Einstein's Equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \hat{G} T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein Tensor, and $\hat{G}$ is the Newtonian constant.

Note:

1. The equation is non-linear in the metric, which we should expect since there are no neutral observers → no superposition of sources as in Maxwell's EM equations (methods used in EM (like method of images) would work in gravity because the equations aren't linear).
2. Alternative:
   $G = g^{\mu\nu} G_{\mu\nu} = R - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = -R = 8\pi \hat{G} T$
   where $\hat{G}$ is the Newtonian constant, and $T = g^{\mu\nu} T_{\mu\nu}$.
   G is the scalar associated with the Einstein tensor (and Newton's constant)
   → $R_{\mu\nu} = 8\pi \hat{G} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$
3. The addition of a term $\Lambda g_{\mu\nu}$ is compatible with our criteria, where Λ is a constant. Λ represents vacuum energy, and is known as the cosmological constant. We will return to this in section 7.
   Einstein called Λ his greatest blunder, but cosmological observations suggest that it is non-zero (but very small).

5.3 Newtonian Limit

Consider the line element

$$ds^2 = e^{2\phi} dt^2 - dx^2$$

with $\phi = \phi(x) << 1$. This ϕ will represent the gravitational potential. The Einstein equations become the Poisson equations in this Newtonian space.

→ $L_{eff} = e^{2\phi} \dot{t}^2 - \dot{x}^2$

$\frac{\partial L_{eff}}{\partial \dot{t}} = 2e^{2\phi} \dot{t}$; $\frac{dL_{eff}}{dt} = 0$ → $\ddot{t} = 0$

$\frac{\partial L_{eff}}{\partial \dot{x}} = -2\dot{x}$; $\frac{\partial L_{eff}}{\partial x} = 2\partial_x \phi \, e^{2\phi} \dot{t}^2$

→ $\ddot{x} + \partial_x \phi e^{2\phi} \dot{t}^2 = 0$

$\Gamma^x{}_{tt} = \partial_x \phi e^{2\phi}$, $\Gamma = 0$ otherwise

For $\phi << 1$,

$\frac{\ddot{x}}{\dot{t}^2} = \frac{\partial^2 x}{\partial t^2} = -\partial_x \phi = -\nabla \phi$

→ Newtonian equations of motion for test particle.

Now consider

$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\gamma\rho} \Gamma^\gamma{}_{\mu\nu} - \Gamma^\rho{}_{\gamma\nu} \Gamma^\gamma{}_{\mu\rho}$

Use $\Gamma^x{}_{tt} = \partial_x \phi + \nabla(\phi^3) \to \Gamma \sim \phi \to \Gamma^2 \sim \phi^2$ (very small)

Therefore only need to calculate the first two terms.

$$R_{\mu\nu} \approx \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho}$$
$$= \partial_t \underbrace{\Gamma^t{}_{\mu\nu}}_{=0} - \partial_\nu \Gamma^x{}_{\mu\nu} - \partial_\nu \underbrace{\Gamma^t{}_{\mu t}}_{=0} - \partial_\nu \underbrace{\Gamma^x{}_{\mu x}}_{=0}$$

Therefore $R_{\mu\nu} = \partial_x \Gamma^x{}_{\mu\nu}$ (only non-zero part).

$$R_{\mu\nu} = \begin{cases} \partial_x{}^2\phi & \mu\nu = tt \\ 0 & otherwise \end{cases}$$

Hence $R_{tt} = \nabla^2\phi$

Now consider

$$T_{\mu\nu} = (P+\rho)U_\mu U_\nu - P g_{\mu\nu}$$
$$T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = U_\mu U_\nu (P+\rho) - \frac{1}{2}(P-\rho) g_{\mu\nu}$$

In the non-relativistic limit, $u_\mu = (1,2)$ and $\rho << P$

$$\rightarrow\ R_{tt} = \nabla^2\phi = 8\pi G\left(T_{tt} - \frac{1}{2} g_{tt} T\right) = 4\pi G\rho$$

Poisson equation $\nabla^2\phi = 4\pi G\rho$, hence $\phi = -\frac{GM}{r}$.

5.4 Gravitational Radiation

Consider a metric with a small perturbation from Minkowski spacetime.

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \quad (\varepsilon << 1)$$

created by a matter distribution $\varepsilon T_{\mu\nu}$.

