The stability of the wall is due to the potential and the gradient balancing each other (Derrick's Theorem). Simple argument:

$$E_{grad} \sim \Delta(\nabla\phi)^2 \sim \Delta\left(\frac{\eta}{\Delta}\right)^2 \sim \frac{\eta^2}{\Delta}$$

$$E_{pot} \sim \Delta V(\phi) \sim \lambda\Delta\eta^4$$

Derrick's Theorem says that for stability, $E_{pot} = E_{grad}$

$$\rightarrow \frac{\eta^2}{\Delta} \approx \Delta\lambda\eta^4$$

$$\rightarrow \Delta \sim \lambda^{-1/2}\eta^{-1} \sim m_\phi^{-1}$$

where m_ϕ is the mass of the ϕ particle. This is the same (ignoring numerical coefficients) as the equation found above.

One can compute the EM tensor (with units $kg\,m^{-3}$);

$$T^\mu{}_\nu = \frac{\lambda\eta^4}{2}\text{sech}^4\left(\frac{x}{\Delta}\right)\text{diag}(1,0,1,1)$$

[NB: $P = -\frac{2}{3}\rho$.] The surface density (with units $kg\,m^{-2}$) is

$$\sigma = \int \rho dx = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\eta^3 \sim \rho\Delta$$

NB:

1. Δ is very small and σ is very large if $\eta \approx 10^{16}\,GeV$.
2. There exists a topologically conserved charge J^0 and current $\underline{j}$, known as the kink number.

   $J^\mu = \varepsilon^{\mu\nu}\partial_\nu\phi$

   where $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$.

   $\partial_\mu J^\mu = \varepsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$

   because $\varepsilon^{\mu\nu}$ is antisymmetric, and the derivative is symmetric, and last semester we showed that this combination must equal 0.

   $\rightarrow \frac{\partial J^0}{\partial t} - \underline{\nabla}\cdot\underline{j} = 0$

   Define $N = \frac{1}{2\eta}\int dx\, J^0$, then $\frac{\partial N}{\partial t} = 0$

   $J^0 = \partial_x\phi \;\rightarrow N = \frac{1}{2\eta}\int dx\frac{d\phi}{dx} = \frac{\phi(x=+\infty)-\phi(x=-\infty)}{2\eta}$

   (There could exist solutions with an increased number of kinks)

4.3.3 Nielson-Oleson Vortex

$$L = D_\mu \Phi D^\mu \bar{\Phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4}\left(|\phi|^2 - \eta^2\right)^2$$

Examples 2, Question 1: there are 2 particles, one scalar $m_\Phi^{-1} = \left(\sqrt{\lambda}\eta\right)^{-1}$ and one vector $m_A^{-1} = \left(\sqrt{2}e\eta\right)^{-1}$ (The Higgs particle).

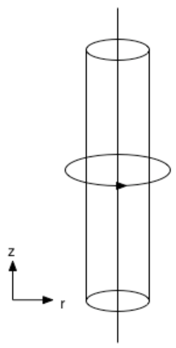
NB: $m_e^{-1} = \dfrac{\hbar}{m_e c}$ is the De Broglie wavelength of the particle.

EOM:

$$D^\mu D_\mu \Phi + \frac{\lambda}{2}\Phi\left(|\phi|^2 - \eta^2\right) = 0$$

$$\partial_\mu F^{\mu\nu} = J^\nu = 2e\,\mathrm{Im}\left\{\bar{\Phi} D^\nu \Phi\right\}$$

[$D^\mu = \partial_\mu - ieA_\mu$] The latter one is the equivalent of the Maxwell laws.

