$$\rightarrow \bar{T}^0_0 = \bar{\rho},$$
$$\bar{T}^i_{\ j} = -\bar{P}\delta^i_{\ j}$$

if $u^\mu = (1,\underline{0})$, and

$$\delta T^0_0 = \delta\rho$$
$$\delta T^i_{\ j} = -\delta P \delta^i_{\ j}$$
$$\delta T^i_{\ 0} = (\bar{\rho}+\bar{P})v^i_{\ j}$$
$$\delta T^0_{\ i} = (\bar{\rho}+\bar{P})v_i$$

where $\delta u^\mu = (0,\underline{v})$

We want to compute the perturbed conservation equation

$$\nabla_\mu T^\mu_{\ \nu} = 0$$

From GR course,

$$\nabla_\mu T^\mu_{\ \nu} = \partial_\mu T^\mu_{\ \nu} - \Gamma^\alpha_{\mu\nu}T^\mu_{\ \alpha} + \Gamma^\mu_{\mu\alpha}T^\alpha_{\ \nu}$$

Consider the cases of $\nu = 0$, $\nu = i$ separately, and we will substitute in $\Gamma^\mu_{\alpha\beta}$, $T^\mu_{\ \nu}$. The zeroth order equation will be

$$\dot{\bar{\rho}} + \frac{3a'}{a}(\bar{\rho}+\bar{P}) = 0\,.$$

For $\nu = 0$,

$$\delta' = -(1+w)\underline{\nabla}\cdot\underline{v} + \frac{1}{2}(1+w)h',$$

where $h = h^i_{\ i}$; $\bar{P} = w\bar{\rho}$.

For $\nu = i$,

$$\underline{v}' + (1-3w)\frac{a'}{a}\underline{v} + \frac{w}{1+w}\underline{\nabla}\delta = 0\,.$$

In order of terms: velocity perturbations; expansion of the universe; perturbed gradient.

Take the divergence of this equation;

$$\underline{\nabla}.\underline{v}' + (1-3w)\frac{a'}{a}(\underline{\nabla}\cdot\underline{v})' + \frac{w}{1+w}\underline{\nabla}^2\delta = 0$$

and the curl

$$(\underline{\nabla}\times\underline{v})' + (1-2w)\frac{a'}{a}(\underline{\nabla}\times\underline{v})' = 0$$

where the curl of grad is zero by definition, which decouples this equation. This part will die off very quickly with the scale factor; $\underline{\nabla}\times\underline{v}\rightarrow 0$ as $a\rightarrow\infty$. This process is splitting up the non-rotational and rotational parts of the velocity respectively; the rotational part will die off very quickly.

Take the fourier transform of these two equations.

$$\underline{\nabla}\cdot\underline{v}\rightarrow\theta = -i\underline{k}\cdot\underline{v}$$
$$\underline{\nabla}\delta\rightarrow i\underline{k}\cdot\delta$$
$$\rightarrow\ \delta' = (1+w)\left(\frac{1}{2}h' - \theta\right)$$
$$\theta' = -(1-3w)\frac{a'}{a}\theta + \frac{w}{1+w}k^2\delta$$

Examples:

1. CDM: $w = 0$; extra simplification is that one can choose $\theta = 0$, which means that the second of the two equations above also decouples, so that the CDM particles are the freely-falling observers of the FRW metric. Then, $\delta'_\mu = \frac{1}{2}h'$, and all we need to know is the evolution of h and we get δ_m.
2. Radiation: $w = \frac{1}{3}$.

   $$\delta_r' = \frac{4}{3}\left[\frac{1}{2}h' - \theta_r\right]$$

   $$\theta_r' = \frac{1}{4}k^2\delta$$

   $$\rightarrow \delta_r'' + \frac{1}{3}k^2\delta_r = \frac{2}{3}h''$$

   If $h'' = 0$, then the radiation exhibits simple harmonic oscillations (responsible for the oscillations in the CMBR power spectrum).

