PC4771 – Gravitation – Lectures 5 & 6

2.5 Tangent Vectors

Consider two curves, $X_1^{\mu}(u)$ and $X_2^{\mu}(u)$

The curves are tangent at $u = u_0$ if:

1. They meet. $X_1^{\mu}(u_0) = X_2^{\mu}(u_0)$
2. They touch, i.e. they don't cross. $\frac{dX_1^{\mu}}{du}(u_0) = \frac{dX_2^{\mu}}{du}(u_0)$

The tangent <u>vector</u> to the curve at some point P with $u = u_0$ is defined to be

$$T^{\mu} = \frac{dX^{\mu}}{du}.$$

A congruence is defined to be a set of curves such that each point in the manifold lies on exactly one curve, e.g. lines of latitude on a sphere S^2.

In another coordinate system,

$$T'^{\mu} = \frac{dX'^{\mu}}{du} = \frac{\partial X'^{\mu}}{\partial X^{\nu}}\frac{dX^{\nu}}{du} = J'^{\mu}_{\nu}T^{\nu}$$

The tangent vectors "live" in the tangent space $T_p(M)$. (P stands for plane, M for manifold). Note that this tangent space depends on where you are on the manifold.

2.6 Vectors, Co-Vectors and Tensors

Any n-tuple (array of numbers) which transform like a tangent vector is a member of $T_p(M)$ and is called a vector. i.e. anything which transforms according to the rules

$$A'^{\mu} = J'^{\mu}_{\nu}A^{\nu} \text{ and } A^{\mu} = J^{\mu}_{\nu}A'^{\nu}$$

is a vector. It can, and generally does, depend on position.

NB: vectors are defined at a point in M and can be thought of as a map $T_P(M) \rightarrow \Re$.

If A^{μ} and B^{μ} are vectors defined at some point P, then consider adding those vectors together $C^{\mu} = \alpha A^{\mu} + \beta B^{\mu}$:

$C'^{\mu} = \alpha A'^{\mu} + \beta B'^{\mu}$ by definition.

$C'^{\mu} = \alpha J'^{\mu}_{\nu}A^{\nu} + \beta J'^{\mu}_{\nu}B^{\nu}$

As both vectors are at the same point, both J'^{μ}_{ν}'s are the same.

$$\begin{aligned} C'^{\mu} &= J'^{\mu}_{\nu}\left[\alpha A^{\nu} + \beta B^{\nu}\right] \\ &= J'^{\mu}_{\nu}C^{\nu} \end{aligned}$$

Hence C^{μ} is also a vector at P.

Geometrically, we have shown that the tangent space $T_P(M)$ is a vector space (i.e. a space which allows the linear addition of vectors within in).

If $\phi = A_{\mu}B^{\mu}$ is a scalar for all vectors B^{μ}, then A_{μ} is a covector.

NB: in many books, including D'Inverno, covectors are called covariant vectors, and vectors are called contravarient vectors.

The question is how does A_{μ} transform under coordinate rotation?

We know that $\phi' = \phi$ because ϕ is a scalar.

$A'_\mu B'^\mu = \phi' = \phi = A_\mu B^\mu$

$A'_\mu J'^\mu_\alpha B^\alpha = A_\alpha B^\alpha$

$\left(A'_\mu J'^\mu_\alpha - A_\alpha\right) B^\alpha = 0$

As this is true for all vectors B^α, we can use

$$B^\alpha = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots$$

hence:

$A'_\mu J'^\mu_\alpha - A_\alpha = 0$

→ $A'_\mu = J^\alpha_\mu A_\alpha$

It is easy to show that:

1. $g_{\mu\nu} A^\nu$ is a covector
2. $g^{\mu\nu} B_\nu$ is a vector

Hence we call $A_\mu = g_{\mu\nu} A^\nu$ and $B^\mu = g^{\mu\nu} B_\nu$. So the metric and its inverse can be used to raise and lower indices.

