EOM:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} - \frac{\nabla^2\phi}{a^2} = 0$$

Pressure and density:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\nabla\phi|^2 + V$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}|\nabla\phi|^2 - V$$

Assume that the universe is somehow dominated by a spatially homogeneous scalar field, i.e.

$$\phi = \phi(t),$$

$$\underline{\nabla} = 0.$$

Hence we have

$$\rho_\phi = \frac{1}{2}\phi^2 + V,$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V.$$

Hence the equation of motion becomes

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi}.$$

We also have from the Friedman equation,

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi} + V\right).$$

If the field is slowly rolling, then the dynamics are dominated by potential energy, i.e. $\dot{\phi}^2 << V$, and the scalar field has an equation of state with $w \approx -1$. (This is a friction-dominated era.) The conditions for "slow-roll" are that $\dot{\phi}^2 << 1$ and $\ddot{\phi} << 1$, and hence

$$3H\dot{\phi} = -\frac{dV}{d\phi},$$

$$H^2 = \frac{8\pi G}{3}V.$$

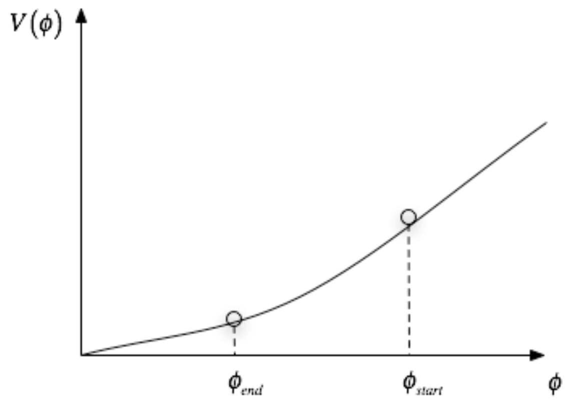
These are called the "slow-roll" equations.

Example potentials:

a. $V(\phi) = \frac{1}{2}m^2\phi^2$ (which would give you the Klein-Gordon equations [see Relativistic Quantum)

b. $V(\phi) = \frac{1}{4}\lambda\phi^4$

c. $V(\phi) = V_0 e^{-\lambda\phi}$



(This can be imagined as a ball rolling down a potential slope, rather than as a scalar field.)

Inflation ends when the field speeds up sufficiently for the slow roll conditions $\left(\ddot{\phi} << 1, \dot{\phi}^2 << 1\right)$ to be violated. More precisely, inflation ends when $\ddot{a}$ passes from >0 to <0, that is, when $\rho_\phi + 3P_\phi = 0$

$$\rightarrow \dot{\phi}^2 = V$$

One can treat t, ϕ, a as equivalent when describing the position on the potential. Another useful description is to use the number of e-foldings from the end of inflation:

$$N = -\int_{t_{end}}^{t} H dt = \int_{\phi_{end}}^{\phi} \frac{H}{\dot{\phi}} d\phi$$

In the slow-roll regime,

$$\frac{H}{\dot{\phi}} = -\frac{8\pi G V}{\frac{dV}{d\phi}}$$

$$\rightarrow N(\phi) = 8\pi G \int_{\phi_{end}}^{\phi} \frac{V}{\frac{dV}{d\phi}} d\phi$$

2.3 Chaotic Inflation

First consider an epoch of inflation starting at $\phi = \phi_{start}$ and ending at $\phi = \phi_{end}$. The slow roll conditions imply that

$$3H\dot{\phi} = -\frac{dV}{d\phi}$$

$$H^2 = \frac{8\pi G}{3} V$$

From these, we can get (by substituting the second equation into the first)

$$\dot{\phi} = -\left(\frac{1}{24\pi G}\right)^{1/2} \frac{V'}{V^{1/2}}$$

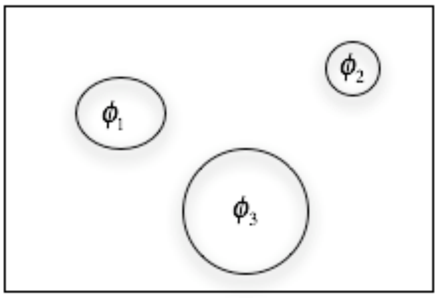
where $V' = \frac{dV}{d\phi}$. This gives $\phi = \phi(t)$. We can also get

$$\frac{1}{a}\frac{da}{d\phi} = -8\pi G \frac{V}{V'}$$

from substituting the first equation into the second. Hence,

$$a(\phi) = a_{start} e^{8\pi G \int_{\phi}^{\phi_{start}} \frac{V}{V'} d\phi}$$

The question is how did the field get to $\phi = \phi_{start}$ in the first place? The philosophy of chaotic inflation is that every value of ϕ is attributed to different regions of the universe.



If ϕ is sufficiently large then inflation takes place, which creates a universe which might lead to our observable universe. This leads to the idea that every possible universe happens somewhere → multiverse.