   For k very small, $k \approx 0$, then $\delta_r'' = \frac{2}{3}h'' = \frac{4}{3}\delta_m''$ from the above equation →

   $\delta_r - \frac{4}{3}\delta_m = A + B\eta$ (a constant + a linear term) on large scales.

   This equation will be important later when we discuss the CMB.

In order to get a closed set of equations, we need to know the evolution of h. We will now do this.

5.3.2 Perturbed Einstein Equations

We will only consider scalar fluctuations, i.e. density waves/fluctuations in the metric. We will reduce the number of free parameters in the 3x3 h_{ij} from 6 to 2, by taking the trace.

$$\rightarrow h_{ij} = \hat{k}_i\hat{k}_j h + 6h_s\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\rho_{ij}\right)$$

where $h = Tr\{h_{ij}\}$, i.e. the trace, and h_s is the anisotropic part of h_{ij}.

NB: we could also discuss vorticity (vector perturbations, i.e. the curl equation above) and gravitational waves (tensor perturbations).

We wish to compute the Einstein Equations at first order.

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = 8\pi G T^\mu{}_\nu$$

We have already computed $T^\mu{}_\nu$. We will now work out the perturbed versions of the Friedman and Raychauduri equations.

00:

$$k^2 h_s - \frac{a'}{2a}h' = 4\pi G a^2 \sum_{i=1}^{N}\bar{\rho}_i\delta_i$$

where the last part is a sum over all the components in the universe.

0i:

$$k^2 h_s' = 4\pi G a^2 \sum_{i=1}^{N} \left(\bar{\rho}_i + \bar{P}_i\right)\theta_i$$

ij :

$$h'' + \frac{2a'}{a}h' - 2k^2 h_s = 24\pi G a^2 \sum_{i=1}^{N} \bar{P}_i \delta_i$$

These encompass the Poisson equation in a slightly more complicated way than usual.

Substitute 00 into ij to get the dynamical equation for h;

$$\rightarrow h'' + \frac{a'}{a}h' = 8\pi G a^2 \sum_{i=1}^{N} \left(\bar{\rho}_i + 3\bar{P}_i\right)\delta_i$$

Now consider a universe containing matter $(w = 0)$ and radiation $\left(w = \frac{1}{3}\right)$.

$$h'' + \frac{a'}{a}h' = \frac{6\omega_r \delta_r}{a^2} + \frac{3\omega_m \delta_m}{a}$$

where $\omega_r = \Omega_r H_0^{\ 2}$ and $\omega_m = \Omega_m H_0^{\ 2}$. We can convert this equation to one for δ_m.
Recall also:

$$\delta_m' = \frac{1}{2}h'$$

$$\delta_r' = \frac{2}{3}h' - \frac{4}{3}\theta_r = \frac{4}{3}\left(\delta_m' - \theta_r\right) \ (1)$$

$$\theta_r = \frac{1}{4}k^2 \delta_r . \ (2)$$

$$\rightarrow \delta_m'' + \frac{a'}{a}\delta_m' = \frac{3\omega_r \delta_r}{a^2} + \frac{3}{2}\frac{\omega_m \delta_m}{a} \ (3)$$

To solve the problem, we need to solve these three coupled equations. Equation (3) is exactly the same equation as with the Newtonian perturbations, with an extra term

In the matter era, $a \approx \frac{1}{4}\omega_m \eta^2$ and $a' \approx \frac{1}{2}\omega_m \eta$.

$$\rightarrow \delta_m'' + \frac{2}{\eta}\delta_m' = \frac{6}{\eta^2}\delta_m + \left(\frac{48\omega_r}{\omega_m^{\ 2}}\right)\frac{\delta_r}{\eta^4} \approx \frac{6}{\eta^2}\delta_m$$

where the last part is the approximation when η is large.
Solution is:

$$\delta_m = A(k)\left(\frac{\eta}{n_i}\right)^2 + B(k)\left(\frac{\eta}{\eta_i}\right)^{-3}$$

The first part is a growing term, the second a decaying term.

