PC4771 – Gravitation – Lectures 3&4

Einstein lift experiment. Observer O in a lift, with light L_1 going vertically down, L_2 going horizontal. Observer O' outside the lift. Cut the lift wire – lift accelerates wrt O'

Time taken for photons emitted by L to reach O is $\delta t = \frac{\delta \ell}{c}$. Standard choice of units: set $c = 1$. Now cut the wire.

A. Consider the light emitted by L_1

The lift is in a freely-falling frame (fff) – which means that O experiences special relativity, i.e. since the EEP holds it is also LIF (Locally Inertial Frame). Therefore no change in frequency is observed by O. However, O' will see the lift moving with speed $\delta u = g\delta t = g\delta\ell$. So the Doppler shift formula for small speeds (for small δu) implies that there is a change in frequency

$$\frac{\delta f}{f} = \delta u = g\delta\ell .$$

If $g = -\frac{\delta\phi}{\delta\ell}$, i.e. the gravitational force is the gradient of the gravitational potential, then

$$\frac{\delta f}{f} = -\delta\phi .$$

This is known as the Gravitational Redshift Formula.

B. Now consider the light emitted by L_2

EEP → lift is in LIF. The coordinates of the light in the lift are

$$(x_\ell, y_\ell) = (t, d),$$

i.e. O sees a straight line.

Coordinates relative to O' on the ground will be

$$\begin{aligned}(x_g, y_g) &= \left(x_\ell, y_\ell - \frac{1}{2}gt^2\right) \\ &= \left(t, d - \frac{1}{2}gt^2\right)\end{aligned}$$

$$y_g = d - \frac{1}{2}g(x_y)^2$$

So the view from the ground is that the photon has a parabolic (i.e. curved) path. But Fermat's principle asserts that the light must take the shortest path between two points – i.e. it extremises

$$S = \int_{x_1}^{x_2} \frac{ds}{dx}dx,$$

where $ds = d\tau$, the proper time.

In Minkowski space, the geodesics (i.e. the paths which extremise S) are straight lines. So one must either drop:

1. Fermat's Principle
2. Minkowski space-time

1.6 Einsteinian Vision of GR
Einstein's choice is to drop Minkowski space-time. He states:

1. Space-time is a psudo-Riemanian manifold with a position dependant metric.
→ the internal of special relativity is given by $ds^2 = g_{\mu\nu}\left(x^\mu\right)dx^\mu dx^\nu$
We will always be able to choose coordinates to make space locally flat, so geodesics are locally straight lines. Only when you talk about very large distances do you see this curvature.
We can classify geodesics as time-like $ds^2 > 0$ (particles), null $ds^2 = 0$ (light) or space-like $ds^2 < 0$ (superluminal particles, which generally do not exist).

2. Energy-momentum (and hence mass) is what defines the curvature of space-time globally.

2. Manifolds & Metrics
2.1 Manifolds
Basic idea is that a manifold is a set of pieces which "look like" open subsets of R^n, but which can be "sewn together" smoothly.
e.g. the maps of the earth analogy – maps of local areas will be R^2, which are locally flat but which can be "sewn together" to make a map of the whole planet (which will be R^3).

Examples:
1. R^n: R^1 is a line, R^2 is a plane, etc.
2. S^n: S^1 is a circle, S^2 sphere, etc.

2.2 Curves and Surfaces
e.g. S^1 in R^2 (a curve), S^2 in R^3 (a surface).

<u>Parametric</u>
A curve is defined by a single parameter u and coordinates within the manifold are $X^\mu = X^\mu\left(u\right)$ where $u = 1,...,n$ (where n is the dimension of the manifold).

E.g. S^1 in R^2:
$x = \cos\theta$
$y = \sin\theta$
θ is the parameter.

A surface is an m-dimensional sub-manifold defined by m parameters $u_1,...,u_m$. The coordinates are $X^\mu = X^\mu\left(u^1,...,u^m\right)$.
$x = \sin\theta\cos\phi$
$y = \sin\theta\sin\phi$
$z = \cos\theta$
θ and ϕ are the parameters.

If $m = n-1$ then the surface is called a "hypersurface".

Constraints

By eliminating the parameters an m-dimensional surface can be defined in terms of $n-m$ constraints.

$f'\left(x^1,...,x^n\right)=0$

…

$f^{(n-m)}\left(x^1,...,x^n\right)=0$

e.g. S^1 in R^2: $x^2+y^2=1$

S^2 in R^3: $x^2+y^2+z^2=1$

2.3 Coordinate Transformations

Consider some coordinates $X'^{\mu}=X'^{\mu}\left(x^1,...,x^n\right)$

$$dx'^{\mu}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}dx^{\nu}$$

$$dx^{\mu}=\frac{\partial x^{\mu}}{\partial x'^{\nu}}dx^{\nu}$$

This is by the chain rule.

Define the Jacobeans:

$$J'^{\mu}_{\ \nu}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

$$J^{\mu}_{\nu}=\frac{\partial x^{\mu}}{\partial x'^{\nu}}$$

Then:

$$dx'^{\mu}=J'^{\mu}_{\ \nu}dx^{\nu}$$

$$dx^{\mu}=J^{\mu}_{\nu}dx'^{\nu}$$

Define the Kroenecker Delta $\delta^{\mu}{}_{\nu}=\begin{cases}1 & \mu=\nu\\ 0 & \mu\neq\nu\end{cases}$

Then $J^{\mu}_{\nu}J'^{\nu}_{\ \rho}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial x'^{\rho}}=\frac{\partial x'^{\mu}}{\partial x'^{\rho}}=\delta^{\mu}{}_{\nu}$.

i.e. J and J' are inverses of each other: $JJ'=I$.

e.g. R^2:

$\left(x^1,x^2\right)=(x,y)$

$\left(x'^1,x'^2\right)=(r,\theta)$

$x=r\cos\theta$

$y=r\sin\theta$

$\tan\theta=\frac{y}{x}$

$r=\sqrt{x^2+y^2}$

$$\frac{dx}{dr} = \cos\theta \qquad \frac{dy}{dr} = \sin\theta$$
$$\frac{dx}{d\theta} = -r\sin\theta \quad \frac{dy}{d\theta} = r\cos\theta$$
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

2.4 Metric & Line Element

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$

$\mu = 0,1,2,3$

Implicitly throughout the course, assume $g_{\mu\nu} = g_{\mu\nu}(x^\mu)$, i.e. depends on position.

