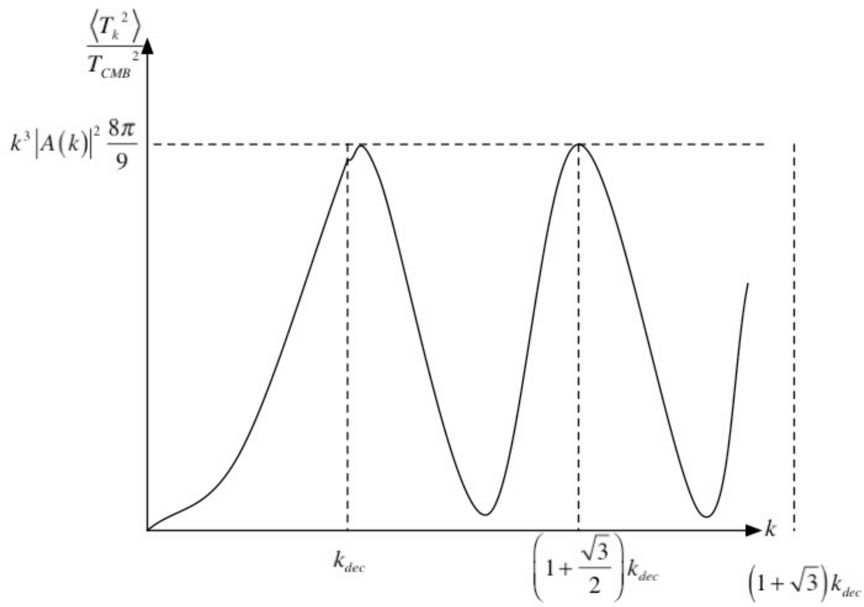
(NB: start of the lecture missing…)

$$\left\langle T_k^{\ 2}\right\rangle = \frac{k^3}{2\pi^2}\frac{\left(AT_k\right)^2}{T_{CMB}^{\ 2}} = \frac{8\pi^2}{9}k^3\left|A(k)\right|^2 \begin{cases} \left(\dfrac{k}{k_{dec}}\right)^4 & k < k_{dec} \\ \cos^2\left(\dfrac{2\pi}{\sqrt{3}}\left(\dfrac{k}{k_{dec}}-1\right)\right) & k > k_{dec} \end{cases}$$

NB: $k^3\left|A(k)\right|^2 =$ constant for universe, i.e. $\eta_\xi = 1$.



Therefore there exists a peak at $k = k_{dec}$ and at subsequent peaks.

6.4.3 Velocity induced Anisotropies

Velocity perturbations induce temperature anisotropies via the Doppler effect.

$$\frac{\Delta T}{T} = \frac{\Delta \nu}{\nu} = \Delta v$$

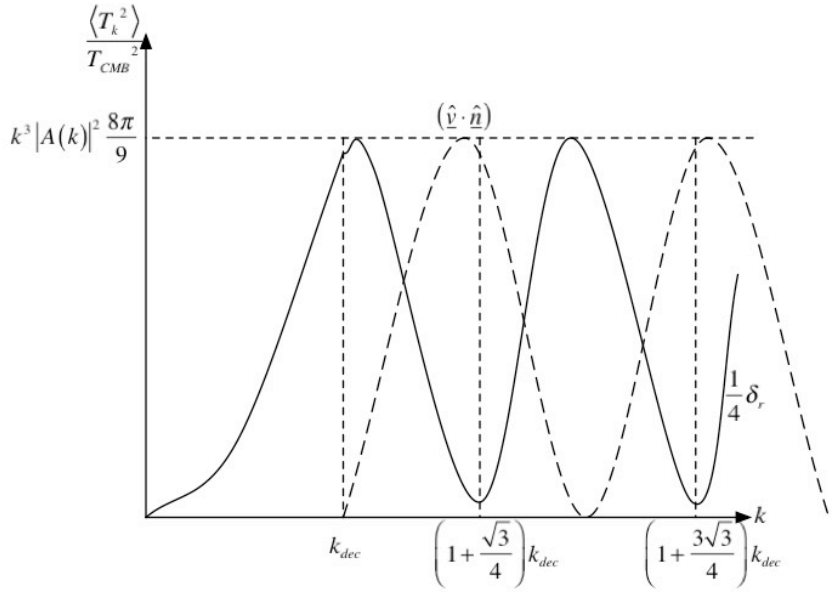
$$\rightarrow \frac{\Delta T_k}{T} = \underline{\hat{n}} \cdot \underline{v}_r\left(k, \eta_{dec}\right)$$

where $\underline{\hat{n}}$ is the photon propagation direction.