From last time, the metric was $ds^2 = a^2\left(\eta_{\mu\nu} + h_{\mu\nu}\right)$, in which we are automatically in conformal time. We can switch back to normal time by the following.

$t = \int a d\eta \approx \frac{1}{12}\omega_m \dot{\eta}^3$, and we have $\eta \propto t^{1/3}$

$$\rightarrow \delta_m = A(k)\left(\frac{t}{t_i}\right)^{2/3} + B(k)\left(\frac{t}{t_i}\right)^{-1}$$

This is the same result as in section 5.2 case 2. Note that we now have a much more general solution, which can deal with radiation, plus other things.

5.4 Adiabatic Perturbations

Recall that

$$\delta_r'' + \frac{1}{3}k^2\delta_r = \frac{4}{3}\delta_m''.$$

For $k = 0$,

$$\delta_r'' = \frac{4}{3}\delta_m''$$

$$\rightarrow \delta_r - \frac{4}{3}\delta_m = A + \beta\eta.$$

Now, $\rho_r \propto a^{-4} \propto n_r^{4/3}$ (the origin of the 4/3 in the above equations) and $\rho_m \propto a^{-3} \propto n_m$ the number density.

$$\delta\rho_r \propto \frac{4}{3}\delta n_r n_r^{1/3} \rightarrow \delta_r = \frac{\delta\rho_r}{\delta r} = \frac{4}{3}\frac{\delta n_r}{n_r}$$

$$\delta\rho_m \propto \delta n_m \rightarrow \delta_m = \frac{\delta n_m}{n_m}$$

Hence,

$$\frac{\delta_r}{\delta_m} = \frac{4}{3}\frac{\delta n_r}{\delta n_m}\frac{n_m}{n_r}.$$

We will assume that the initial perturbations are adiabatic, that is,

$$\frac{n_r}{n_m} = const., \text{ and hence } \frac{\delta n_r}{\delta n_m} = const.$$

then

$$\delta_r(t_i) = \frac{4}{3}\delta_m(t_i) \text{ or in conformal time, } \delta_r(\eta_i) = \frac{4}{3}\delta_m(\eta_i).$$

The equations of motion are compatible with this condition being imposed on super-horizon scales.

In fact, inflationary perturbations are adiabatic, and are also called curvature perturbations since $h_s \neq 0$.

5.5 Power Spectrum and Transfer Functions

We have shown that (denoting "the perturbation in CDM" by just "CDM"):

1. CDM grows during the matter era
2. CDM stops growing at Λ-domination.
3. No growth takes place during the radiation era – the Mezaros effect.

Although it was not explicitly stated, the no-growth in the radiation era is only valid on sub-horizon scales. Here, we will discuss the growth of perturbations in the radiation era, before working out the transfer function $T(k)$ which relates

$\delta_m(t_0) = T(k)\delta_m(t_i)$. The important issue is whether the perturbations are super-horizon $(k\eta << 1)$ or sub-horizon $(k\eta >> 1)$.

The equation of motion that we are dealing with is

$$\delta_m'' + \frac{a'}{a}\delta_m' = \frac{3\omega_r\delta_r}{a^2} + \frac{3}{2}\frac{\omega_m\delta_m}{a}$$

In general, this will need to be supplemented by one for δ_r, however we will make two suppositions which will mean that we don't need to do this.
When $\eta < \eta_{eq}$

$$\rightarrow a \approx \sqrt{\omega_r}\eta$$

$$\rightarrow \delta_m'' + \frac{a'}{a}\delta_m' = \frac{3}{\eta^2}\delta_r + \frac{3}{2}\frac{\omega_m}{\sqrt{\omega_r}\eta}\delta_m$$

This last term will be small, and hence

$$\delta_m'' + \frac{1}{\eta}\delta_m' = \frac{3}{\eta^2}\delta_r$$

The perturbations in m are sourced by those in r.

