PC4771 – Gravitation – Lectures 7 & 8

3. Connection & Tensor Calculus
3.1 Covarient Differentiation & Torsion
Notation:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \; ; \; \partial'_\mu = \frac{\partial}{\partial x'^\mu}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} \; ; \; \partial'^\mu = \frac{\partial}{\partial x'_\mu}$$

This will denote ordinary partial differentiation.

If ϕ is a scalar, then:

$$\partial'_\mu \phi' = \frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial \phi}{\partial x^\alpha}$$
$$= J_\mu{}^\alpha \partial_\alpha \phi$$

This implies that $\partial_\mu \phi$ is a covector, and similarly $\partial^\mu \phi$ is a vector.

Now compute the partial derivative of a covector

$$\partial'_\mu A'_\nu = J^\alpha_\mu \partial_\alpha \left[J^\beta_\nu A_\beta \right]$$
$$= J^\alpha_\mu J^\beta_\nu \partial_\alpha A_\beta + \left(J^\alpha_\mu \partial_\alpha J^\beta_\nu \right) A_\beta$$
$$\partial'_\mu A'^\nu = J^\alpha_\mu \partial_\alpha \left[J'^\nu_\beta A^\beta \right]$$
$$= J^\alpha_\mu J'^\nu_\beta \partial_\alpha A^\beta + \left(J^\alpha_\mu \partial_\alpha J'^\nu_\beta \right) A^\beta$$

In each case the first term is what one would expect of $\partial_\mu A_\nu$ and $\partial_\mu A^\nu$ were tensors of type $\binom{0}{2}$ and $\binom{1}{1}$ respectively.

$$J^\alpha_\mu \partial_\alpha J^\beta_\nu = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x'^\nu} \right) = \frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\nu} \neq 0$$

→ $\partial_\mu A_\nu$ and $\partial_\mu A^\nu$ are not tensors!

Consider two points P and Q with coordinates x^μ and $x^\mu + \delta x^\mu$ respectively, and a vector $A^\nu(x)$ defined along the curve joining them.
From Taylor's Theorem:

$$A^\nu(x+\delta x) = A^\nu(x) + \delta x^\beta \partial_\beta A^\nu(x) + O(2)$$
$$= A^\nu(x) + \delta A^\nu(x) + O(2)$$

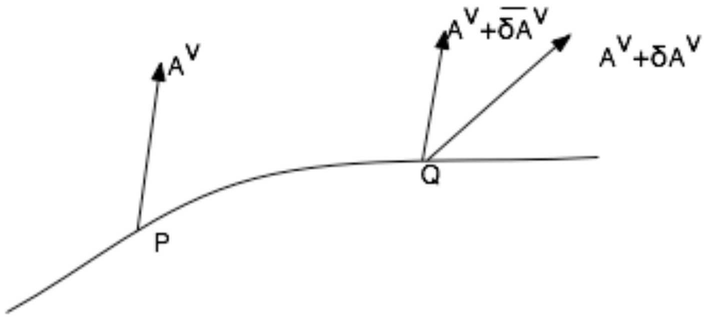
Ignore $O(2)$ → $\delta A^\nu = \delta x^\beta \partial_\beta A^\nu = A^\nu(x+\delta x) - A^\nu(x)$, which is the difference between tensors defined at two distinct points on the manifold.

Under a coordinate transformation:

$$A'^\nu(x+\delta x) = J'^\nu_\alpha(x+\delta x) A^\alpha(x+\delta x)$$
$$A'^\nu(x) = J'^\nu_\alpha(x) A^\nu(x)$$

→ clearly $A^{\nu}(x+\delta x)-A^{\nu}(x)$ is not a vector defined at P.

In order to define a tensorial derivative we must compare vectors at the same point in the manifold. Hence we introduce the concept of parallel transport via $\overline{\delta A}^{\mu}$.