We will assume that $V(\phi_{start}) = M_{pl}{}^4$, and ϕ_{end} is defined by the condition that $\dot{\phi}^2 = V(\phi_{end})$. Assuming slow-roll,

$$\frac{1}{24\pi G} V'(t_{end}) = V^2(\phi_{end})$$

Hence we can calculate ϕ_{start}, ϕ_{end}, $\phi(t)$ and $a(\phi)$.

Example (a)

$$V(\phi) = \frac{\lambda}{4}\phi^4$$

From this, we can calculate

$$V' = \lambda\phi^3$$

$$\dot{\phi} = -\left(\frac{4\lambda}{24\pi G}\right)^{1/2}\phi = -\left(\frac{\lambda}{6\pi G}\right)^{1/2}\phi$$

$$\phi = \phi_{start} e^{-\left(\frac{\lambda}{6\pi}\right)^{1/2} M_{pl} t}$$

$$a(\phi) = a_{start} e^{\pi G\left(\phi_{start}{}^2 - \phi^2\right)}$$

$$V(\phi_{start}) = \frac{\lambda}{4}\phi_{start}{}^4 = M_{pl}{}^4$$

$$\rightarrow \phi_{start} = \left(\frac{4}{\lambda}\right)^{1/4} M_{pl}$$

$$\frac{1}{24\pi G}\lambda^2{\phi_{end}}^6=\frac{\lambda^2}{16}{\phi_{end}}^8$$

$$\rightarrow\phi_{end}=\left(\frac{2}{3\pi}\right)^{1/2}M_{pl}$$

We can calculate the number of e-foldings from the expression for $a(\phi)$ from above;

$$N_{tot}=\pi G\left({\phi_{start}}^2-{\phi_{end}}^2\right)=2\pi\lambda^{-1/2}-\frac{2}{3}$$

If we require that $N_{tot}>60$, then we need $\lambda >\approx \frac{1}{100}$ (where $>\approx$ denotes that it is approximately less than), which says that $\phi_{start}>M_{pl}$!

Example (b)

$$V(\phi)=\frac{1}{2}m^2\phi^2$$

From this, we can calculate

$$V'=m^2\phi$$

$$\ddot{\phi}=-\sqrt{\frac{1}{12\pi}}mM_{pl}$$

$$\rightarrow\phi=\phi_{start}-\sqrt{\frac{1}{12\pi}}mM_{pl}t$$

$$a(\phi)=a_{start}e^{2\pi G\left({\phi_{start}}^2-\phi^2\right)}$$

$$V(\phi_{start})=\frac{1}{2}m^2{\phi_{start}}^2={M_{pl}}^4$$

$$\rightarrow\phi_{start}=\sqrt{2}\left(\frac{M_{pl}}{m}\right)M_{pl}$$

$$\frac{1}{24\pi G}m^4{\phi_{end}}^2=\frac{1}{4}m^4{\phi_{end}}^4$$

$$\rightarrow\phi_{end}=\sqrt{\frac{1}{6\pi}}M_{pl}$$

Hence

$$N_{tot}=2\pi G\left({\phi_{start}}^2-{\phi_{end}}^2\right)=4\pi\left(\frac{M_{pl}}{m}\right)^2-\frac{1}{3}$$

As we need $N_{tot}>60$,

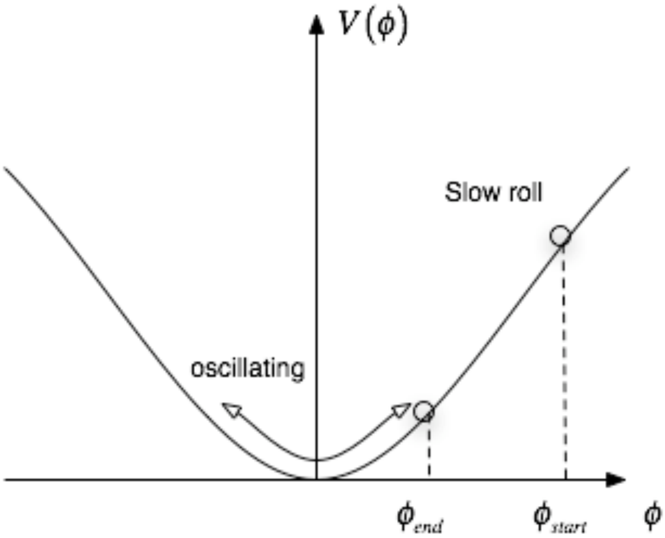
$$m<\approx\left(\frac{\pi}{15}\right)^{1/2}M_{pl}$$

Hence $\phi_{start}\geq M_{pl}$!

We see that requiring $N>60$ constrains either λ or m. We are basically making the potential sufficiently flat for $w=-1$ for long enough.

2.4 Reheating
At the end of inflation, we need to create a radiation-dominated universe. The process by which this happens is known as reheating.

At $\phi = \phi_{end}$, the field ϕ begins to oscillate.



From the point of view of this course, we will assume that reheating takes place instantaneously (This may or may not be a good approximation – it depends on many things). The energy stored in ϕ at $\phi = \phi_{end}$ is thermalised directly into relativistic species.

