PC4771 – Gravitation – Lectures 11 & 12

Consider coordinates (t, r, θ) and (t', r', θ') where:

$$t' = t$$
$$r' = r$$
$$\theta' = \theta - \omega t$$

$$J' = \begin{pmatrix} 1 & 0 & -\omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; J = \begin{pmatrix} 1 & 0 & \omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -r^2 \end{pmatrix}; \; g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{r^2} \end{pmatrix}$$

$$\left[ds^2 = dt^2 - dr^2 - r^2 d\theta^2 \right]$$

$$g' = JgJ^T = \begin{pmatrix} 1 - r^2\omega^2 & 0 & -r^2\omega \\ 0 & -1 & 0 \\ -r^2\omega & 0 & -r^2 \end{pmatrix}$$

$$(g')^{-1} = \begin{pmatrix} 1 & 0 & \omega \\ 0 & -1 & 0 \\ -\omega & 0 & \omega^2 - \frac{1}{r^2} \end{pmatrix}$$

$$\Gamma'^r{}_{ij} = \frac{1}{2} g'^{rk} \left| -\partial_k g'_{ij} + \partial_i g'_{kj} + \partial_j g'_{ik} \right|$$
$$= \frac{1}{2} g'^{rr} \left| -\partial_r g'_{ij} + \partial_i g'_{rj} + \partial_j g'_{ir} \right|$$
$$= -\frac{1}{2} g'^{rr} \partial_r g'_{ij}$$

$$\Gamma'^r{}_{ij} = -\frac{1}{2}(-1)\frac{\partial}{\partial r}\left[\begin{pmatrix} 1 - r^2\omega^2 & 0 & -r^2\omega \\ 0 & -1 & 0 \\ -r^2\omega & 0 & -r^2 \end{pmatrix} \right]$$
$$= \begin{pmatrix} -r\omega^2 & 0 & -r\omega \\ 0 & -1 & 0 \\ -r\omega & 0 & -r \end{pmatrix}$$

Now work out the geodesic equation.

$$\ddot{r} + \Gamma'^r{}_{ij}\dot{x}^i \dot{x}^j = 0$$

→ $\ddot{r} = r\omega^2 \dot{t}^2 + 2r\omega \dot{t}\dot{\theta} + r\dot{\theta}^2$

→ $\dfrac{\ddot{r}}{\dot{t}^2} = rw^2 + 2rw\dfrac{\partial \theta}{\partial t} + r\left(\dfrac{\partial \theta}{\partial t}\right)^2$

rw^2 is the centrifugal force - $\underline{\omega} \times (\underline{\omega} \times \underline{r})$

$2rw\frac{\partial \theta}{\partial t}$ is the Coriollis force $2(\underline{\omega} \times \underline{v})$

4. Curvature

Consider $\nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi$, the second order differential commutator for a scalar.

$\nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

where $A_\mu = \nabla_\mu \phi = \partial_\mu \phi$

$= \partial_\mu A_\nu - \Gamma^\gamma{}_{\mu\nu} A_\gamma - \left\{\partial_\nu A_\mu - \Gamma^\gamma{}_{\nu\mu} A_\gamma\right\}$

$= \underbrace{\left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right)}_{0}\phi + \underbrace{\left(\Gamma^\gamma{}_{\nu\mu} - \Gamma^\gamma{}_{\mu\nu}\right)}_{0}\partial_\gamma \phi$

The first part is zero as partial derivatives commute.

The second part is zero as $\Gamma^\gamma{}_{\nu\mu} = \Gamma^\gamma{}_{\mu\nu}$

→ $\nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi = 0$

→ covariant derivates commute for a scalar.