The vectors live in the tangent space $T_P(M)$ of the manifold M, and the covectors live in the dual-space $T_P^*(M)$. This space is geometrically orthogonal to $T_P(M)$.

Note: vectors and covectors are defined as some point in the manifold. One can add vectors together if they are defined at the same point since $T_P(M)$ and $T_P^*(M)$ are vector spaces. But one cannot do this if they are defined at different points.

We now generalize and define an $\binom{a}{b}$ tensor at some point P in M to be a map

$$\underbrace{T_P(\mu) \times \ldots \times T_P(\mu)}_{a} \times \underbrace{T_P^*(\mu) \times \ldots \times T_P^*(M)}_{b} \to R$$

which transforms according to:

$$A'^{\mu_1 \ldots \mu_a}{}_{\nu_1 \ldots \nu_b} = \underbrace{J'^{\mu_1}_{\alpha_1} \ldots J'^{\mu_a}_{\alpha_a}}_{a} \underbrace{J^{\beta_1}_{\nu_1} \ldots J^{\beta_b}_{\nu_b}}_{b} A^{\alpha_1 \ldots \alpha_a}{}_{\beta_1 \ldots \beta_b}$$

e.g. special cases:

- Vectors are $\binom{1}{0}$ tensors
- Covectors are $\binom{0}{1}$ tensors
- Scalars are $\binom{0}{0}$ tensors
- $g_{\mu\nu}$ is a $\binom{0}{2}$ tensor
- $g^{\mu\nu}$ is a $\binom{2}{0}$ tensor
- δ^μ_ν is a $\binom{1}{1}$ tensor.

Important points:

1. If $A^{\mu_1 \dots \mu_a}{}_{\nu_1 \dots \nu_a} = B^{\mu_1 \dots \mu_a}{}_{\nu_1 \dots \nu_a}$ is true in one coordinate system or frame, then it is true in all.
2. If $A^{\mu_1 \dots \mu_a}{}_{\nu_1 \dots \nu_a} = 0$ in one frame, it is zero in all.
3. Two terms of the same "type" can be added together at some point P in M. If A and B are $\binom{2}{0}$ tensors, then $C^{\mu\nu} = A^{\mu\nu} + B^{\mu\nu}$ is a tensor of type $\binom{2}{0}$.
   Proof:
   $$C'^{\mu\nu} = A'^{\mu\nu} + B'^{\mu\nu}$$
   $$= J'^{\mu}_{\alpha} J'^{\nu}_{\beta} A^{\alpha\beta} + J'^{\mu}_{\alpha} J'^{\nu}_{\beta} B^{\alpha\beta}$$
   This is true because $A^{\mu\nu}$ and $B^{\mu\nu}$ are tensors.
   $$C'^{\mu\nu} = J'^{\mu}_{\alpha} J'^{\nu}_{\beta} \left(A^{\alpha\beta} + B^{\alpha\beta} \right)$$
   $$= J'^{\mu}_{\alpha} J'^{\nu}_{\beta} C^{\alpha\beta}$$
   → $C^{\mu\nu}$ is a $\binom{2}{0}$ tensor.
4. If $C = AB$, where A is a tensor of type $\binom{a}{b}$ and B is a tensor of type $\binom{c}{d}$, then C is a tensor of type $\binom{a+c}{b+d}$.
   $$C^{\mu_1 \dots \mu_a \alpha_1 \dots \alpha_c}{}_{\nu_1 \dots \nu_a \beta_1 \dots \beta_d} = A^{\mu_1 \dots \mu_a}{}_{\nu_1 \dots \nu_b} B^{\alpha_1 \dots \alpha_c}{}_{\beta_1 \dots \beta_d}$$
   See Questions Sheet 2, Number 6.
5. A tensor of type $\binom{a}{b}$ becomes of type $\binom{a-2}{b}$ upon contraction with the metric, and of type $\binom{a}{b-2}$ when contracted with the inverse metric.
   e.g. $T^{\mu\nu\rho}{}_{\alpha\beta}$ which is a $\binom{3}{2}$ tensor, then $g_{\mu\nu} T^{\mu\nu\rho}{}_{\alpha\beta}$ is of type $\binom{1}{2}$ and $g^{\alpha\beta} T^{\mu\nu\rho}{}_{\alpha\beta}$ is of type $\binom{3}{0}$.
6. One can define the inner product of two vectors T^{μ} and S^{ν} to be $d(S,T) = S_{\mu} T^{\mu} = g_{\mu\nu} T^{\mu} S^{\nu}$ which is a scalar.
   The length of a vector is $\|T\| = \left(d(T,T) \right)^{1/2}$, and the angle between S and T is defined by $\cos\theta = \dfrac{d(T,S)}{\|T\| \|S\|}$.