The vector A^{ν} evaluated at Q is $A^{\nu}+\delta A^{\nu}$ parallelly transported to Q is $A^{\nu}+\overline{\delta A}^{\nu}$, where $\overline{\delta A}^{\nu}=-\Gamma^{\nu}{}_{\alpha\beta}A^{\beta}\delta x^{\alpha}$, and $\Gamma^{\nu}{}_{\kappa\beta}$ is known as the connection. It parameterizes the effects of parallel transport.
Now we define the covariant derivative to be:

$$\nabla_{\mu}A^{\nu}(x)=\lim_{\delta x^{\mu}\to 0}\left\{\frac{A^{\nu}(x+\delta x)-\left[A^{\nu}(x)+\overline{\delta A}(x)\right]}{\delta x^{\mu}}\right\}$$

$$=\frac{\delta x^{\beta}\partial_{\beta}A^{\nu}+\Gamma^{\nu}_{\alpha\beta}A^{\beta}\delta x^{\alpha}}{\delta x^{\mu}}$$

$$=\partial_{\mu}A^{\nu}+\Gamma^{\nu}_{\mu\beta}A^{\beta}$$

We demand that :

1. $\nabla_{\mu}A^{\nu}$ is a tensor of type $\binom{1}{1}$
2. Linearity: $\nabla_{\mu}\left(\alpha A^{\nu}\to\beta B^{\nu}\right)=\alpha\nabla_{\mu}A^{\nu}+\beta\nabla_{\mu}B^{\nu}$
3. Liebnitz rule holds:
   $$\nabla_{\mu}\left(A^{\nu}B^{\gamma}\right)=B^{\gamma}\left(\nabla_{\mu}A^{\nu}\right)+A^{\nu}\left(\nabla_{\mu}B^{\gamma}\right)$$
   $$\nabla_{\mu}\left(A^{\nu}B_{\gamma}\right)=B_{\gamma}\left(\nabla_{\mu}A^{\nu}\right)+A^{\nu}\left(\nabla_{\mu}B_{\gamma}\right)$$

Consider $\phi=A_{\nu}B^{\nu}$ to be a scalar for all vectors B^{ν}.

$$\partial_{\mu}\phi=B^{\nu}\partial_{\mu}A_{\nu}+A_{\nu}\partial_{\mu}B^{\nu}$$

$$\partial_{\mu}\phi=\nabla_{\mu}\phi$$

$$\nabla_{\mu}\phi=B^{\nu}\nabla_{\mu}A_{\nu}+A_{\nu}\nabla_{\mu}B^{\nu}$$

$$=B^{\nu}\nabla_{\mu}A_{\nu}+A_{\nu}\left\{\partial_{\mu}B^{\nu}+\Gamma^{\nu}_{\mu\beta}B^{\beta}\right\}$$

$$=B^{\nu}\left\{\nabla_{\mu}A_{\nu}+\Gamma^{\alpha}_{\mu\nu}A_{\alpha}\right\}+A_{\nu}\partial_{\mu}B^{\nu}$$

→ $B^{\nu}\left[\nabla_{\mu}A_{\nu}-\partial_{\mu}A_{\nu}+\Gamma^{\alpha}_{\mu\nu}A_{\alpha}\right]=0$

→ $\nabla_{\mu}A_{\nu}=\partial_{\mu}A_{\nu}-\Gamma^{\alpha}_{\mu\nu}A_{\alpha}$

Similarly, one can deduce:

$$\nabla_\mu A^{\alpha_1\ldots\alpha_a}{}_{\beta_1\ldots\beta_b} = \partial_\mu A^{\alpha_1\ldots\alpha_a}{}_{\beta_1\ldots\beta_b}$$
$$+\sum_{c=1}^{a}\Gamma^{\alpha_c}_{\mu\gamma}A^{\alpha_1\ldots\alpha_{c-1}\gamma\alpha_{c+1}\ldots\alpha_a}{}_{\beta_1\ldots\beta_b}$$
$$-\sum_{c=1}^{a}\Gamma^{\gamma}_{\mu\beta_c}A^{\alpha_1\ldots\alpha_{c-1}}{}_{\beta_1\ldots\beta_{c-1}\gamma\beta_{c+1}\ldots\beta_b}$$

for a tensor of type $\binom{a}{b}$.

e.g.:

