$R_{tr} = 0$ → $\dot{\lambda} = 0$ → $\lambda = \lambda(r)$.
Using (1) in (3) (see handout):
→ $\nu' + \lambda' = 0$ → $\nu + \lambda = f(t)$ (constant of integration).
This constant of integration can be set to zero by allowing a re-parameterization of the time coordinate.
→ $\nu = -\lambda$.
Therefore (4) → $e^{\nu}(1 + r\nu') = 1$
→ $(re^{\nu})' = 1$
→ $re^{\nu} = r + c$
→ $e^{\nu} = 1 + \frac{c}{r}$
Hence, this is the solution.
→ $ds^2 = \left(1 + \frac{c}{r}\right)dt^2 - \left(1 + \frac{c}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
This is the Schwarzschild solution without a definition for the integration constant c.
By comparing this to the Newtonian spacetime,
$ds^2 \approx (1 + 2\phi)dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
where $\phi = -\frac{GM}{r}$, we can identify $c = -2GM$ and the solution represents the vacuum solutions to the Einstein equations outside an object of mass M.
→ $ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
$g_{\mu\nu} = diag\left(1 - \frac{2GM}{r}, \left(1 - \frac{2GM}{r}\right)^{-1}, -r^2, -r^2\sin^2\theta\right)$
We define the Schwarzschild mass to be M and the Schwarzschild radius to be $r = 2GM$.
Note the sign of g_{rr} and g_{tt} as one goes through $r = 2GM$, which will be called the Event Horizon.

Examples:

	Mass (kg)	Physical radius R (m)	r_s (m)	$\frac{r_s}{R}$
e^-	10^{-30}	2×10^{-15}	10^{-57}	5×10^{-43}
Earth	10^{24}	5×10^{6}	$0.1m$	2×10^{-9}
Sun	2×10^{30}	5×10^{8}	2×10^{3}	2×10^{-6}
Galaxy	10^{42}	10^{21}	10^{15}	10^{-6}
Neutron star	10^{30}	10^{4}	10^{3}	0.15
Black hole	-	-	-	≥ 1

6.1 Birkhoff's Theorem
Consider a spherically symmetric mass distribution in Newton's theory of gravity.
$\nabla^2\phi = 4\pi G\rho$ (satisfies Poisson equation)
F,

For $\nabla^2\phi = 4\pi G\rho$,

$$\rho(r) = \begin{cases} \rho_0 & r < R \\ 0 & r > R \end{cases}$$

→ $$\phi = \begin{cases} -\dfrac{A}{r} & r > R \\ 4\pi\displaystyle\int_0^r \dfrac{m(r)}{r^2}dr & r < R \end{cases}$$

where $m(r) = \int_0^r r^{1/2}\rho_0(r')dr$

→ $A = GM_{tot}$ where $M_{tot} = 4\pi\int_0^R r^{1/2}\rho_0(r')dr'$

→ $\phi = -\dfrac{GM_{tot}}{r}$ for $r > R$.

The exterior field is independent of the mass distribution so long as it is spherically symmetric (NB: this is also true for symerically symmetric charge distribution in EM) – this is called Newton's theorem.

We have proved a similar theorem in GR:

Birkhoff's Theorem states:
Spherically symmetric vacuum solution in the exterior of a mass distribution must be static, and the solution is the Schwarzschild solution.
c.f. last week, need quadropole to generate gravity radiation.
→ spherically symmetric oscillations of a mass distribution cannot affect the exterior gravitational field.