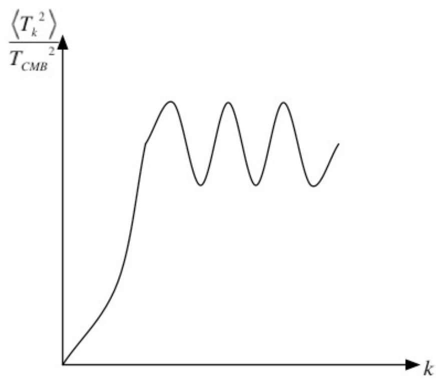
However, we know that $\underline{v}_r = \underline{\hat{k}} \cdot \dfrac{\theta_r}{k}$, therefore

$$\frac{\left\langle T_k^{\ 2}\right\rangle}{T_{CMB}^{\ 2}} = \begin{cases} 0 & k < k_{dec} \\ \dfrac{8\pi^2}{3}k^3\left|A(k)\right|^2\left(\underline{\hat{k}}\cdot\underline{\hat{n}}\right)^2\sin^2\left(\dfrac{2\pi}{\sqrt{3}}\left(\dfrac{k}{k_{dec}}-1\right)\right) & k > k_{dec} \end{cases}$$

$\left(\underline{\hat{k}}\cdot\underline{\hat{n}}\right) \approx \dfrac{1}{3}$ when averaged over the whole sky.

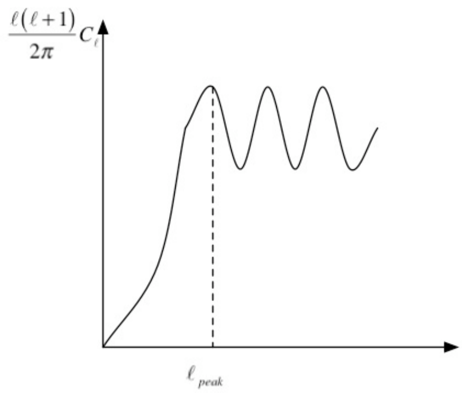


In reality, the velocity induced anisotropies are suppressed relative to the density induced anisotropies.



Project onto Q-space via

$$\frac{\ell(\ell+1)}{2\pi} C_\ell \approx \left\langle T_k^{\,2} \right\rangle\Big|_{k=\frac{1}{\eta_0}}$$



→ peak at $\ell = 2\pi \frac{\eta_0}{\eta_{dec}} \left(1 + \frac{\sqrt{3}}{2}(n-1)\right)$, with $n = 1, 2, 3, \ldots$.

For a universe with $\Omega_m = 1$, $h = 0.65$, then $\frac{\eta_0}{\eta_{dec}} \approx 45$, and hence

$\ell_{peak} = 280\left(1 + \frac{\sqrt{3}}{2}(n-1)\right)$, therefore $\ell_{peak} = 280$ at $n = 1$.

Actually is $\ell \approx 250$.

6.4.5 Angular diameter degeneracy

Consider a radial null geodesic, $ds = 0$, $d\theta = d\phi = 0$, and is approaching from $r = r_e$, $t = t_e$ with observer at $r = r_0$ and $t = t_0$.

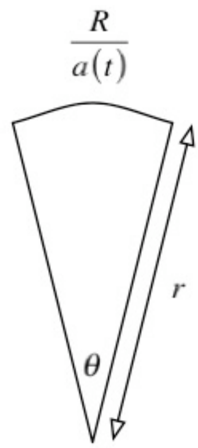
$$\eta_0 - \eta_e = \int_{t_0}^{t_e} \frac{dt}{a(t)} = -\int_{r_e}^{r_0=0} \frac{dr'}{\left(1 - kr'^2\right)^{1/2}} = \begin{cases} \sin^{-1}(r_e) & k = 1 \\ r_e & k = 0 \\ \sinh^{-1}(r_e) & k = -1 \end{cases}$$

The coordinate distance to the point of emission is

$$r = S_k\left(\int_{t_0}^{t_e} \frac{dt}{a(t)}\right),$$

where $S_k(x) = \begin{cases} \sin x & k = 1 \\ x & k = 0 \\ \sinh x & k = -1 \end{cases}$.