However, this is not true for a vector, covector or tensor (see sheet)

$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A^\rho = \left(\partial_\mu \Gamma^\rho{}_{\alpha\nu} - \partial_\nu \Gamma^\rho{}_{\alpha\mu} + \Gamma^\rho{}_{\gamma\mu}\Gamma^\gamma{}_{\alpha\nu} - \Gamma^\rho{}_{\gamma\nu}\gamma^\gamma{}_{\alpha\mu}\right) A^\alpha$

$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A_\rho = \left(\partial_\nu \Gamma^\alpha{}_{\rho\mu} - \partial_\mu \Gamma^\alpha{}_{\rho\nu} + \Gamma^\gamma{}_{\rho\mu}\Gamma^\alpha{}_{\gamma\nu} - \Gamma^\gamma{}_{\rho\nu}\Gamma^\alpha{}_{\gamma\mu}\right) A_\alpha$

If we define:

$R^\rho{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\rho{}_{\alpha\nu} - \partial_\nu \Gamma^\rho{}_{\alpha\mu} + \Gamma^\rho{}_{\gamma\mu}\Gamma^\gamma{}_{\alpha\nu} - \Gamma^\rho{}_{\gamma\nu}\Gamma^\gamma{}_{\alpha\mu}$

then:

$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A^\rho = R^\rho{}_{\alpha\mu\nu} A^\alpha$

and

$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A_\rho = R^\alpha{}_{\rho\nu\mu} A_\alpha$

$R^\rho{}_{\alpha\mu\nu}$ is a $\binom{1}{3}$ tensor.

$R_{\beta\alpha\mu\nu} = g_{\beta\rho} R^\rho{}_{\alpha\mu\nu}$ is a $\binom{0}{4}$ tensor.

We can define the Ricci tensor:

$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = g_{\alpha\beta} R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}$

and the Ricci scalar

$R = R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta} = g_{\alpha\beta} R^{\alpha\beta}$

4.2 Symmetries of Riemann tensor and the Bianchi Identity

Consider a local inertial frame: $\Gamma^\mu{}_{\alpha\beta} = 0$ at a point (NB: $\partial_\rho \Gamma^\mu{}_{\alpha\beta} \neq 0$), then

$R^\rho{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\rho{}_{\alpha\nu} - \partial_\nu \Gamma^\rho{}_{\alpha\mu}$

Now substitute $\Gamma^\rho{}_{\alpha\nu} = \frac{1}{2} g^{\rho\gamma}\left(-\partial_\gamma g_{\alpha\nu} + \partial_\alpha g_{\gamma\nu} + \partial_\nu g_{a\gamma}\right)$, then:

$R^\rho{}_{\alpha\mu\nu} = \frac{1}{2} g^{\rho\gamma}\left[\partial_\mu \partial_\alpha g_{\gamma\nu} + \partial_\nu \partial_\gamma g_{\alpha\mu} - \partial_\mu \partial_\gamma g_{\alpha\nu} - \partial_\nu \partial_\alpha g_{\gamma\mu}\right]$

→ $R_{\gamma\alpha\mu\nu} = \frac{1}{2}\left[\partial_\mu\partial_\alpha g_{\gamma\nu} + \partial_\nu\partial_\gamma g_{\alpha\mu} - \partial_\mu\partial_\gamma g_{\alpha\nu} - \partial_\nu\partial_\alpha g_{\gamma\mu}\right]$

We can show that in this LIF (Locally Inertial Frame):

1. $R_{\gamma\alpha\mu\nu} = -R_{\gamma\alpha\nu\mu}$ (true $\forall$ connections)
2. $R_{\gamma\alpha\mu\nu} = -R_{\alpha\gamma\mu\nu}$
3. $R_{\gamma\alpha\mu\nu} = R_{\mu\nu\gamma\alpha}$
4. $R_{\gamma\alpha\mu\nu} + R_{\gamma\nu\alpha\mu} + R_{\gamma\mu\nu\alpha} \equiv 3R_{\gamma[\alpha\mu\nu]} = 0$

But these equations are tensorial, so they are true in all frames.

