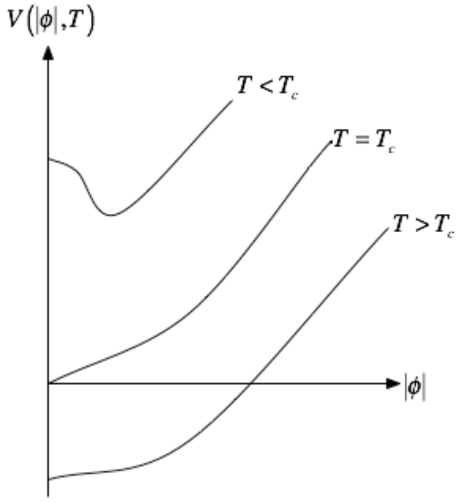


If $\Phi = |\phi| e^{i\theta}$, then;

1. $|\phi|$ has dynamics associated with the effective potential (free energy). $|\phi|$ slips from $|\phi| = 0$ to $|\phi| = \eta$, that is from the symmetric phase into the broken phase.
2. θ has no potential – there exists large spatial variations in θ.

3.4.2 Correlation Length

The fluctuations in θ are uncorrelated on scales larger than ξ. This is known as the correlation length. The exact dynamics of $\xi = \xi(t)$ are complicated and also depend on the details of the phase transition. However, crucially one can bound it to be less than the particle horizon distance by causality.

→ $\xi(t) < d_H(t) \propto t$ for $a \propto t^p$

4. Topological Defects

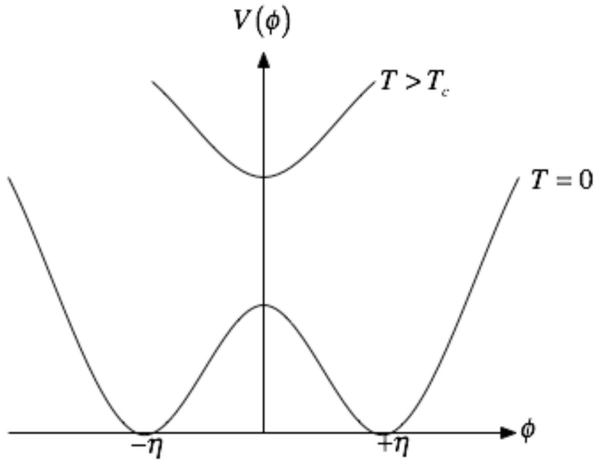
4.1 Kibble Mechanism

4.1.1 Discrete Goldstone Model in 2D

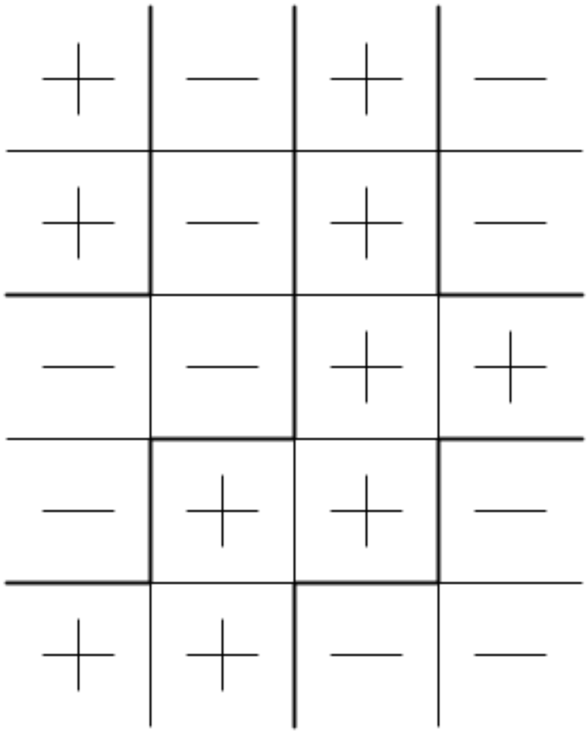
Consider

$$V_{eff}(\phi, T) = \frac{\lambda}{4}\left(\phi^2 - \eta_{eff}^{\ 2}(T)\right)^2,$$

with $\eta_{eff}^{\ 2} = \eta^2 \left(1 - \dfrac{T^2}{T_c^{\ 2}}\right)$.



Random fluctuations → choice of vacuum $\phi = \pm\eta$ is spatially dependent, and there exists correlation length $\xi < d_H(t) \sim \psi$. Causally disconnected regions will make different choices of vacuum.



(NB: lines not necessarily straight)

In 2D, line defects separate regions with different vacua. In 3D these would be 2D surfaces, i.e. sheets.

In the transition region between the vacua, there is a topological defect, in this case it is known as a *domain wall*.