NB: $\int_{t_e}^{t_0} \frac{dt'}{a(t')} = \int_0^z \frac{dz'}{H(z')}$

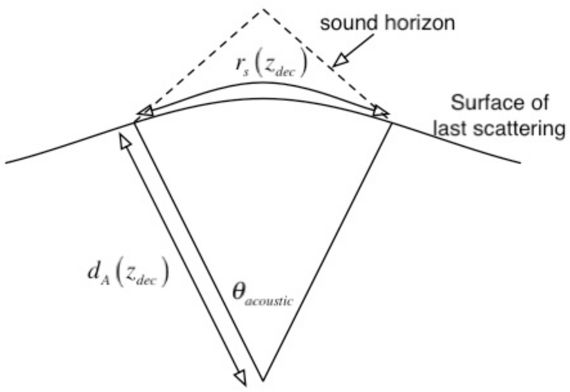


The angular diameter distance d_A is defined by the Euclidean relation $R = d_A\theta$, where R is the actual size of the source, and θ is its observed angular size.

$$\frac{R}{a(t)} = r(z)\theta$$

$$\rightarrow d_A(z) = \frac{r(z)}{1+z}$$

What we are seeing in the CMB anisotropies is a projection of a fixed scale. The fixed scale is the so-called sound horizon for acoustic waves from the Big Bang.



$r_s\left(z_{dec}\right)$ is the sound horizon at decoupling $= C_s a_{dec}\eta_{dec} \approx \frac{1}{\sqrt{3}} a_{dec}\eta_{dec}$.

$d_A\left(z_{dec}\right)$ is the angular diameter distance to decoupling. $= \frac{1}{1+z_{dec}}\int_0^{z_{dec}} \frac{dz'}{H(z')}$ in a flat universe $\approx a_{dec}\left(\eta_0 - \eta_{dec}\right)$.

$$\theta_{acoustic} = \frac{r_s\left(z_{dec}\right)}{d_A\left(z_{dec}\right)} = \frac{1}{\sqrt{3}}\frac{\eta_{dec}}{\eta_0 - \eta_{dec}} \approx \frac{1}{\sqrt{3}}\frac{\eta_{dec}}{\eta_0} \quad \left(\eta_0 >> \eta_{dec}\right)$$

$$\ell_{acoustic} = \frac{\pi}{\theta_{acoustic}} = \pi\sqrt{3}\frac{\eta_0}{\eta_{dec}}$$

Note: this is the same combination as we got before.

In a non-flat universe, we must take into account the curvature.

$$d_A(z) = \frac{1}{1+z_{dec}} S_k\left(\int_0^z \frac{dz'}{H(z')}\right) = a_{dec} S(k)\left(\eta_0 - \eta_{dec}\right)$$

$$\rightarrow \theta_{acoustic} = \frac{1}{\sqrt{3}}\frac{\eta_{dec}}{S_k\left(\eta_0 - \eta_{dec}\right)}$$

In an open universe with $\Lambda = 0$, we have $\frac{H^2}{H_0^{\,2}} = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2}$, where $\Omega_m + \Omega_r + \Omega_k = 1$, $\Omega_r \approx 0 \ \left(<< \Omega_m, \Omega_k\right)$.

$$\begin{aligned} H(z) &= H_0\sqrt{\Omega_m (1+z)^3 + \Omega_k (1+z)^2} \\ &= H_0 (1+z)\sqrt{\Omega_k + \Omega_m (1+z)} \\ &= H_0 (1+z)\sqrt{1+\Omega_m z} \end{aligned}$$

Therefore, $\theta_{acoustic} = \frac{1}{\sqrt{3}} \frac{\eta_{dec}}{\sinh\left(\frac{1}{H_0}\int_0^{z_{dec}} \frac{dz}{1+z\left(1+\Omega_m z\right)}\right)}$

6.4.6 Photon Diffusion Damping

The treatment so far is valid in the tight coupling regime, but as one approaches the epoch of last scattering the tight coupling approximation breaks down.