1. $\binom{0}{2}$ tensor $A_{\alpha\beta}$:
   $$\nabla_\mu A_{\alpha\beta} = \partial_\mu A_{\alpha\beta} - \Gamma^\gamma_{\mu\alpha}A_{\gamma\beta} - \Gamma^\gamma_{\mu\beta}A_{\alpha\gamma}$$
2. $\binom{2}{0}$ tensor $A^{\alpha\beta}$:
   $$\nabla_\mu A^{\alpha\beta} = \partial_\mu A^{\alpha\beta} + \Gamma^\alpha_{\mu\gamma}A^{\gamma\beta} - \Gamma^\beta_{\mu\gamma}A^{\alpha\gamma}$$
3. $\binom{1}{1}$ tensor $A^\alpha{}_\beta$:
   $$\nabla_\mu A^\alpha{}_\beta = \partial_\mu A^\alpha{}_\beta + \Gamma^\alpha_{\mu\gamma}A^\gamma{}_\beta - \Gamma^\gamma_{\mu\beta}A^\alpha{}_\gamma$$

Since we have demanded the covariant differentiation is tensorial, we can deduce a transformation law for $\Gamma^\mu_{\alpha\beta}$.

Recall: $\partial'_\mu A'_\nu = J^\alpha_\mu J^\beta_\nu \partial_\alpha A_\beta + \left(J^\alpha_\mu \partial_\alpha J^\beta_\nu\right)A_\beta$

If $\nabla_\mu A_\nu$ is a $\binom{0}{2}$ tensor than

$$\nabla'_\mu A'_\nu = J^\alpha_\mu J^\beta_\nu \nabla_\alpha A_\beta = J^\alpha_\mu J^\beta_\nu\left(\partial_\alpha A_\beta - \Gamma^\gamma_{\alpha\beta}A_\gamma\right)$$
$$\nabla'_\mu A'_\nu = \partial'_\mu A'_\nu - \Gamma'^\gamma_{\mu\nu}A'_\gamma$$
$$= J^\alpha_\mu J^\beta_\nu \partial_\alpha A_\beta + \left(J^\alpha_\mu \partial_\alpha J^\beta_\nu\right)A_\beta - \Gamma'^\rho_{\mu\nu}J^\gamma_\rho A_\gamma$$

Hence we have

$$-J^\alpha_\mu J^\beta_\nu \Gamma^\gamma_{\alpha\beta}A_\gamma = \left(J^\alpha_\mu \partial_\alpha J^\gamma_\nu\right)A_\gamma - \Gamma'^\rho_{\mu\nu}J^\gamma_\rho A_\gamma$$

We can cancel off the A_γ's:

$$-J^\alpha_\mu J^\beta_\nu \Gamma^\gamma_{\alpha\beta} = J^\alpha_\mu \partial_\alpha J^\gamma_\nu - \Gamma'^\rho_{\mu\nu}J^\gamma_\rho$$

Hence:

$$\Gamma'^\rho_{\mu\nu} = J'^\rho_\gamma J^\alpha_\mu J^\beta_\nu \Gamma^\gamma_{\alpha\beta} + J'^\rho_\gamma J^\alpha_\mu \partial_\alpha J^\gamma_\nu$$

A connection which satisfies this transformation law is known as an affine connection.

Note:

1. $\Gamma^\mu_{\alpha\beta}$ is not a tensor of type $\binom{1}{2}$.
2. If we define the torsion $T^\mu_{\alpha\beta} = \Gamma^\mu_{[\alpha\beta]} = \frac{1}{2}\left[\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}\right]$ then:
   $$T'^\mu_{\alpha\beta} = J'^\mu_\rho J^\sigma_\alpha J^\lambda_\beta T^\rho_{\sigma\lambda} + J'^\mu_\rho J\partial_\sigma J^\rho_\beta - J'^\mu_\rho J^\sigma_\beta \partial_\sigma J^\rho_\alpha$$
   $$= J'^\mu_\rho J^\sigma_\alpha J^\lambda_\beta T^\rho_{\sigma\lambda}$$
   → $T^\mu_{\alpha\beta}$ is a tensor.

If $T^{\mu}_{\alpha\beta} = 0$ in one frame then it is so in all frames, and the connection symmetric and said to be torsion free.

3.2 Affine Geodesics

Consider a curve parameterized by u with tangent vector $T^{\mu} = \frac{dx^{\mu}}{du}$, which connects two points P $\left(x^{\mu}(u)\right)$ and Q $\left(x^{\mu}(u+\delta u)\right)$ separated by δu. An affine geodesic is a curve defined by the parallelly transported tangent vector $\left(T^{\mu}(u) - \Gamma^{\mu}_{\alpha\beta}T^{\beta}\delta x^{\alpha}\right)$ from P $\left(T^{\mu}(u)\right)$ being parallel to that defined at Q $\left(T^{\mu}(u+\delta u)\right)$.