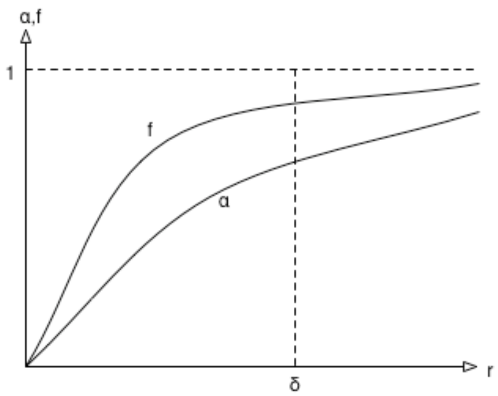


Cylindrical Symmetrical Solution

$$\Phi = \eta e^{i\theta} f(r)$$

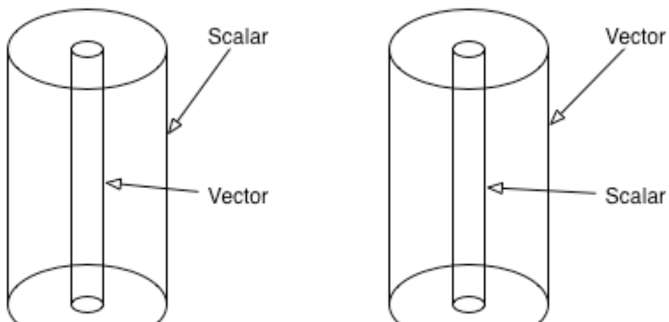
$$A_\theta = -\frac{\eta\alpha(r)}{er}$$

where $n \equiv$ winding number $\equiv$ topological charge.
Boundary conditions: $f(0) = \alpha(0) = 0$ and $f(r) \to 1, \alpha(r) \to 1$ as $r \to \infty$.



There are two flux tubes, scalar and vector, with widths $\delta_s \sim m_s^{-1}$ and $\delta_A \sim m_A^{-1}$, corresponding to the Compton wavelength of the particles.

Define $\beta = \left|\dfrac{m_A^{-1}}{m_S^{-1}}\right|^2 = \dfrac{\lambda}{2e^2}$.

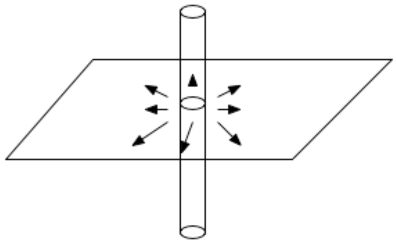


Left: $\beta < 1$: Type 1 regime, $m_s^{-1} > m_A^{-1}$.

Right: $\beta > 1$: Type 2 regime, $m_s^{-1} < m_A^{-1}$

NB:

1. Type 1 and Type 2 are almost analogous to the type 1 and 2 regimes of superconductors. This is no coincidence since the Landau-Ginsburg theory of superconductors is the non-relativistic limit of the Abelian-Higgs model.
2. Typically the core radius of the strings is very small, $\delta \sim \eta^{-1}$, whereas the mass per unit length $\mu \sim \eta^2$ is very large.
    a. From large distances, we can treat the string as line distribution of mass.
    b. The gravitational properties are governed by the parameter $G\mu \sim 10^{-6}$ for $\eta \sim 10^{16} GeV$. This can lead to density fluctuations, gravitational lensing and many other interesting effects.
3. The magnetic flux associated with the vortexes is quantized.



$$\Phi_B = \int \underline{B} \cdot d\underline{S} = \int \underline{A} \cdot d\underline{\ell} = \frac{2\pi n}{e}$$

4.3.4 Monopoles

Consider $\underline{\phi} = (\phi_1, \phi_2, \phi_3)$.

$$L = \frac{1}{2}\partial_\mu \underline{\phi} \partial^\mu \underline{\phi} - \frac{1}{4}\lambda\left(\underline{\phi}^2 - \eta^2\right)^2$$

Break global $SO(3) \to SO(2)$. Vacuum manifold is

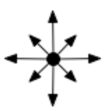
$$M = \left\{\underline{\phi} : |\phi|^2 = \eta^2\right\} \cong S^2 .$$

$\pi_2\left(S^2\right) = \mathbb{Z}$ → monopoles

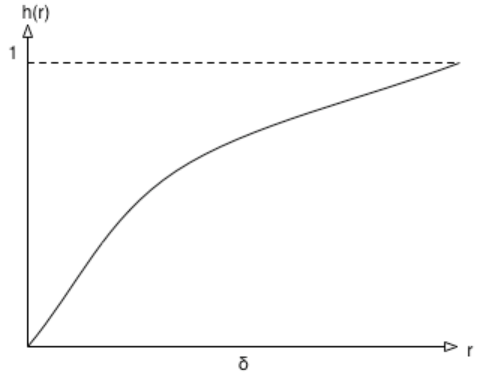
Aside: unfortunately, by Derrick's Theorem global monopoles are unstable in 3D, but actually the gauge equivalent is stable and is known as the "t'Hooft-Polyakov" monopole.

