PC4771 – Gravitation – Lectures 9 & 10

3.3 Metric Connection & Metric Geodesign

Recall: Fermatt's principle – light rays extremise the integral $S=\int_a^b dt$.

Convert to relativity – extremise $S=\int_a^b ds$.

→ $S=\int_a^b \left(g_{\alpha\beta} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \right)^{1/2} d\tau$ is extremised, where we use $d\tau = \left|ds^2\right|^{1/2}$ is the proper time.

→ compute the Euler-Lagrange (E-L) equations with

$L=\left| g_{\alpha\beta} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \right| = L\left(\tau, x^\mu, \frac{dx^\mu}{d\tau} \right)$.

For the whole calculation, see the earlier handout.

$$\frac{dL}{dx^\mu} = \frac{\partial_\mu g_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}{L}$$

$$\frac{dL}{d\left(\frac{\partial x^\mu}{\partial \tau}\right)} = \frac{g_{\alpha\beta}\delta^\alpha{}_\mu \frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\delta^\beta{}_\mu}{L}$$

NB: $\frac{\partial L}{\partial \tau}=0$ → $L-\frac{dx^\mu}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial x^\mu}{\partial \tau}\right)} = const.$ → $L-2L=const$ → $L=const.$

E-L:

$$\partial_\mu \left(\frac{\partial L}{\partial\left(\frac{\partial x^\mu}{d\tau}\right)} \right) - \frac{\partial L}{\partial x^\mu} = 0$$

→ $\frac{d^2x^\mu}{d\tau^2} + \left\{ {}^{\nu}_{\alpha\beta} \right\} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$

where $\left\{ {}^{\nu}_{\alpha\beta} \right\} = \frac{1}{2} g^{\mu\nu}\left(-\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} \right)$

This would look like the affine geodesic equation if $\Gamma^\nu_{\alpha\beta} = \left\{ {}^{\nu}_{\alpha\beta} \right\}$.

$\left\{ {}^{\nu}_{\alpha\beta} \right\}$ is known as the Christoffel connection or the Metric Connection.

Now consider an affine connection with $\nabla_\alpha g_{\mu\nu} = 0$. Then (see sheet):

$$\Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\mu\nu} = g^{\alpha\rho}\left(-\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha}\right) + 2\left(T_\nu{}^\rho{}_\mu + T_\mu{}^\rho{}_\nu\right)$$

If the connection is torsion free, i.e. $T^\mu{}_{\alpha\beta} = 0$, then

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\alpha\rho}\left(-\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha}\right) = \left\{ {\rho \atop \mu\nu} \right\}$$

If the metric is covariently conserved $\nabla_\alpha g_{\mu\nu} = 0$, and $T^\mu{}_{\alpha\beta} = 0$ then the affine connection is the Christoffel connection and the connection of parallel transport can be defined in terms of the metric.

Notes:

1. $T^\nu \nabla_\mu \left[d(T,A) \right] = 0$ (see section 3.2) → parallel transport of A^μ involves the angle between it and the tangent vector remaining constant.
2. $ds^2 = d\tau^2 \left[d(T,T) \right]$ where $T^\mu = \dfrac{dx^\mu}{du}$ → the interval can be classified as spacelike, timelike or null since the interval ds^2 is conserved.

From now onwards we will use the Christoffel connection as the definition of the connection.

3.4 Locally inertial coordinates

Let P be some point in the manifold. Without loss of generality, the coordinates of P can be chosen such that $x^\mu = 0$, i.e. it is at the origin.

Consider the coordinate transformation.

$x'^\rho = x^\rho + \frac{1}{2} Q^\rho_{\mu\nu} x^\mu x^\nu$ for some array of numbers $Q^\rho_{\mu\nu}$.

$$\rightarrow J'^\rho_\gamma = \frac{\partial x'^\rho}{\partial x^\gamma} = \delta^\rho{}_\gamma + \frac{1}{2} Q^\rho_{\mu\nu}\left[\delta^\mu{}_\gamma x^\nu + x^\mu \delta^\nu{}_\gamma \right]$$

$$\partial_\alpha J'^\rho{}_\gamma = \frac{1}{2}\left[Q^\rho_{\gamma\alpha} + Q^\rho_{\alpha\gamma} \right]$$

Under a coordinate transformation, $\Gamma'^\rho_{\mu\nu} = J'^\rho_\gamma J^\alpha_\mu J^\beta_\nu \Gamma^\gamma_{\alpha\beta} + J'^\rho_\gamma J^\alpha_\mu \partial_\alpha J^\gamma_\nu$