6.3 Dynamics in the Schwarzschild Radius

Use $L_{eff} = \left(1-\dfrac{2GM}{r}\right)dt^2 - \left(1-\dfrac{2GM}{r}\right)^{-1}dr^2 - r^2\left(\theta^2 + \sin^2\theta d\phi^2\right)$

$$\left(\frac{ds}{dt}\right)^2 = \begin{cases} 1 & timelike \\ 0 & null \\ -1 & spacelike \end{cases}$$

$\dfrac{\partial L_{eff}}{\partial \dot{t}} = 2\left(1-\dfrac{2GM}{r}\right)\dot{t}$, $\dfrac{\partial L_{eff}}{\partial t} = 0$

$\dfrac{\partial L_{eff}}{\partial \dot{r}} = -2\left(1-\dfrac{2GM}{r}\right)^{-1}\dot{r}$, $\dfrac{\partial L_{eff}}{\partial u} \neq 0$

$\dfrac{\partial L_{eff}}{\partial \dot{\theta}} = -2r^2\dot{\theta}$, $\dfrac{\partial L_{eff}}{\partial \theta} = -2r^2\cos\theta\sin\theta d\phi^2$

$\dfrac{\partial L_{eff}}{\partial \dot{\phi}} = -2r^2\dot{\phi}$, $\dfrac{\partial L_{eff}}{\partial \phi} = 0$

$\dfrac{\partial L_{eff}}{\partial t} = 0$ → 3 first integrals:

1. $\left(1-\frac{2GM}{r}\right)\dot{t}=E$

   (as $r\to\infty$, $E=\dot{t}=\gamma=\left(1-v^2\right)^{1/2}$ → Energy)

2. $r^2\phi=L$ Angular Momentum

3. $L_{eff}=const=L=\begin{cases}1 & timelike\\ 0 & null\\ -1 & spacelike\end{cases}$

→ $L=\left(1-\frac{2GM}{r}\right)\dot{t}-\left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^2-r^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)$

If $\theta=\frac{\pi}{2}$, $\dot{\theta}=0$ initially then $\ddot{\theta}=\dddot{\theta}=...=0$. So, we can set $\theta=\frac{\pi}{2}$ (basically means we're at the equator).

→ $\frac{E^2-\dot{r}^2}{1-\frac{2GM}{r}}-\frac{h^2}{r^2}=L$

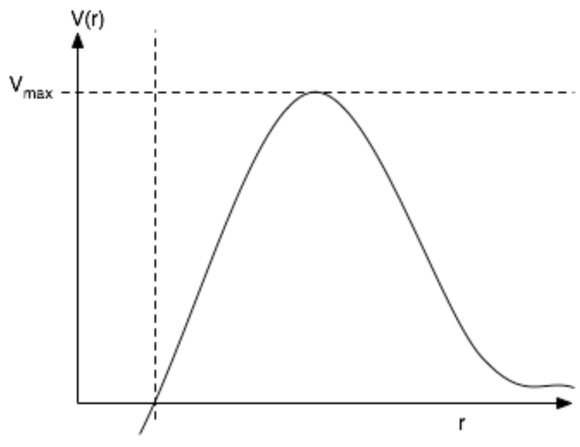
Hence,

$\dot{r}^2=E^2-V(r)$, where $V(r)=\left(L+\frac{h^2}{r^2}\right)\left(1-\frac{2GM}{r}\right)=\left(L+\frac{h^2}{r^2}\right)\left(1-\frac{r_s}{r}\right)$

Consider various cases:

(i) $L=0$, $h\neq 0$

$V(r)=\frac{h^2}{r^2}\left(1-\frac{r_s}{r}\right)$



$V_{\max}=\frac{8h^2}{27r_s^2}$

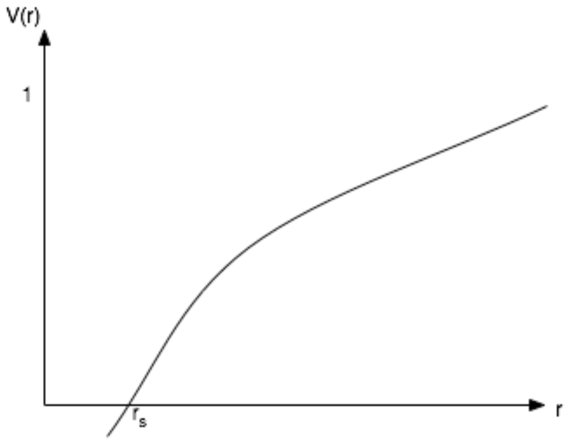
Possibilities:

1: $E^2<V_{\max}$, a photon from ∞ is repelled and one from $r\sim r_s$ moving out falls back in.