The global monopole solution is known as the "Hedgehog" solution.

$$\underline{\phi}(\underline{r}) = \eta h(r)\hat{\underline{r}}$$



The arrows point in all directions, creating a "hedgehog"-like appearance.



where $\delta \sim \eta^{-1}$.

The gauge equivalent has a magnetic monopole associated with it:

$$\Phi_B = \int \underline{\nabla} \cdot \underline{B} \, d\underline{V} = \int \underline{B} \cdot d\underline{S} = \frac{2\pi n}{e}$$

as $\underline{B} = -\frac{1}{e}\frac{\hat{r}}{r}$, where $e \equiv$ electromagnetic coupling constant, and $n \equiv$ the winding number. It also a mass

$$m = \frac{4\pi\eta}{e}.$$

NB: Maxwell's theory says $\underline{\nabla} \cdot \underline{B} = 0$, i.e. no magnetic monopoles.

4.4 Simple Models for defect evolution

4.4.1 Monopoles

Monopoles evolve like point particles if one assumes that monopole-antimonopole annihilation is very inefficient.

→ $\rho_{monopole} \propto a^{-3}$

where a is the scale factor. If $G \to SU(3) \times SU(2) \times U(1)$, then monopoles form at $t = t_f$. Now assume that the correlation length $\xi \sim t_f$, and that time (i.e. one monopole per horizon volume).

$$\rho_{monopoles}(t_f) = \frac{M}{\xi^3} \approx \frac{M}{t_f^{\,3}}$$

$$\rho_{monopoles}(t) = \rho_{monopoles}(t_f)\left(\frac{a_f}{a}\right)^3 \propto t^{-3/2}$$

for $a \propto t^{1/2}$. Now $\rho_{crit} \propto t^{-2}$ during both radiation and matter eras.

$$\left[\rho \propto H^2 \propto \begin{Bmatrix} a^{-4} & rad \\ a^{-3} & matter \end{Bmatrix} \propto t^{-2}\right]$$

Hence the density of monopoles increases relative to radiation. If $\rho_{monopoles}(t_f) >> \rho_{matter}(t_f)$, then $\rho_{monopoles}$ will come to dominate the universe.

NB: we have observed no magnetic monopoles in the universe.
→ there exists a bound on η known as the Parker bound (see example sheet).
This is called the monopole problem, and it was the original motivation for inflation. If $a \propto e^{Ht}$ then $\rho_{monopole}(t) \to 0$ very quickly. If the reheat temperature $T_{reheat} < \eta$ then the phase transition does not happen again after inflation.

4.4.2 Dimensional scaling vs. Self-Similar scaling

The evolution of strings and domain walls is more complicated. Let us first assume that the defects are static (i.e. not interacting).
If $L \propto a$, $A \propto a^2$ and $V \propto a^3$, then

$$\rho_{domain\,walls} = \frac{\sigma A}{V} \propto \frac{1}{a}$$

$$\to w = -\frac{2}{3} \quad \left(w = \frac{P}{\rho} \right)$$

$$\rho_{string} = \frac{\mu L}{V} \propto \frac{1}{a^2}$$

$$\to w = -\frac{1}{3}$$

This is called "dimensional scaling".

However, this ignores the possibility of:

1. Interactions
2. Work against the expansion due to $v \neq 0$

It is <u>possible</u> that the defect network evolves towards a self-similar scaling regime, where $L \propto t$.

$$\to \rho_{string} \propto t^{-2}, \; \rho_{domain\,walls} \propto t^{-1}$$

4.4.3 Domain Walls

If $\rho_{domain\,walls} \propto a^{-1}$ or $\propto t^{-1}$, then this grows relative to the background → domain wall problem, and a bound on η or a necessity for inflation.