$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad (1)$$
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (2)$$

Define $H = \frac{\dot{a}}{a}$.

Friedman equation becomes:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Ray-chauduri Equation becomes:

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P)$$

H is known as the Hubble parameter. $H(t_0) = H_0 \equiv$ Hubble constant $\approx 70 km\, s^{-1}\, Mpc^{-1}$.

Notes:

1. If $\rho + 3P > 0$ (i.e. the strong energy condition), then $\ddot{a} < 0$ and the universe is decelerating.
2. Conservation of energy: $\nabla^\mu T_{\mu\nu} = 0$ (3) → $\dot{\rho} + 3H(\rho + P) = 0$

However, (2) + (3) implies (1); (1) + (3) implies (2), so these equations are not independent. So we have thee unknown variables and two equations. To solve these, then we need to define something like an equation of state – a relation between the pressure and density – to give us a total of three equations.

7.4 Solution in the radiation and matter eras

For matter, $P = 0$ → $\dot{\rho} + 3H\rho = 0$ → $\rho \propto \frac{1}{a^3}$.

For radiation, $P = \frac{1}{3}\rho$ → $\dot{\rho} + 4H\rho = 0$ → $\rho \propto \frac{1}{a^4}$.

Hence:

$$\rho = \frac{\rho_m(t_0)}{a^3} + \frac{\rho_r(t_0)}{a^4}$$

where we have set $a(t_0) = 1$ and t_0 is the time at the present day.

If we define the critical density to be

$$\rho_{crit} = \frac{3H_0^{\ 2}}{8\pi G}$$
$$\Omega_X = \frac{\rho_X(t_0)}{\rho_{crit}}$$

where $X = m$ for matter, and $X = r$ for radiation, then the Freedman equation becomes:

$$H^2 = H_0^{\ 2}\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right) - \frac{k}{a^2}$$

Furthermore, define

$$\Omega_k = -\frac{k}{H_0^{\,2}}$$

such that the curvature acts dynamically like an extra component of matter with equation of state $P_k = -\frac{1}{3}\rho_k$. $\Omega_k > 0$ is called an open universe, and $\Omega_k < 0$ is a closed universe. Hence:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2}$$

with the constraint that $\Omega_m + \Omega_r + \Omega_k = 1$.

Sometimes it is sensible to work in conformal time. Defining

$$H' = \frac{a'}{a}$$

where $a' = \frac{da}{d\eta}$, we have

$$\frac{H'^2}{H_0^{\,2}} = \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_k$$

Hence,

$$\dot{a} = H_0\sqrt{\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_k}$$

$$a' = H_0\sqrt{\Omega_m a + \Omega_r + \Omega_k a^2}$$

NB: the age of the universe is given by

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}}$$

Various cases

t_{eq} is the time when the matter and radiation components are equal. $\rho_m\left(t_{eq}\right) = \rho_r\left(t_{eq}\right)$.

t_{curve} is when the curvature and matter components are equal. $\rho_k\left(t_{curve}\right) = \rho_m\left(t_{curve}\right)$.

1. Radiation domination

$t < t_{eq}$

$a \propto t^{1/2} \propto \eta$

2. Matter domination

$t_{eq} < t < t_{curve}$

$a \propto t^{2/3} \propto \eta^2$

3. Curvature domination

$t > t_{curve}$

$a \propto t \propto \exp\left(\sqrt{\Omega_k H_0^{\,2}}\,\eta\right)$

Age of the universe

$$t_0 = \int_0^1 \frac{da}{\dot{a}} = \frac{1}{H_0}\int_0^1 \frac{da}{\sqrt{\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_k}}$$

Hence,

$$H_0 t_0 = \int_0^1 \frac{a\,da}{\sqrt{\Omega_r + \Omega_r a + \Omega_k a^2}}$$

If $\Omega_m = 1$, $\Omega_r = 0$ and $\Omega_k = 0$, then $H_0 t_0 = \frac{2}{3}$ → $t_0 \approx \frac{2}{3}\frac{1}{H_0}$.

A. Radiation & Matter only

$$H_0\eta = \int_0^a \frac{da}{\sqrt{\Omega_r + \Omega_m a}} = \frac{1}{\Omega_m}\left(\sqrt{\Omega_r + \Omega_m a} - \sqrt{\Omega_r}\right)$$

$$\rightarrow\ a = \sqrt{\Omega_r H_0^{\ 2}}\,\eta + \frac{1}{4}\Omega_r H_0^{\ 2}\eta^2$$

$$\rightarrow\ t = \frac{1}{2}\sqrt{\Omega_r H_0^{\ 2}}\,\eta^2 + \frac{1}{2}\left(\Omega_r H_0^{\ 2}\right)\eta^3$$

t_{eq} is the time of equal-matter radiation

→ $1 = \frac{\rho_m}{\rho_r} = \frac{\Omega_m}{\Omega_r} a_{eq}$

→ $a_{eq} = \frac{\Omega_r}{\Omega_m}$

For $a \ll a_{eq}$, $a \propto t^{1/2}$.

For $a \gg a_{eq}$, $a \propto t^{2/3}$.