2: $E^2=V_{\max}$, an unstable circular orbit

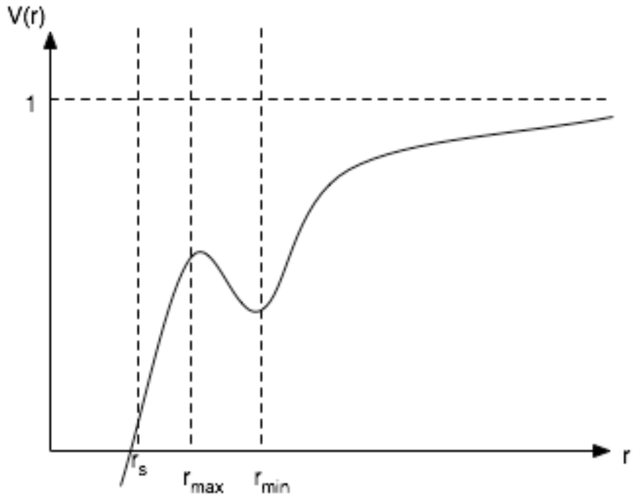
3: $E^2>V_{\max}$, photon falls in and escapes.

(ii): $L = 1$, $h < \sqrt{3} r_s$



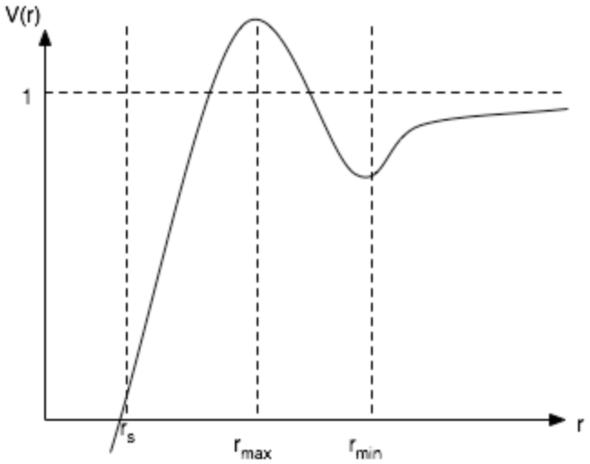
Possibilities:
1: If $E^2 < 1$, a particle from $r \sim r_s$ falls back in
2: If $E^2 > 1$, the same particle escapes.

(iii) $L = 1$, $\sqrt{3} r_s < h < 2 r_s$



Stable orbit at $r = r_{\min}$
Unstable orbit at $r = r_{\max}$

(iv) $L = 1$, $h > 2 r_s$

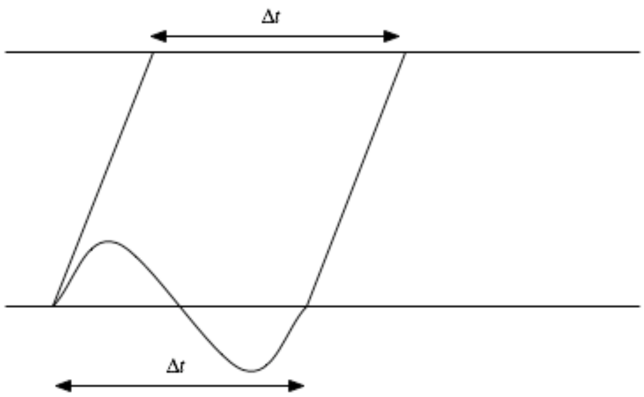


6.4 Gravitational Redshift

Consider the t-r point of the metric.