$\delta x^{\alpha} = T^{\mu}\delta u$

Through Taylor expansion, $T^{\mu}(u+\delta u) = T^{\mu}(u) + \delta u \frac{dT^{\mu}}{du} + O(2)$.

Therefore, the curve between P and Q is an affine geodesic if:

$$T^{\mu}(u) + \frac{dT^{\mu}}{du}\delta u = (1+\lambda\delta u)\left[T^{\mu}(u) - \Gamma^{\mu}_{\alpha\beta}T^{\alpha}(u)T^{\beta}(u)\delta u\right]$$

The coefficient $(1+\lambda\delta u)$ is 1 if $\delta u = 0$.

→ $\frac{dT^{\mu}}{du} + \Gamma^{\mu}_{\alpha\beta}T^{\alpha}T^{\beta} = \lambda T^{\mu}$

if one ignores $(\delta u)^2$.

Now $T^{\mu} = \frac{dx^{\mu}}{du}$, therefore:

$$\frac{d^2x^{\mu}}{du^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{du}\frac{dx^{\beta}}{du} = \lambda\frac{dx^{\mu}}{du}$$

Also:

$$\begin{aligned}
T^{\nu}\nabla_{\nu}T^{\mu} &= T^{\nu}\left\{\partial_{\nu}T^{\mu} + \Gamma^{\mu}_{\nu\alpha}T^{\alpha}\right\} \\
&= \frac{dx^{\nu}}{du}\frac{\partial T^{\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\alpha}T^{\nu}T^{\alpha} \\
&= \frac{dT^{\nu}}{du} + \Gamma^{\mu}_{\nu\alpha}T^{\nu}T^{\alpha}
\end{aligned}$$

→ the affine geodesic equation can be written as

$T^{\nu}\nabla_{\nu}T^{\mu} = \lambda T^{\mu}$.

It is said to be affinely parameterized if $\lambda = 0$. Then u is an affine parameter.

Let us now generalize this concept to an arbitrary vector. If A^{μ} is a vector then it is parallelly transported along the curve if $T^{\nu}\nabla_{\nu}A^{\mu} = 0$, i.e. $\frac{dA^{\mu}}{du} + \Gamma^{\mu}_{\alpha\beta}T^{\alpha}A^{\beta} = 0$.

→ the value of $A^{\mu}(u)$ is the solution of the above differential equation along the curve with initial conditions $A^{\mu}(0)$.

Consider the inner product of $d(T, A)$ of a parallelly transported vector A^{μ} along some curve which is affinely parameterized.

i.e. $T^{\nu}\nabla_{\nu}T^{\mu}=0$ and $T^{\nu}\nabla_{\nu}A^{\mu}=0$.

$T^{\nu}\nabla_{\nu}\left(d\left(T,A\right)\right)=T^{\nu}\nabla_{\nu}\left(g_{\alpha\beta}T^{\alpha}A^{\beta}\right)$

$=T^{\alpha}A^{\beta}T^{\nu}\nabla_{\nu}g_{\alpha\beta}+g_{\alpha\beta}A^{\beta}\left(T^{\nu}\nabla_{\nu}T^{\alpha}\right)+g_{\alpha\beta}T^{\alpha}\left(T^{\nu}\nabla_{\nu}A^{\beta}\right)$

The second term is 0 because it is geodesic, while the third term is 0 because it is parallelly transported.

→ the inner product (i.e. norms and angles) is preserved along a geodesic if the metric is covariently conserved, i.e. $\nabla_{\nu}g_{\alpha\beta}=0$.

Note that parallel propagation is path dependent.
e.g. take a sphere. Consider two points P and Q on the equator, with R being the north pole. The points P and Q are separated by some angle θ, and can be individually joined to R along the surface by a “great circle”.
PQR is pointing upwards at P, upwards at Q, and horizontally backwards (1) at R.
PR is pointing upwards at P, and at an angle (2) at R.
The angle between (1) and (2) is θ.