B. Curvature and Matter only

$$H_0\eta = \int_0^a \frac{da}{\sqrt{\Omega_m a + \Omega_k a^2}}$$

NB: $\Omega_m + \Omega_k = 1$

$\Omega_k = 0$ → $a = \frac{1}{4}H_0^{\ 2}\eta^2$ ($\Omega_m = 1$)

Otherwise, substitute $a + \frac{\Omega_m}{2\Omega_k} = \frac{\Omega_m}{2\Omega_k}x$.

$$H_0\eta = \int_1^{1+\frac{2\Omega_k}{\Omega_m}a} \frac{dx}{\sqrt{\Omega_k\left(x^2 - 1\right)}}$$

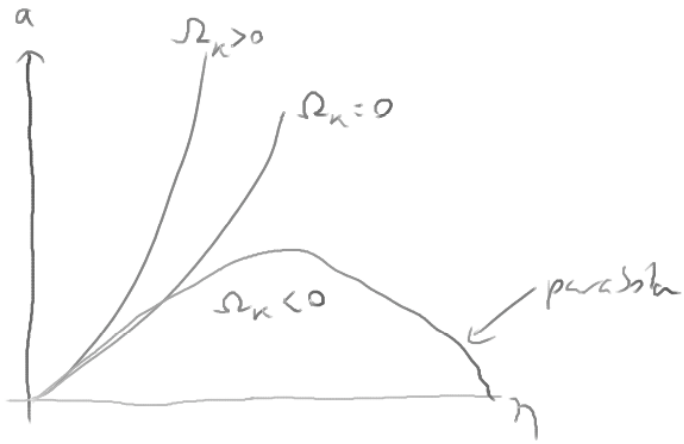
If $\Omega_k > 0$:

$$a = \frac{\Omega_m}{2\left(1-\Omega_m\right)}\left(\cosh\left(\sqrt{1-\Omega_m}\,H_0\eta\right) - 1\right)$$

$$\approx \frac{1}{4}H^2\eta^2 + O\left(\eta^3\right)$$

If $\Omega_k < 0$

$$a = \frac{\Omega_m}{2(1-\Omega_m)}\left(1-\cos\left(\sqrt{\Omega_m - 1}H_0\eta\right)\right)$$
$$\approx \frac{1}{4}H^2\eta^2 + O\left(\eta^3\right)$$



$\Omega_k > 0$: Open
$\Omega_k = 0$: Flat (Parabola)
$\Omega_k < 0$: Closed

Closed universe recollapses in a finite time $\eta = \dfrac{2\pi}{\sqrt{\Omega_m - 1}} H_0^{-1}$.

7.5 Cosmological Constant
When we derived the formulation of the Einstein equation, we ignored the possibility of a linear term in the metric which fulfilled all the relevant criteria. Einstein considered this possibility in order to create a static universe and later described it as his greatest blunder. It was shown that it was unstable to perturbations. At the present time there is evidence that it exists.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Either $G_{\mu\nu} \to G_{\mu\nu} - \Lambda g_{\mu\nu}$, i.e. a change to the geometrical formulation of GR, or

$T_{\mu\nu} \to T_{\mu\nu} + \dfrac{\Lambda}{8\pi G} g_{\mu\nu}$, i.e. a change to the matter constant.

If $T_{\mu\nu}{}^{\Lambda} = \dfrac{\Lambda}{8\pi G} g_{\mu\nu}$ then it corresponds to a fluid with $P = -\rho$, which violates the strong energy condition.

Consider a universe containing pressureless matter and Λ, which is flat $(k = 0)$.
Friedman Equation:

$$H^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3}$$

Raychauduri Equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3}.$$

→ $\frac{H^2}{H_0^{\ 2}} = \frac{\Omega_m}{a^3} + \Omega_\Lambda$ → for late times, $H = const$ → $a \propto e^{Ht}$. Also, when Λ dominates, $\frac{\ddot{a}}{a} > 0$ and the universe accelerates.

7.6 Cosmological Redshift

Remove the constraint that $a(t_0) = 1$. Consider a radial null geodesic: $ds = d\theta = d\phi = 0$, i.e. a signal of light from a distant source.

$$\to \frac{dt}{a(t)} = d\eta = \pm \frac{dr}{\sqrt{1-kr^2}}$$

where the $+$ corresponds to a receding geodesic, and the $-$ corresponds to an approaching geodesic. Consider it to be approaching us, at $r = 0$ and $t = t_0$, from $r = r_e$ and $t = t_e$.

$$\to \int_{t_e}^{t_0} \frac{dt'}{a(t')} = -\int_{r_e}^{0} \frac{dr'}{\left(1-kr^2\right)} = \begin{cases} \sin^{-1} r_e & k = +1 \\ r_e & k = 0 \\ \sinh^{-1} r_e & k = -1 \end{cases}$$

Consider two signals sent from the same point at t_e and $t_e + dt_e$, and received at t_0 and $t_0 + dt_0$. Since they are at the same coordinate distance,

$$\int_{t_e+dt_e}^{t_0+dt_0} \frac{dt'}{a(t')} = \int_{t_e}^{t_0} \frac{dt'}{a(t')}$$

$$\to \int_{t_0}^{t_0+dt_0} \frac{dt'}{a(t')} = \int_{t_e}^{t_e+dt_e} \frac{dt'}{a(t')}$$

and hence if $dt_0 << 1$,

$$\frac{dt_0}{a(t_0)} = \frac{dt_e}{a(t_e)}$$

For static observers, $dt = d\tau$,

$$\frac{d\tau_0}{d\tau_e} = \frac{a(t_0)}{a(t_e)}$$

$$\frac{\nu_e}{\nu_0} = \frac{a(t_0)}{a(t_e)}$$

If we define the redshift to be $1+z = \frac{a(t_0)}{a(t_e)}$, then:

$$\nu_0 = \frac{\nu_e}{1+z}$$

$$\lambda_0 = \lambda_e(1+z)$$

$$T_0 = \frac{T_e}{1+z}$$

The last one of these is considering it to be a black body, e.g. the CMB…

NB: $\rho_\gamma \propto T_\gamma^{\ 4} \propto (1+z)^4 \propto a^{-4}$.