$ds^2 = g_{tt}(r)dt^2 - g_{rr}(r)dr^2$

Frequency of radiation $\nu \propto \frac{1}{\Delta t} \propto \frac{1}{ds}$



The top line is $r = r_1$, with $g_{tt}(r_1)$. The bottom line is $r = r_2$, with $g_{tt}(r_2)$.

At $r = r_1$, $\Delta\tau_1 = \sqrt{g_{tt}(r_1)}\Delta t$

At $r = r_2$, $\Delta\tau_2 = \sqrt{g_{tt}(r_2)}\Delta t$

→ $\frac{v_1}{v_2} = \frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{g_{tt}(r_2)}{g_{tt}(r_1)}}$

e.g. Schwarzschild $g_{tt} = 1 - \frac{2GM}{r}$.

$$\frac{v_1}{v_2} = \left(\frac{1 - \frac{2GM}{r_2}}{1 - \frac{2GM}{r_1}} \right)^{1/2} \approx 1 + \frac{GM}{r_1} - \frac{GM}{r_2} \approx 1 + \varphi_1 + \varphi_2$$

→ radiation is redshifted as it climbs out of gravitational potentials.

6.5 Light Deflection

Define $u = \frac{r_0}{r}$ → $\dot{u} = -\frac{\dot{r}}{r_0}u^2$; $\dot{\phi} = \frac{hu^2}{r_0^{\ 2}}$

→ $\frac{du}{d\phi} = -\frac{r_0\dot{r}}{h}$

Henc,

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2 r_0^{\ 2}}{h^2} - \frac{r_0^{\ 2}}{h^2}V(r)$$

(by substituting $\dot{r}^2 = r^2 - V(r)$)

$$\left(\frac{du}{d\phi}\right)^2 = 1 - u^2 + \varepsilon u^3$$

where $r_0 = \frac{h}{E}$, $\varepsilon = \frac{r_s}{r_0}$

NB: $\varepsilon \begin{cases} = 0 & \textit{solution is easy} \\ \neq 0 & \textit{solution is difficult} \end{cases}$

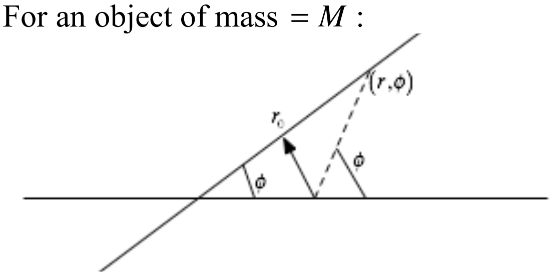
We will solve the equation as a power series in t.

$u = u_0 + \varepsilon u_1 + O(t^2)$

→ $\left(\frac{du}{d\phi}\right)^2 = \left(\frac{du_0}{d\phi}\right)^2 + 2\varepsilon\frac{du_1}{d\phi}\frac{du_0}{d\phi} + O(t^2)$

1st order: $\left(\frac{du_0}{d\phi}\right)^2 = 1 - u_0^{\ 2}$ → $u_0 = \sin(\phi - \phi_0)$. $u = \frac{r_0}{r}$ → $r\sin(\phi - \phi_0) = r_0$ (straight line)

For an object of mass $= M$:



To zeroth order, there is no deflection. WLOG set $\phi_0 = 0$.

$O(t)$:

$$2\frac{du_0}{d\phi}\frac{du_1}{d\phi} = 2u_0u_1 + u_0^{\ 3}$$

→ $\frac{d}{d\phi}\left(\frac{u_1}{\cos\phi}\right) = \frac{1}{2}\frac{\sin^3\phi}{\cos^2\phi} = \frac{1}{2}(\sec\phi\tan\phi - \sin\phi)$

→ $\frac{u_1}{\cos\phi} = \frac{1}{2}\left(\frac{1}{\cos\phi} + \cos\phi\right) + A$

Therefore to order ε the solution is

$$u(\phi) = \sin\phi + \varepsilon\left[A\cos\phi + \frac{1}{2}(1 + \cos^2\phi)\right] + O(t^2).$$