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V = \frac{3}{2}V$$

at $\phi = \phi_{end}$. Renormalization,

$$\rho_\phi = \rho_R = \frac{\pi^2}{30} N T_R^{\ 4}$$

where T_R is the temperature at the start of the radiation era, and N is the number of relativistic species. This is typically about 100 (within the standard model – around 106.75 degrees of freedom).

NB: we are assuming that the radiation era is 'born' in thermal equilibrium.

Hence,

$$T_R = \left(\frac{45}{\pi^2} \frac{V(\phi_{end})}{N} \right)^{1/4}$$

For example, in $\frac{\lambda}{4}\phi^4$ inflation, $V(\phi_{end}) = \frac{\lambda}{9\pi^2} M_{pl}^{\ 4}$

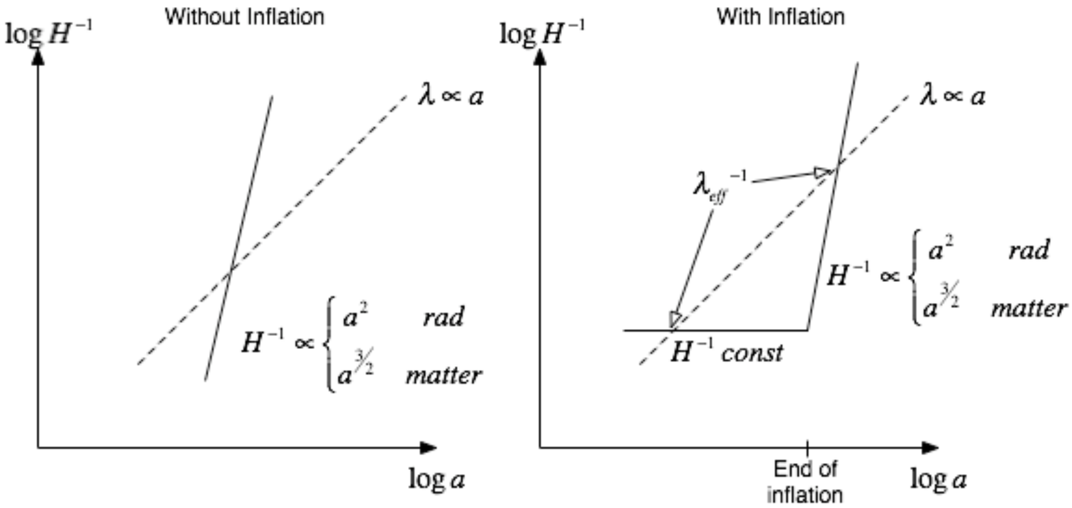
$$\rightarrow T_R \approx \left(\frac{5}{\pi^4} \frac{\lambda M_{pl}^{\ 4}}{N} \right)^{1/4}$$

2.5 Perturbations from Inflation

Consider the evolution of the Hubble radius.

$$H^{-1} \propto \rho^{-1/2} \propto \begin{cases} a^2 & \textit{radiation era} \\ a^{3/2} & \textit{matter era} \\ const & \textit{exponential inflation} \\ a^n & (n<1)\,\textit{during power law inflation} \end{cases}$$

and a physical wavelength $\lambda \propto a$.



Therefore in an inflationary universe, a particular wavelength crosses the Hubble radius twice, once during inflation $\lambda > H^{-1}$ and once during the subsequent radiation / matter era, when it becomes $\lambda < H^{-1}$.

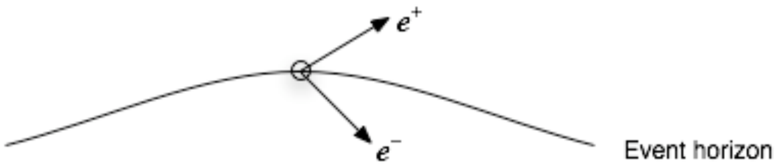
Fourier modes $\left(k = \dfrac{2\pi}{\lambda}\right)$ satisfy the 'first out last back' principle. Fluctuations just entering the horizon today went outside the horizon 60 e-foldings from the end of inflation.

NB: We will show (in section 5) that super-horizon perturbations, i.e. those with $\lambda > H^{-1}$, are 'frozen' i.e. they do not grow. So the amplitude of perturbations when they cross the horizon during the radiation and matter eras is equal to the amplitude when they went outside the horizon during inflation.

Aside

Quantum Mechanics at an event horizon:

We have e^+e^- pair creation at the event horizon.



This leads to Hawking Radiation. The Schwarzschild spacetime has a temperature that is inversely proportional to mass,

$$T \propto \frac{1}{M} .$$

We are borrowing energy from the gravitational field, as opposed to an electric field or the like.

Gibbons-Hawking Effect

Every spacetime with a horizon has a temperature. De-Sitter space, which is the spacetime for $w = -1$, also has a temperature

$$T = \frac{H}{2\pi}$$

where H is the Hubble parameter.