<u>Super-horizon</u>

For adiabatic fluctuations, $\delta_r = \frac{4}{3}\delta_m$ on super-horizon scales.

$$\rightarrow \delta_m'' + \frac{1}{\eta}\delta_m' = \frac{4}{\eta^2}\delta_m$$

If we assume that $\delta_m \propto \eta^p$, then

$$p(p-1) + p - 4 = 0 \rightarrow p = \pm 2$$

So

$$\delta_m = A(k)\left(\frac{\eta}{\eta_i}\right)^2 + B(k)\left(\frac{\eta}{\eta_i}\right)^{-2}.$$

Again, the first part is the growing mode, the second the decaying mode. We always ignore the decaying mode.

→ super-horizon fluctuations grow in the radiation era, just as they do in the matter era.

<u>Sub-horizon</u>

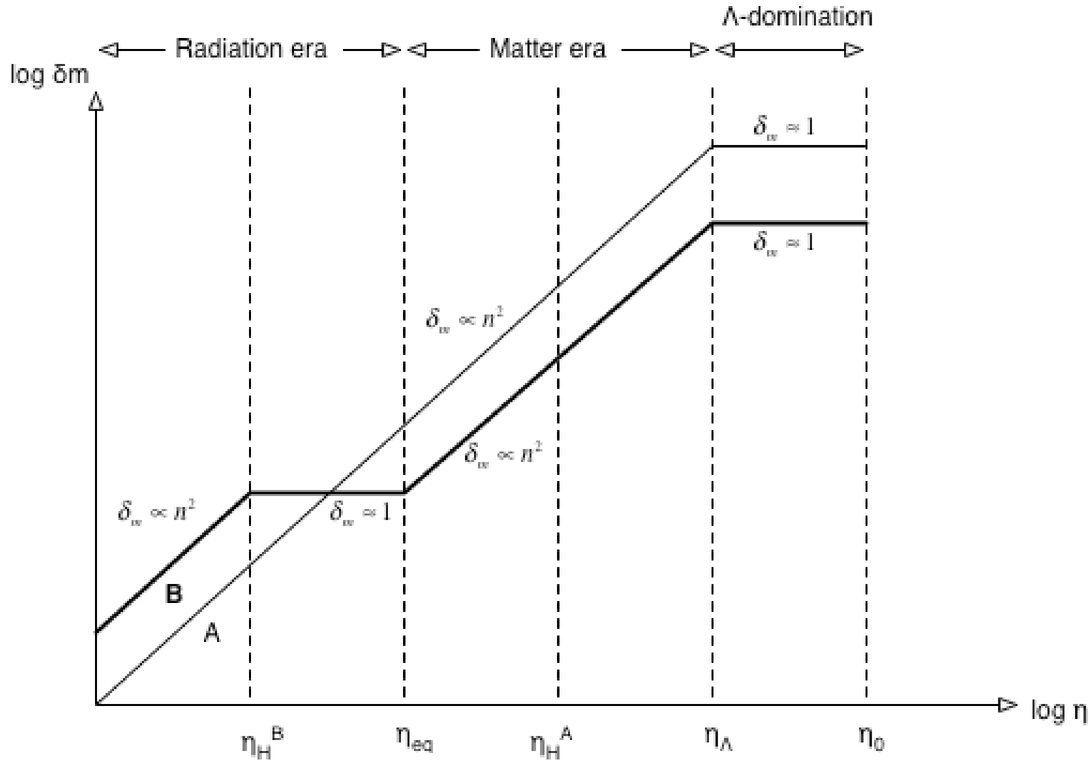
We will discuss the radiation fluctuations in the section on the CMB. Suffice to say that on sub-horizon scales, $\delta_r'' + \frac{1}{3}k^2\delta_r \approx 0$ → δ_r oscillates, and hence the average of $\langle\delta_r\rangle \approx 0$ (not exactly true, with interesting effects, but we will ignore that here).

$$\rightarrow \delta_m'' + \frac{1}{\eta}\delta_m' = 0$$

$$\rightarrow \delta_m = A(k)\log\eta + B(k)$$

The linear part $B(k)$ associates itself with the growing solution, while the logarithmic part $A(k)$ associates itself with the decaying solution.