The mean free path of the photons is no longer zero, and the equation for δ_r gets modified to:

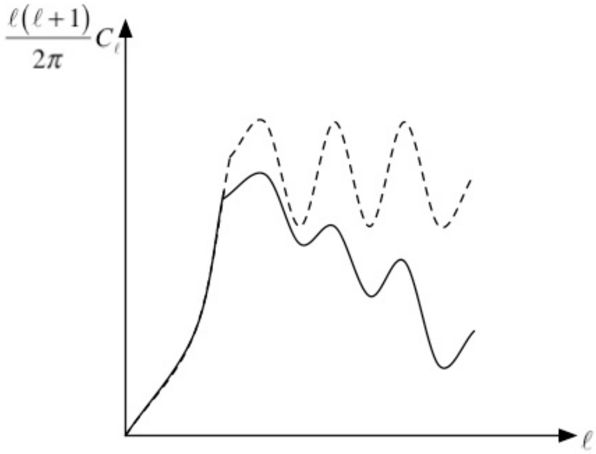
$$\delta_r'' + 2k^2\eta_*\delta_r' + \frac{1}{3}k^2\delta_r = 0$$

where η_* is the timescale associated with the increase in the mean free path.

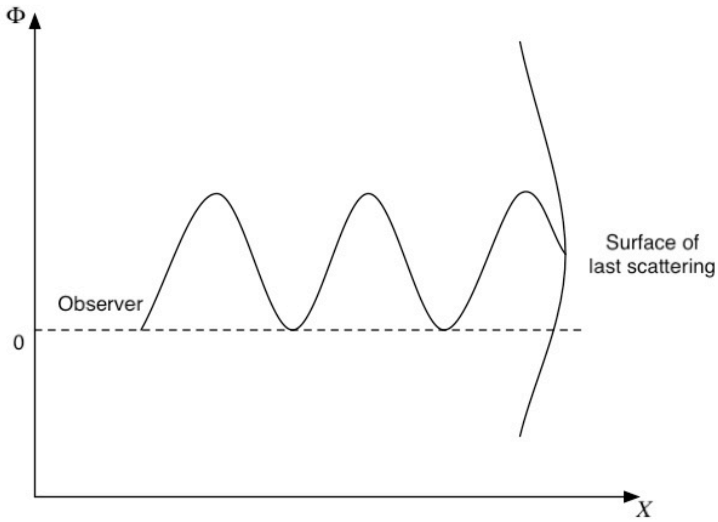
$$\rightarrow \delta_r = e^{-k^2\eta_*\eta}\left(\alpha\cos\left(\frac{k}{\sqrt{3}}\left(1-3k^2\eta_*^{\,2}\right)\eta\right) + \beta\sin\left(\frac{k}{\sqrt{3}}\left(1-3k^2\eta_*^{\,2}\right)\eta\right)\right)$$

Therefore $\delta_r\left(\eta_{dec}\right) = e^{-\frac{k^2}{k_0^{\,2}}}\delta_r\left(\eta_{dec}\right)\Big|_{tight\ coupling}$.

If we argue the modulation is the peak heights, where $k_0^{-2} = \eta_* \eta_{dec}$, we get exponential suppression by $e^{-\left(\frac{\ell}{1000}\right)^2}$.



6.5 Perturbations due to Gravitational Redshift
As the photons travel towards us from the surface of last scattering, they will experience gravitational redshift due to the varying gravitational potentials along the line of sight.



There are two calculations:

$$\left.\frac{\Delta T}{T}\right|_k = \frac{1}{3}\Phi\left(k, \eta_{dec}\right) + 2\int_{\eta_{dec}}^{\eta_0} \dot{\Phi} d\eta$$

where the first part is due to the potential at the last scattering surface, which the photons only see part of, and the second is due to the time variations of Φ.

In a static potential, the photons go up and down → temperature fluctuations cancel.

These effects are known as the Sachs-Wolfe effects (second is called the ISW effect – Integrated S-W effect).

In a hot, matter-dominated universe, $\dot{\Phi} = 0$, but $\dot{\Phi} \neq 0$ during Λ-domination.