4.4.4 Cosmic Strings

For non-interacting strings,

$$P_{str} = \frac{1}{3}\left(2\left\langle v^2 \right\rangle - 1\right)\rho_{str}$$

where $\left\langle v^2 \right\rangle$ is the RMS velocity of the strings.

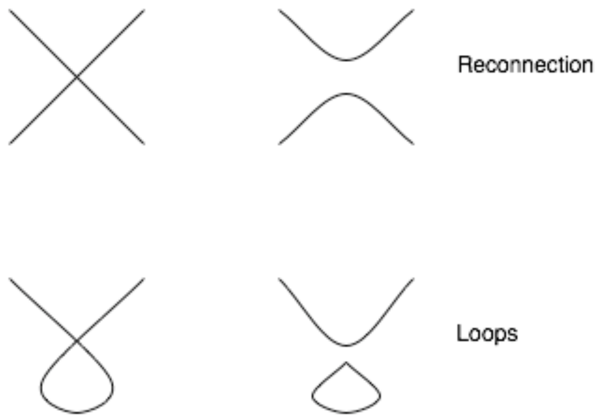
$$\rightarrow \dot{\rho}_{str} = -\frac{2\dot{a}}{a}\left(1 + \left\langle v^2 \right\rangle\right)\rho_{str}$$

So:

- If $\left\langle v^2 \right\rangle = 0$, then $\rho_{str} \propto a^{-2}$, $w = -\frac{1}{3}$.
- If $\left\langle v^2 \right\rangle = \frac{1}{2}$, then $\rho_{str} \propto a^{-3}$, $w = 0$
- If $\left\langle v^2 \right\rangle = 1$, then $\rho_{str} \propto a^{-4}$, $w = \frac{1}{3}$.

During the matter era, $\left\langle v^2 \right\rangle \approx \frac{1}{2}$. Then ρ_{str} / ρ_{crit} is a constant → self-similar scaling, i.e. the strings expand at the same time as matter. However, if $\left\langle v^2 \right\rangle \neq \frac{1}{2}$, then the string density grows relative to the background. This is the case in the radiation era, *in the absence of interactions*.

Fortunately, there exists a natural mechanism for the network to lose the extra energy in order to maintain scaling.



If you have a pair of strings crossing, then they will change partners. This is called reconnection. If a string crosses itself, then it produces a loop. Loops are unstable, and they decay away into radiation.

5. Structure Formation

5.1 Power Spectrum Measures

The density perturbation (or contrast) is defined by $\delta(\underline{x},t)$ (which is *not* a delta function).

$$\delta(\underline{x},t) = \frac{\delta_\rho(\underline{x},t)}{\bar{\rho}} = \frac{\rho(\underline{x},t) - \bar{\rho}(t)}{\bar{\rho}(t)}$$

where $\bar{\rho} = \bar{\rho}(t)$ is the background density. In a flat universe,

$$H^2 = \frac{8\pi G}{3}\bar{\rho}$$

We expand $\delta(\underline{x},t)$ in terms of comoving Fourier modes in a large volume V.

$$\delta(\underline{x},t)=\frac{V}{(2\pi)^3}\sum_{\underline{k}}\delta_{\underline{k}}(t)e^{-i\underline{k}\cdot\underline{x}}$$

NB: in the infinite volume limit, $V\delta_{\underline{k}}(t)\to\hat{\delta}_{\underline{k}}(t)$ the Fourier Transform.
Hence,

$$\frac{1}{V}\int\delta(\underline{x},t)e^{i\underline{k}\cdot\underline{x}}\,d^3\underline{x}=\frac{1}{(2\pi)^3}\sum_{\underline{k}'}\delta_{\underline{k}'}\int e^{i(\underline{k}-\underline{k}')\cdot\underline{x}}d^3\underline{x}=\sum_{k}\delta_{\underline{k}}$$

The correlation function $\xi(\underline{r})$ $(=\xi(r)$ in an isotropic universe) is defined to be

$$\begin{aligned}
\xi(\underline{r})&=\frac{1}{V}\int d^3\underline{x}\,\delta(\underline{x})\delta(\underline{x}+\underline{r})\\
&=\frac{1}{V}\left(\frac{V}{(2\pi)^3}\right)^2\sum_{\underline{k}',\underline{k}''}\delta_{\underline{k}'}\delta_{\underline{k}''}e^{i\underline{k}'\cdot\underline{r}}\int e^{-i(\underline{k}'+\underline{k}'')\cdot\underline{x}}d^3\underline{x}\\
&=\frac{V}{(2\pi)^3}\sum_{\underline{k}',\underline{k}''}\delta_{\underline{k}'}\delta_{\underline{k}''}e^{i\underline{k}''\cdot\underline{r}}\delta(\underline{k}'+\underline{k}'')\\
&=\frac{V}{(2\pi)^3}\sum_{\underline{k}'}\delta_{\underline{k}'}\delta_{-\underline{k}'}e^{-i\underline{k}'\cdot\underline{r}}\\
&=\frac{V}{(2\pi)^3}\sum_{\underline{k}}\left|\delta_{\underline{k}}\right|^2e^{-i\underline{k}\cdot\underline{r}}
\end{aligned}$$