Now;

$$J'^\rho_\gamma J^\gamma_\nu = \delta^\rho{}_\nu$$

$$J^\gamma_\nu \partial_\alpha J'^\rho_\gamma + J'^\rho_\gamma \partial_\alpha J^\gamma_\nu = 0$$

$$\rightarrow \begin{aligned} \Gamma'^\rho_{\mu\nu} &= J'^\rho_\gamma J^\alpha_\mu J^\beta_\nu \Gamma^\gamma_{\alpha\beta} - J^\alpha_\mu J^\gamma_\nu \partial_\alpha J'^\rho_\gamma \\ &= J'^\rho_\gamma J^\alpha_\mu J^\beta_\nu \Gamma^\gamma_{\alpha\beta} - J^\alpha_\mu J^\gamma_\nu Q^\rho_{\gamma\alpha} \end{aligned}$$

At P, $x^\mu = 0$ → $J^\gamma_\nu = J'^\nu_\gamma = \delta^\gamma_\nu$, and hence:

$$\Gamma'^\rho_{\mu\nu} = \Gamma^\rho_{\mu\varpi} - Q^\rho_{\mu\nu}$$

If we choose $Q^\rho_{\alpha\beta} = \Gamma^\rho_{\alpha\beta}\big|_P$, then $\Gamma'^\rho_{\mu\nu}\big|_P = 0$ → the connection in the primed coordinate system is zero at P.

→ we can always choose coordinates to be inertial at a point P in the manifold, and at that point $\Gamma^\mu_{\alpha\beta} = 0$ and $\partial_\mu = \nabla_\mu$ - only at that point, not globally.

3.5 Isometrics and Killing's Equation

Consider a coordinate transformation $x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}$, where ε is small.

$$dx'^{\mu} = J'^{\mu}_{\alpha} dx^{\alpha} = \left(\delta^{\mu}{}_{\alpha} + \varepsilon d_{\alpha}\xi^{\mu}\right) dx^{\alpha}$$

$$\begin{aligned} ds^2(x') &= g_{\mu\nu}(x') dx'^{\mu} dx'^{\nu} \\ &= g_{\mu\nu}(x+\varepsilon\xi)\left(\delta^{\mu}{}_{\alpha} + \varepsilon\partial_{\alpha}\xi^{\mu}\right)\left(\delta^{\nu}{}_{\beta} + \varepsilon\partial_{\beta}\xi^{\nu}\right) dx^{\alpha} dx^{\beta} \\ &= \left(g_{\mu\nu} + \varepsilon\xi^{\rho}\partial_{\rho} g_{\mu\nu} + O\left(\varepsilon^2\right)\right)\left(\delta^{\mu}{}_{\alpha} + \varepsilon\partial_{\alpha}\xi^{\mu}\right)\left(\delta^{\nu}{}_{\beta} + \varepsilon\partial_{\beta}\xi^{\nu}\right) dx^{\alpha} dx^{\beta} \\ &\approx ds^2(x) + \varepsilon\left[\xi^{\rho}\partial_{\rho} g_{\mu\nu}\delta^{\nu}_{\beta}\delta^{\mu}_{\alpha} + g_{\mu\nu}\delta^{\nu}_{\beta}\partial_{\alpha}\xi^{\mu} + g_{\mu\nu}\delta^{\mu}_{\alpha}\partial_{\beta}\xi^{\nu}\right] dx^{\alpha} dx^{\beta} + O\left(\varepsilon^2\right) \end{aligned}$$

Therefore:

$ds^2(x1) = ds^2(x) + O\left(\varepsilon^2\right)$ if $\xi^{\rho}\partial_{\rho} g_{\alpha\beta} + g_{\mu\beta}\partial_{\alpha}\xi^{\mu} + g_{\alpha\nu}\partial_{\beta}\xi^{\nu} = 0$ .which is the equation for ξ^{μ} to be an isometry.

Now choose a point P and use locally inertial coordinates at that point, i.e. $\partial = \nabla$.

$\xi^{\rho}\nabla_{\rho} g_{\alpha\beta} + g_{\mu\beta}\nabla_{\alpha}\xi^{\mu} + g_{\mu\nu}\nabla_{\beta}\xi^{\nu} = 0$ at P.

→ $\nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} = 0$ if $\nabla_{\alpha} g_{\mu\nu} = 0$.

This is known as Killing's Equation.

NB:

1. $\nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} = 0$ is a tensorial equation – it is true in all frames.
2. Moreover the choice of P was arbitrary, so we could go through and do this at every point in the manifold.

→ $\nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} = 0$ is an identity in the isometry.

If ξ_{μ} is an isometry, then for T^{μ} a tangent vector:

$$T^{\nu}\nabla_{\nu}\left(T^{\mu}\xi_{\mu}\right) = \xi_{\mu} T^{\nu}\nabla_{\nu} T^{\mu} + T^{\mu} T^{\nu}\nabla_{\nu}\xi_{\mu}$$

If the curve is an affinely parameterized geodesic then $T^{\nu}\nabla_{\nu} T^{\mu} = 0$.

→ $T^{\nu}\nabla_{\nu}\left(T^{\mu}\xi_{\mu}\right) = 0$.

→ $T^{\mu}\xi_{\mu}$ is constant along affinely parameterised geodesics.

3.6 Computing Christoffel Symbols & Geodesics

See handout.