<u>Notation</u>

Symmetrization and antisymmetrization

We will from time to time use the permutation of indices:

( ) will mean Symmetrization

[ ] will mean antisymmetrization.

e.g.:

1. $\binom{0}{2}$ tensor $A_{\mu\nu}$:
   $$A_{(\mu\nu)} = \frac{1}{2} \left(A_{\mu\nu} + A_{\nu\mu} \right)$$
   $$A_{[\mu\nu]} = \frac{1}{2} \left(A_{\mu\nu} - A_{\nu\mu} \right)$$

2. $\binom{0}{3}$ tensor $A_{\mu\nu\rho}$

$$A_{(\mu\nu\rho)} = \frac{1}{3!}\left[A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} + A_{\nu\mu\rho} + A_{\rho\nu\mu} + A_{\mu\rho\nu}\right]$$

$$A_{[\mu\nu\rho]} = \frac{1}{3!}\left[A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\nu\mu\rho} - A_{\rho\mu\nu} - A_{\mu\rho\nu}\right]$$

(Cyclic permutations, minus the anticyclic permutations)

Consider a few examples:

1.
$$A_{(\mu\nu\rho)}x^\mu x^\nu x^\rho = \frac{1}{3!}\left[A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} + A_{\mu\rho\nu} + A_{\nu\mu\rho} + A_{\rho\nu\mu}\right]x^\mu x^\nu x^\rho$$
$$= A_{\mu\nu\rho}x^\mu x^\nu x^\rho$$

2. $A_{[\mu\nu\rho]}x^\mu x^\nu x^\rho = 0$

3.
$$A_{(\mu\nu)}A^{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)\frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right)$$
$$= \frac{1}{4}A_{\mu\nu}A^{\mu\nu} + A_{\nu\mu}A^{\mu\nu} - A_{\mu\nu}A^{\nu\mu} - A_{\nu\mu}A^{\nu\mu}$$
$$= 0$$

2.7 Conformal Metrics

Two metrics $g_{\mu\nu}$ and $G_{\mu\nu}$ are conformally related if there exists a possibly spatially dependant function $\Omega\left(X^{\mu\nu}\right)$ such that

$$G_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

This is of interest because:

1. $ds_g{}^2 = g_{\mu\nu}dx^\mu dx^\nu$

   $ds_G{}^2 = G_{\mu\nu}dx^\mu dx^\nu$

   Hence $ds_G{}^2 = \Omega^2 ds_g{}^2$

   → the null curves $\left(ds^2 = 0\right)$ of both metrics are the same.

2. Consider two vectors A^μ and B^μ, then the angle between them is

$$\cos\theta_g = \frac{g_{\mu\nu}A^\mu B^\nu}{\left(g_{\mu\nu}A^\mu A^\nu\right)^{1/2}\left(g_{\mu\nu}B^\mu B^\nu\right)^{1/2}}$$

$$\cos\theta_G = \frac{G_{\mu\nu}A^\mu B^\nu}{\left(G_{\mu\nu}A^\mu A^\nu\right)^{1/2}\left(G_{\mu\nu}B^\mu B^\nu\right)^{1/2}}$$

   → $\cos\theta_G = \cos\theta_g$

   So conformal transformations preserve angles.