and the power spectrum $P(k)=|\delta_k|^2$, which we have shown is the Fourier Transform of the correlation function.

We have already computed $P_i(k)$ from inflation. This section explains how to compute $P(k)$ at the present day using linear perturbation theory. This can be quantified (in linear perturbation theory) by the Transfer Function $T(k)$, which is defined by

$$\delta_k(t_0)=T(k)\delta_k(t_i).$$
$$\to P(k)=|T(k)|^2P_i(k)$$

5.2 Overview of Structure Formation
Matter content of the universe:
Definite:

- Baryons (& leptons), i.e. protons, neutrons, electrons, positrons, etc.
  $\Omega_b h^2\approx0.02$ (Big Band Nucleosynthesis considerations)
- Photons & Neutrinos – "Radiation"
  $\Omega_r h^2\approx4.2\times10^{-5}$ (from CMB temperature)

Hypothesized:

- Cold Dark Matter (CDM)
  - Weakly interacting particles which only interact gravitationally with normal matter, e.g. WIMPs (Weakly Interacting Massive Particles), Axions.

- Hot Dark Matter (HDM)
    - o Massive neutrinos
- Dark Energy: Λ, Quintessence, etc.

At present, we believe that $\Omega_{CDM}h^2 \approx 0.1$, $\Omega_{HDM}h^2 << 1$, $\Omega_{DE} \approx 0.7$.

Using linear perturbation theory in the Newtonian limit, one can deduce that

$$\ddot{\delta}_M + \frac{2\dot{a}}{a}\dot{\delta}_M = \delta_M\left[4\pi G\rho_M - \frac{{c_s}^2k^2}{a^2}\right]$$

The derivatives are with respect to time; δ_M is the density contrast in the matter and ${c_s}^2 = \frac{dP}{d\rho}$ is the sound speed in the fluid.

For CDM, $P = 0$ and hence ${c_s}^2 = 0$. For HDM and baryons, there exists pressure due to collisions, therefore ${c_s}^2 \neq 0$. For radiation, ${c_s}^2 = 1/3$.

1. $\dot{a} = 0$, $a = 1$

$$\ddot{\delta}_M = \delta_M\left[4\pi G\rho_M - {c_s}^2k^2\right]$$

$$\rightarrow \delta_M \propto e^{\sqrt{4\pi G\rho_M - {c_s}^2k^2}\,t}$$

For $k > \frac{\sqrt{4\pi G\rho_M}}{c_s}$ → oscillatory solutions

For $k < \frac{\sqrt{4\pi G\rho_M}}{c_s}$ → Exponential growth

(NB: we will ignore decaying solutions)

→ there exists a length scale, known as the Jean's Length,

$$\lambda_J = \frac{1}{2\pi k_J} = c_s\sqrt{\frac{\pi}{G\rho_M}}$$

where:

$\lambda > \lambda_J$, density fluctuations grow exponentially (natural propensity of fluctuations

$\lambda < \lambda_J$, density fluctuations oscillate due to pressure support.

e.g. $c_s = 0$ → $\lambda_J = 0$ and density fluctuations grow on all scales.

2. ${c_s}^2 = 0$, $a \propto t^{2/3}$ → $\rho_m \approx \frac{1}{6\pi G t^2}$

$$\rightarrow \ddot{\delta}_M + \frac{4}{3t}\dot{\delta}_M - \frac{2}{3t^2}\delta_M = 0$$

Solution is $\delta_M \propto t^P$

$$\rightarrow p(p-1) + \frac{4}{3}p - \frac{2}{3} = 0$$

$$\rightarrow p = 2/3, -1$$

There is one growing solution, where $\delta_M \propto t^{2/3} \propto a \propto \frac{1}{1+z}$, and one decaying solution, where $\delta_M \propto \frac{1}{t}$.
Decaying solutions are singular at $t = 0$, so we ignore them because we have assumed δ_M is small.

