\phi_1, \phi_2, \ldots, \phi_N)$$

is a vector field.

$$L = \frac{1}{2}\partial_\mu \underline{\Phi} \cdot \partial^\mu \underline{\Phi} - V(\underline{\Phi})$$

where $\partial_\mu \underline{\Phi} \cdot \partial^\mu \underline{\Phi} = \partial_\mu \phi_i \partial^\mu \phi_i = \dot{\underline{\Phi}}^2 - |\nabla \underline{\Phi}|^2$, i.e. summations over both the field index i and the Lorentz index μ.

This is $SO(N)$ symmetric if $V(g\Phi) = V(\Phi)\forall g \in SO(N)$.

NB: $N = 2$

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu \underline{\Phi} \cdot \partial^\mu \underline{\Phi} - V(\underline{\Phi}) \\ &= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2) \end{aligned}$$

Consider a scalar field $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$:

$$\begin{aligned} L &= \partial_\mu \overline{\Phi} \partial^\mu \Phi - V(\Phi) \\ &= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2) \end{aligned}$$

→ $SO(2)$ is equivalent to $U(1)$.

(4) All symmetries we have discussed so far are global symmetries, that is the elements of the symmetry groups are constant in space and time. We will consider a simple example of a gauge (or local) symmetry which corresponds to a complex scalar coupled to electromagnetism.

This is known as the Abelian Higgs Model.

$$L = D_\mu \Phi D^\mu \overline{\Phi} - V(|\Phi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $D_\mu = \partial_\mu - ieA_\mu \equiv$ covarient derivative, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor (of EM). $A_\mu \equiv$ Gauge field $\equiv (\phi, \underline{A})$.

This model is $U(1)$ symmetric under the following operations:

1. $\Phi \rightarrow \Phi' = e^{i\alpha}\Phi$
2. $A_\mu \rightarrow A_\mu' = A_\mu + \frac{1}{e}\partial_\mu \alpha$ where $\alpha = \alpha(\underline{x}, t)$.

$$\begin{aligned} D_\mu' \Phi' &= (\partial_\mu - ieA_\mu')\Phi' \\ &= \partial_\mu (e^{i\alpha}\Phi) - ie\left(A_\mu + \frac{1}{e}\partial_\mu \alpha\right) e^{i\alpha}\Phi \\ &= e^{i\alpha} D_\mu \Phi \end{aligned}$$

$$\begin{aligned} F_{\mu\nu}' &= \partial_\mu A_\nu' - \partial_\nu A_\mu' \\ &= \partial_\mu \left(A_\nu + \frac{1}{e}\partial_\nu \alpha\right) - \partial_\nu \left(A_\mu + \frac{1}{e}\partial_\mu \alpha\right) \\ &= F_{\mu\nu} \end{aligned}$$

→ Lagrangian is $U(1)$ symmetric.

3.2 Spontaneous Symmetry Breaking

QFT usually selects the lowest energy state to be the vacuum. For the Klein-Gordon potential $\left(V(\phi) = \frac{1}{2}m^2\phi^2\right)$, $\phi = 0$ is the unique lowest energy state. But what happens if there are two or more with the same value?