Since $g^{\mu\nu} g_{\mu\rho} = \delta_\rho^{\mu + O(\varepsilon^2)}$, we must have that

$$g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon h^{\mu\nu}$$

hence,

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\varepsilon\left(-\partial^\mu h_{\alpha\beta} + \partial_\alpha h_\beta{}^\mu + \partial_\beta h_\alpha{}^\mu\right) + O(\varepsilon^2)$$

and

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$$
$$= \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\mu \Gamma^\rho{}_{\nu\rho} + O(\varepsilon^2)$$
$$= \frac{1}{2}\varepsilon\left(-\Box h_{\mu\nu} + \partial_\mu\left(\partial_\rho h^\rho{}_\nu\right) + \partial_\nu\left(\partial_\rho h^\rho{}_\mu\right) - \partial_\mu\partial_\nu h\right) + O(\varepsilon^2)$$

where $\Box = \partial_\rho \partial^\rho$ and $h = h^\rho{}_\rho$

Now consider the effect of a coordinate transformation.

$$x'^\mu = x^\mu + \varepsilon\xi^\mu$$
$$\rightarrow\ J'^\mu_\alpha = \delta^\mu_\alpha + \varepsilon\partial_\alpha\xi^\mu \text{ and } J^\mu_\alpha = \delta^\mu_\alpha - \varepsilon\partial_\alpha\xi^\mu + O(\varepsilon^2)$$

If $h'_{\mu\nu}$ is the metric perturbation in the new coordinate system, then

$$\eta_{\mu\nu} + \varepsilon h'_{\mu\nu} = J^\alpha_\mu J^\beta_\nu\left(\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}\right) = \eta_{\mu\nu} + \varepsilon\left(h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu\right) + O(\varepsilon^2)$$

→ physical processes should be invariant under the "gauge transformation"

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu.$$

(c.f. $A'_\mu = A_\mu + \nabla_\mu \phi$ in ED)

Consider $\partial^\mu h'_{\mu\nu} - \frac{1}{2}\partial_\nu h' = \partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h - \Box\xi_\nu$

→ by a gauge transformation we can set $\partial^\mu h'_{\mu\nu} - \frac{1}{2}\partial_\nu h' = 0$ if we choose

$\Box\xi_\nu = \partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h$

Hence, we can always choose the de-Donder gauge where $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. Note that this is similar to the choice of the Lorentz gauge in EM.

→ $R_{\mu\nu} = -\frac{1}{2}\varepsilon\Box h_{\mu\nu} \to \Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$

This is the wave equation for the generation and propagation of gravitational waves. It is very similar to that for dipole EM radiation.

Consider free wave-like solutions with $T_{\mu\nu} = 0$, i.e. $\Box h_{\mu\nu} = 0$.

Solutions $h_{\mu\nu} = e_{\mu\nu}e^{ikx}$

$e_{\mu\nu}$ is the polarization tensor. $kx = k_\mu x^\mu$.

$\Box h_{\mu\nu} = 0$ → $k_\mu k^\mu = 0$ i.e. k is null.

$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0$ → $p_{\mu\nu}k^\mu = \frac{1}{2}k_\nu e^\mu{}_\mu$ (*)

$e_{\mu\nu}$ has 10 degrees of freedom (4x4 symmetric tensor), but * is 4 equations → only 6 degrees of freedom (maximum).

Moreover, for a wave of the form $e_{\mu\nu}$, a further gauge transformation can be made, in this form: $e'_{\mu\nu} = e_{\mu\nu} + \alpha_\mu k_\nu + \alpha_\nu k_\mu$ using $\xi_\mu = i\alpha_\mu e^{ik\cdot x}$, in which α_μ is arbitrary and * is unchanged.

Since α_μ is a 4-vector, this removes 4 degrees of freedom. → 2 degrees of freedom for free gravitational waves, i.e. two polarization states.

Consider an x-directed wave $k^0 = k' = k$, $k^2 = k^3 = 0$, then

$$h_{\mu\nu} = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & h_t & h_x \\ & h_x & h_t \end{array}\right) e^{ik\cdot x}$$

$h_{\mu\nu} = h_t e^t{}_{\mu\nu} + h_x e^x{}_{\mu\nu}$

where $e^t{}_{\mu\nu} = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}\right)$, $e^x{}_{\mu\nu} = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 1 \\ & 1 & 0 \end{array}\right)$

In contrast to EM radiation, gravitational radiation is generated by the mass quadropole: Power $P = \frac{G}{5}\left|\dddot{I}_{ij}\right|$, where $I_{ij} = \int e\left(x_i x_j - \frac{1}{3}\delta_{ij}\underline{x}^2\right)d^3x$ is the quadrapole of the matter distribution $\rho(\underline{x})$.