3. Λ domination: $a \propto e^{\sqrt{\frac{\Lambda}{3}}t}$

$\rho_M \to 0$ and $a \to \infty$ very quickly.

$$\ddot{\delta}_M + 2\sqrt{\frac{\Lambda}{3}}\dot{\delta}_M = 0$$

→ constant solution, hence exponential expansion suppresses the growth of perturbations.

4. Mezaros Effect $\left(c_s^{\,2} = 0\right)$

$\rho = \rho_r + \rho_M$

$P = \frac{1}{3}\rho_r$

$k = 0$ (flat universe)

The first two give $H^2 = \frac{8\pi G}{3}\left(\rho_r + \rho_M\right)$, $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(2\rho_r + \rho_m\right)$.

Define $y = \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}$.

$$H^2 = \frac{8\pi G\rho_m}{3}\left(1 + \frac{1}{y}\right)$$

$$\ddot{a} = -\frac{4\pi G\rho_M}{3}\left(1 + \frac{2}{y}\right)$$

Change variables from t to y.

e.g. $\frac{\partial}{\partial t} = \frac{\dot{a}}{a_{eq}}\frac{\partial}{\partial y}$, $\frac{\partial^2}{\partial t^2} = \frac{\ddot{a}}{a_{eq}}\frac{\partial}{\partial y} + \left(\frac{\dot{a}}{a_{eq}}\right)^2\frac{\partial^2}{\partial y^2}$

$$\to \frac{\partial^2}{\partial y^2}\delta_M + \frac{2+3y}{y(1+y)}\frac{\partial}{\partial y}\delta_M = \frac{3}{2y(1+y)}\delta_M$$

This is the Mezaros Equation. There exists a solution with $\frac{\partial^2}{\partial y^2}\delta_M = 0$.

$\to \frac{\partial}{\partial y}\delta_M = \frac{3}{2+3y}\delta_M \to \delta_M \propto \frac{2}{3} + y$ (which satisfies $\frac{\partial^2}{\partial y^2}\delta_M = 0$)

→ Density fluctuations are constant during the radiation era $\left(y \ll 1\right)$ and then they grow like $y\left(\propto a\right)$ in the matter era, as found in section 2.

→ density fluctuations in CDM start to grow at t_{eq}.

5.3 Relativistic Perturbation Theory

We will perturb a flat FRW universe with a (background) metric $\bar{g}_{\mu\nu} = a^2(n)\eta_{\mu\nu}$ (NB: time is conformal) with line element

$$ds^2 = a^2(n)\left[\eta_{\mu\nu} + h_{\mu\nu}\right]dx^\mu dx^\nu$$

and we will assume that , that is, linear perturbations.

In relativistic perturbation theory, one can make a choice of gauge which fixes the coordinates: we will use the Synchronous Gauge with

$$h_{00} = h_{0i} = 0$$

which corresponds to perturbations seen by a comoving freely-falling observer.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2\left(\eta_{\mu\nu} + h_{\mu\nu}\right)$$

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu} = a^2\left(\eta^{\mu\nu} - h^{\mu\nu}\right)$$

One can compute the Christoffel symbols,

$$\Gamma^0_{00} = \frac{a'}{a}$$

$$\Gamma^0_{0i} = \Gamma^i_{00} = 0$$

$$\Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} - \frac{1}{2}h_{ij}' - \frac{a'}{a}h_{ij}$$

$$\Gamma^i_{0j} = \frac{a'}{a}\delta_{ij}' - \frac{1}{2}h^i{}_j{}'$$

$$\Gamma^i_{jk} = -\frac{1}{2}\left(\partial_k h^i{}_j + \partial_j h^i{}_k - \partial^i h_{jk}\right)$$

5.3.1 Perturbed Fluid Equations

Consider a perfect fluid with energy-momentum tensor

$$\bar{T}^\mu_\nu = \left(\bar{\rho} + \bar{P}\right)u^\mu u_\nu - \bar{P}\delta^\mu{}_\nu$$

when perturbed, this becomes

$$\begin{aligned} T^\mu{}_\nu &= \bar{T}^\mu_\nu + \delta T^\mu{}_\nu \\ &= \left(\bar{\rho} + \bar{P}\right)u^\mu u_\nu - \bar{P}\delta^\mu{}_\nu + \left(\delta\rho + \delta P\right)u^\mu u_\nu + \left(\bar{\rho} + \bar{P}\right)\left(\delta u^\mu u_\nu + u^\mu \delta u_\nu\right) - \delta P \delta^\mu{}_\nu \end{aligned}$$

where $u^\mu \delta u_\mu = 0$, i.e. the perturbation is perpendicular to the fluid flow.