So, therefore, let us consider the evolution of two modes (A) and (B).
(A): $k < k_{eq}$
(B): $k > k_{eq}$
where $k_{eq} = \frac{2\pi}{\lambda_{eq}}$, $\lambda_{eq} = 16\left(\delta_m h^2\right)^{-1} Mpc$, see QS1.
For either of the modes consider $\eta_H =$horizon crossing time, and define η_Λ to be the point of Λ domination.



Therefore modes which come inside the horizon during the radiation era are suppressed relative to those which came inside during the matter era.

Transfer Functions

We need to work out how much each of these modes grows during each epoch.

$k > k_{eq}$:

$$T(k) = 1 \cdot \left(\frac{\eta_\Lambda}{\eta_{eq}}\right)^2 \cdot 1 \cdot \left(\frac{\eta_H}{\eta_i}\right)$$

where the first part is from the Λ-era, the second from the subhorizon matter era, the third from the subhorizon radiation era, and the last from the super-horizon radiation era.

$$\left(\frac{\eta_H}{\eta_{eq}}\right)^2 = \left(\frac{k_{eq}}{k}\right)^2 = \left(\frac{k}{k_{eq}}\right)^{-2}.$$

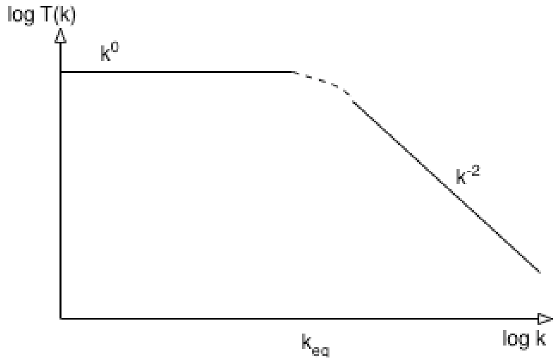
$$\rightarrow T(k) = \left(\frac{\eta_\Lambda}{\eta_i}\right)^2 \left(\frac{k}{k_{eq}}\right)^{-2}$$

$\underline{k < k_{eq}}$

$$T(k) = 1 \cdot \left(\frac{\eta_\Lambda}{\eta_i}\right)^2$$

where the first part is the Λ-era, and the second part the superhorizon and subhorizon growth.

$$\rightarrow T(k) = \left(\frac{\eta_\Lambda}{\eta_i}\right)^2 \begin{cases} 1 & k < k_{eq} \\ \left(\frac{k}{k_{eq}}\right)^{-2} & k > k_{eq} \end{cases}$$

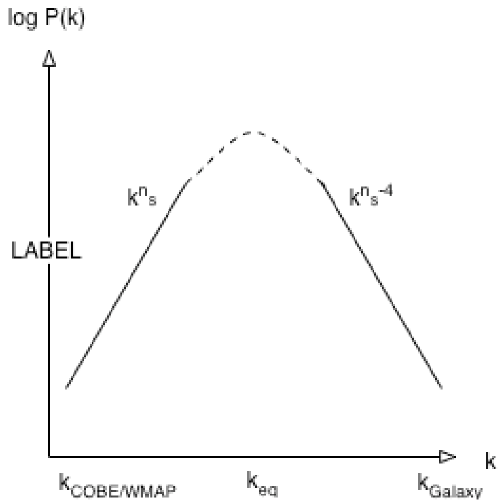


The dashed part represents that which hasn't been calculated here; it is a logical extrapolation.

If $P_i(k) \propto k^{n_s}$ (where $n_s = 1$ corresponds to scale invariant), then today,

$$P(k) \propto k^{n_s} \text{ for } k < k_{eq}$$

$$P(k) \propto k^{n_s - 4} \text{ for } k > k_{eq}$$



Measurements of $P(k)$ are made at $k_{COBE/WMAP}$ using the CMB and at k_g using redshift surveys of galaxies, providing a diagnostic of this theory.