$g_{\mu\nu}$ is the metric, ds^2 is the line element.

NB: by this definition $g_{\mu\nu}$ is symmetric.

Under coordinate transformations, $dx^\mu = J^\mu_\nu dx'^\nu$ and $dx'^\mu = J'^\mu_\nu dx^\nu$

→ $ds^2 = g_{\mu\nu} J^\mu_\alpha dx'^\alpha J^\nu_\beta dx'^\beta = g'_{\alpha\beta} dx'^\alpha dx'^\beta$

→ $g'_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}$

Or in index notations, $g' = JgJ^T$

e.g. 2D Euclidian to plane polars:

2D Euclidian $ds^2 = dx^2 + dy^2$ → $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

NB: $J = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$ from before.

$$J' = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

→ $ds^2 = dr^2 + r^2 d\theta^2$

(Simpler way for this case:

$x = r\cos\theta$ → $dx = dr\cos\theta - r\sin\theta d\theta$

$y = r\sin\theta$ → $dy = dr\sin\theta + r\cos\theta d\theta$

$dx^2 + dy^2 = dr^2 + r^2 d\theta^2$)

What does the metric mean? It defines the measure of distances within the manifold.
e.g. $ds^2 = a^2(t)\left[dx^2 + dy^2\right]$ which governs the expansion of a 2D universe.

Define $g^{\mu\nu}$ as the inverse of the metric, such that $g_{\mu\nu} g^{\nu\beta} = \delta_\mu{}^\beta$, or in matrix representation $gg^{-1} = I$.

e.g. in plane polars

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \text{ and } g^{-'} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

Note that we have assumed that the determinant $\det\left(g_{\mu\nu}\right) \neq 0$.

Consider the line element between two points P with coordinates x^μ and Q with coordinates $x^\mu + dx^\mu$.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \begin{pmatrix} dx^1 & \cdots & dx^n \end{pmatrix} \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix}$$

$$= d\underline{x}^\dagger g d\underline{x}$$

g is a symmetric matrix at each point in the manifold. At each point, the matrix g can be diagonalised. Note that this cannot be done globally in general.

$g = P^T \Lambda P$

where P is the rotation matrix ($PP^T = I$) and Λ is the diagonal matrix

→ $ds^2 = dx^T P^T \Lambda P dx = \left(P d\underline{x}\right)^T \Lambda P d\underline{x}$

Redefine $d\underline{u} = P d\underline{x}$ then $ds^2 = d\underline{u}^T \Lambda d\underline{u}$

Moreover, one can rescale coordinates

$\underline{u} \to \alpha \underline{u}$, $\alpha = \dfrac{1}{\sqrt{\lambda_i}}$

→ $g = diag(1,...,1,-1,...,-1,0,...,0)$

→ locally spacetime is flat.

If s is the number of $+1$'s, and t is the number of -1's, then $s-t$ is known as the signature of the metric, and $s+t$ is its rank.

If $t = 0$, the metric is Euclidean or Riemannian.

If $s = 1$, the metric is Lorentzian or pseudo-Riemannian.

$ds^2 = dx^2 + dy^2$ is Riemannian.

$ds^2 = dt^2 - dx^2$ is Pseudo-Riemannian.

General Relativity is concerned with Pseudo-Riemannian spaces.

(NB: if $\underline{u} = P\underline{x}$, then if $P(\underline{x})$ $d\underline{u} \neq P d\underline{x}$, which reiterates the earlier point that this procedure can only be done locally.)

Consider the sphere.

$x = \sin\theta\cos\theta$

$y = \sin\theta\sin\phi$

$z = \cos\theta$

So $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Look at a line encircling the north pole, $\theta = \theta_0$.

Let us compute the radius, circumference and area of the "circle" $\theta = \theta_0$.

Along the radius, $d\phi = 0$ so $ds = d\theta$. So the radius is $r = \int_{\theta=0}^{\theta=\theta_0} ds = \theta_0$.
Along the circumference $d\theta = 0$, so $ds = \sin\theta_0 d\phi$. So the circumference is $C = \int_{\phi=0}^{\phi=2\pi} ds = 2\pi \sin\theta_0$.
Now consider the infinitesimal area element $dA = d\theta \sin\theta d\phi$. The area is hence $A = \int_0^{2\pi} d\phi \int_0^{\theta_0} d\theta \sin\theta = 2\pi\left(1 - \cos\theta_0\right)$.
Hence $\frac{C}{2\pi R} = \frac{\sin\theta_0}{\theta_0}$ and $\frac{A}{\pi R^2} = \frac{2\left(1-\cos\theta_0\right)}{\theta_0^{\,2}}$, i.e. they are not equal to 1 as they would in Euclidean space.

NB: $\frac{C}{2\pi R} \to 1$ and $\frac{A}{\pi R^2} \to 1$ as $\theta_0 \to 0$. This is a consequence of the space being locally flat.