4.1.2 U(1) Goldstone Model in 2D
Now consider the potential

$$V_{eff}(|\phi|, T) == \frac{\lambda}{4}\left(|\phi|^2 - \eta_{eff}{}^2\right)^2$$

where $\Phi = |\phi| e^{i\theta} \in \mathbb{C}$. The choice of the vacuum is spatially dependent → the phase θ is spatially dependent – this is more difficult to visualize.

Consider discrete phases $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ (phases 0, 1 and 2 respectively), and look for 2π windings on a discrete lattice. Assign a phase (0, 1, 2) to each vertex of the plaquette.

0	1	2	1
1	0	X 0	X 1
2	1	X 2	0
1	2	1	X 0
0	X 0	2	1

We want to look for situations like

0		1
	X	
0		2

, which gives a 2π winding (marked with an X on the above diagram), or

0		1
1		2

, which goes round by π and then goes back.

There exists a vortex when the field winds around 2π. In 3D, these vortices become what we shall call cosmic strings.

4.1.3 Defect density created during a 2nd order transition

- Domain walls have dimension 2 – they are sheet-like.
- Cosmic strings have dimension 1 – they are line-like.
- Monopoles have dimension 0 – they are point-like.

Let d be the dimension of the defect. If ξ is the correlation length then one would expect one defect of size ξ^d in a volume ξ^3, i.e.;

- One domain wall of area ξ^2 in volume ξ^3
- One string of length ξ in volume ξ^3
- One monopole in volume ξ^3

4.2 Defect Classification and Homotopy Theory

Consider a symmetry breaking transition $G \to H$ with vacuum manifold $M = \frac{G}{H}$.

The possibility of defect solutions is governed by the non-trivial homotopy of the vacuum manifold.

e.g.

1. Discrete Goldstone model. $M = \{+\eta, -\eta\}$, which is disconnected.
2. $U(1)$ Goldstone model. $M = \{\Phi : |\phi| = n\} \cong S^1$, which is not simply connected.

More generally we quote the conditions in terms of the Homotopy Group. Letting I represent the Identity,

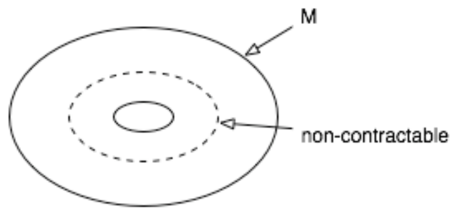
- $\pi_0(M) \neq I$ → domain walls
- $\pi_1(M) \neq I$ → cosmic strings
- $\pi_2(M) \neq I$ → monopoles

NB:

- $\pi_0\left(\frac{G}{H}\right) \neq I$ if $\frac{G}{H}$ is disconnected, i.e. $\frac{G}{H} = \{\square, \square\}$
- $\pi_1\left(\frac{G}{H}\right) \neq I$ if $\frac{G}{H}$ is not simply connected, i.e. $\frac{G}{H} = \{(\bigcirc)\}$ (which is meant to represent a 2D torus…)

More generally, $\pi_N(M) \neq I$ if there exists curves of dimension N which are not contractible.

e.g. $N = 1$



Useful Relations

$$\pi_1(G) = \begin{cases} Z & G = U(1), SO(2) \\ Z_2 & G = SO(N), N > 2 \\ I & Most\ other\ groups \end{cases}$$

$$\pi_2(G) \cong I \text{ for all simple groups}$$

$$\pi_N(S^N) = Z$$

where $Z = \{integers\}$. $U(1) \cong S^1$, $SU(2) \cong S^3$.

Theorems

$$\pi_1\left(\frac{G}{H}\right) \cong \pi_0(H) \text{ if } \pi_0(G) = \pi_1(G) = I \text{ (i.e. simply connected)}$$

$$\pi_2\left(\frac{G}{H}\right) \cong \pi_1(H) \text{ if } \pi_0(G) = \pi_1(G) = I \text{ (i.e. simply connected)}$$

Classical situations for the formation of defects

1. Domain Walls
   These form when a discrete group is broken, e.g. $\mathbb{Z}_N \to I$
2. Strings
   When $U(1)$ is broken, e.g. $U(1) \to I$
3. Monopoles
   When a $U(1)$ subgroup is created, e.g. $G \to H \times U(1)$

NB: monopoles are, therefore, part of the Standard Model, $G \to SU(3) \times SU(2) \times U(1)$ (the $U(1)$ is EM). But Maxwell's laws say that there are no magnetic monopoles since $\underline{\nabla} \cdot \underline{B} = 0$, and also we don't see any.

4.3 Field Theory Solutions

4.3.1 Dynamical Stability: Derrick's Theorem

The topological conditions are necessary but not sufficient for the existence of solutions, but dynamical stability must also be taken into account.