Section 6 - Schwartzchild solution

6.1 - Spherically Symmetric Vacuum Solution

Vacuum → $T_{\mu\nu} = 0$ → $R_{\mu\nu} = 0$ except at one point.

Spherically symmetric with area of the 2-sphere $4\pi r^2$.

→ $ds^2 = A(r,T)dT^2 - 2B(r,T)dTdr - C(r,T)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$

Consider a change of timescale $T = T(t,r)$

→ $dT = \dot{T}dt + T'dr$, where $\dot{T} = \dfrac{\partial T}{\partial t}$ and $T' = \dfrac{\partial T}{\partial r}$.

Therefore

$ds^2 = A\dot{T}^2 + 2\dot{T}\left(AT' - B\right)dtdr - \left(C + 2BT' - AT'^2\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$

If we choose $AT' = B$ and set coefficient of $dt^2 = e^{v(t,r)}$, and coefficient of $dr^2 = -e^{\lambda(r,t)}$ → most general spherically symmetric metric, i.e.

$ds^2 = e^{v(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - r^2\left(d\theta^2 + \sin^2\alpha d\phi^2\right)$

→ $g_{\mu\nu} = diag\left(e^v, -e^\lambda, -r^2, -r^2\sin^2\alpha\right)$

$g^{\mu\nu} = diag\left(e^{-v}, e^{-\lambda}, -\dfrac{1}{r^2}, -\dfrac{1}{r^2\sin^2\alpha}\right)$

To compute the Christoffel symbols, use

$L_{eff} = \left(\dfrac{ds}{d\tau}\right)^2 = e^v\dot{t}^2 - e^\lambda\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\alpha\dot{\phi}^2\right)$

For t:

$\dfrac{\partial L_{eff}}{\partial \dot{t}} = 2e^v\dot{t}$; $\dfrac{\partial L_{eff}}{\partial t} = \dfrac{\partial v}{\partial t}e^v\dot{t}^2 - \dfrac{\partial \lambda}{\partial t}e^\lambda\dot{r}^2$

→ $\ddot{t} + \dfrac{1}{2}\dfrac{\partial v}{\partial t}\dot{t} + \dfrac{\partial v}{\partial r}\dot{r}\dot{t} + \dfrac{1}{2}\dfrac{\partial \lambda}{\partial t}e^{\lambda - v}\dot{r}^{-2} = 0$

$\Gamma^t{}_{tt} = \dfrac{1}{2}\dfrac{\partial v}{\partial t}$; $\Gamma^t{}_{rt} = \dfrac{1}{2}\dfrac{\partial v}{\partial r}$; $\Gamma^t{}_{rr} = \dfrac{1}{2}e^{\lambda - v}\dfrac{\partial \lambda}{\partial t}$

Similarly for (r), (θ) and (ϕ).

Then compute $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho\Gamma^\rho{}_{\mu\nu} - \partial_\nu\Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\gamma\rho}\Gamma^\gamma{}_{\mu\nu} - \Gamma^\rho{}_{\gamma\nu}\Gamma^\gamma{}_{\mu\rho}$, from which we can derive:

$$R_{tt} = \frac{1}{2}e^{v-\lambda}\left(v'' + \frac{1}{2}v'\left(v' - \lambda'\right) + \frac{2v'}{r}\right) + \dot{\lambda}\left(\dot{v} - \dot{\lambda}\right) - \frac{1}{2}\ddot{\lambda}$$

$$R_{tr} = \frac{\dot{\lambda}}{2r}$$

$$R_{rr} = \frac{1}{2}e^{v-\lambda}\left(\ddot{\lambda} - \frac{1}{2}\dot{\lambda}\left(\dot{v} - \dot{\lambda}\right)\right) - \frac{1}{2}\left(v'' + \frac{1}{2}v'\left(v' - \lambda'\right) - \frac{2\lambda'}{r}\right)$$

$$R_{\theta\theta} = 1 - e^{-\lambda}\left(1 + \frac{1}{2}r\left(v' - \lambda'\right)\right)$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$$