Assume one has a localized solution $\phi(\underline{x})$ in D dimensions such that the energy density tends to zero at infinity so that the energy is finite.

$$
\begin{aligned}
E &= \int d^D \underline{x} \left\{ \frac{1}{2} |\nabla \phi|^2 + V(\underline{\phi}) \right\} \\
&= I_2 + I_0
\end{aligned}
$$

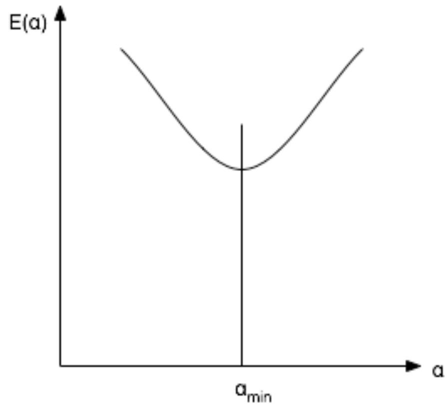
where I_2 represents the gradient (2 because of the 2 derivatives), and I_0 represents the potential. Now consider a scaling transformation $x \to x' = \alpha x$, whence the energy is given by

$$
\begin{aligned}
E(\alpha) &= \int d^D \underline{x}' \left\{ \frac{1}{2} |\nabla' \phi|^2 + V(\underline{\phi}) \right\} \\
&= \alpha^D \left\{ \frac{1}{\alpha^2} I_2 + I_0 \right\} \\
&= \alpha^{D-2} I_2 + \alpha^D I_0
\end{aligned}
$$

$\underline{D = 1}$

$$
\begin{aligned}
E(\alpha) &= \frac{I_2}{\alpha} + \alpha I_0 \\
\frac{dE}{d\alpha} &= -\frac{I_2}{\alpha^2} + I_0
\end{aligned}
$$

$\frac{dE}{d\alpha} = 0$ when $\alpha = \sqrt{\frac{I_2}{I_0}}$, and the minimum energy is $E = 2\sqrt{I_0 I_2}$.



$\underline{D=2}$

$$E(\alpha)=I_2+\alpha^2 I_0$$

This is scale-free if $I_0=0$.

$\underline{D \geq 3}$

$E(\alpha)\to 0$ as $\alpha \to 0$, so there are no stable solutions with finite energy if $D\geq 3$.

Ways out of this are:

1. $D=1$ (e.g. domain walls)
2. Gauge fields, $F_{\mu\nu}F^{\mu\nu}\sim \dfrac{1}{\alpha}$ if $D=3$, which can create stable solutions
3. 4th order derivatives
4. Things need not have finite energy.

4.3.2 Domain Walls

$$L=\frac{1}{2}\left(\partial_\mu \phi\right)^2 - V(\phi)$$

$$\Box\phi+\frac{dV}{d\phi}=0$$

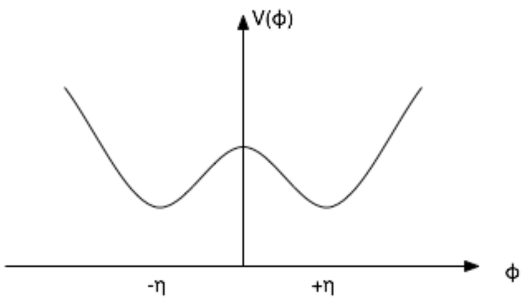
is the equation of motion, where $\Box=\dfrac{\partial^2}{\partial t^2}-\nabla^2$. The energy-momentum tensor is

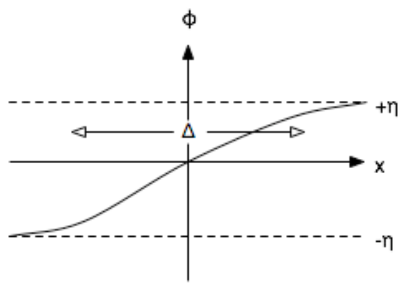
$$T_{\mu\nu}=\partial_\mu\phi\partial^\mu\phi-g_{\mu\nu}L$$

$$\rightarrow \rho=T^0{}_0=\frac{1}{2}\dot{\phi}^2+\frac{1}{2}(\nabla\phi)^2+V(\phi)$$

where the terms on the right are the kinetic energy, the gradient of the energy, and the potential energy.

If $V(\phi)=\dfrac{\lambda}{4}\left(\phi^2-\eta^2\right)^2$, then there exists a stable (topologically and dynamically) solution sometimes known as the ϕ^4-kink, which interpolates between the two vacua.





The solution is $\phi(x)=\eta\tanh\left(\dfrac{x}{\Delta}\right)$. Substitution into the EOM yields $\Delta=\left(\dfrac{2}{\lambda}\right)^{1/2}\eta^{-1}$, and this is the thickness of the wall.

At $x=\pm\infty$, $\rho=0$. At $x=0$, $\rho_{PE}=\rho_{Grad}=\dfrac{\lambda^4}{4}\eta^4$.