These are the symmetries of the Riemann tensor.

$$\begin{aligned} R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} &= g^{\alpha\beta} R_{\alpha\mu\beta\nu} \\ &= g^{\alpha\beta} R_{\beta\nu\alpha\mu} \\ &= g^{\beta\alpha} R_{\beta\nu\alpha\mu} \\ &= R_{\nu\mu} \end{aligned}$$

→ the Ricci tensor is symmetric, as a result of the symmetries of the Riemann tensor.

Note:

1. In n spacetime dimensions the number of components of the Riemann tensor is naively n^4, but the symmetries make it actually $\frac{1}{12}n^2\left(n^2-1\right) = \begin{cases} 1 & n=2 \\ 6 & n=3 \\ 20 & n=4 \end{cases}$
2. These symmetries are only true for a metric connection since we have used the Christoffel form for the connection.

Once again in an LIF:

$\nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\mu R_{\gamma\alpha\nu\beta} + \nabla_\nu R_{\gamma\alpha\beta\mu} = \partial_\beta R_{\gamma\alpha\mu\nu} + \partial_\mu R_{\gamma\alpha\nu\beta} + \partial_\nu R_{\gamma\alpha\beta\mu}$

Substitute $R_{\gamma\alpha\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\alpha g_{\gamma\nu} + \partial_\nu\partial_\gamma g_{\alpha\mu} - \partial_\mu\partial_\gamma g_{\alpha\nu} - \partial_\nu\partial_\alpha g_{\gamma\mu}\right)$

Then

$\nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\mu R_{\gamma\alpha\nu\beta} + \nabla_\nu R_{\gamma\alpha\beta\mu} = 0$

i.e. $3\nabla_{[\beta} R_{\mu\nu]\gamma\alpha} = 0$

Again this is a tensorial equation, known as the Bianchi Identity, and is true in all frames.

Now contract with $g^{\gamma\mu}$

→ $\nabla_\beta R_{\alpha\nu} + \nabla_\mu R^{\mu}{}_{\alpha\nu\beta} + \nabla_\nu R^{\gamma}{}_{\alpha\beta\gamma} = 0$

→ $\nabla_\beta R_{\alpha\nu} + \nabla_\mu R^{\mu}{}_{\alpha\nu\beta} - \nabla_\nu R_{\alpha\beta} = 0$

and contract with $g^{\alpha\beta}$

→ $\nabla^\alpha R_{\alpha\nu} + \nabla_\mu R^{\mu}{}_{\nu} - \nabla_\nu R = 0$

→ $\nabla^\mu\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] = 0$

This is known as the contracted Bianchi Identity.

4.3 Round trips by Parallel Transport

If A^μ is parallelly transported along a curve then

$$\frac{dA^\mu}{du} = -\Gamma^\mu{}_{\alpha\beta} A^\alpha \frac{dx^\beta}{du}$$

The change in A^μ around a closed curve is given by

$$\Delta A^\mu = \oint du \frac{dA^\mu}{du} = -\oint \Gamma^\mu{}_{\alpha\beta} A^\alpha \frac{dx^\beta}{du} du$$

Consider a closed curve. $x^\rho(0) = 0$. At some point $x^\rho(u)$.

1. Through a Taylor expansion:
$$A^\alpha(u) = A^\alpha(0) + u\frac{dA^\alpha}{du}(0) + \dots$$
$$= A^\alpha(0) - u\Gamma^\alpha{}_{\gamma\rho}(0) A^\gamma(0) \frac{dx^\rho}{du}(0) + \dots$$
$$x^\rho = u\frac{dx^\rho}{du}(0) + \dots$$
→ $A^\alpha(u) = A^\alpha(0) - \Gamma^\alpha{}_{\gamma\rho}(0) A^\gamma(0) x^\rho(0) + \dots$
2. Through another Taylor expansion:
$$\Gamma^\mu{}_{\alpha\beta}(x) = \Gamma^\mu{}_{\alpha\beta}(0) + x^\rho(u)\partial_\rho \Gamma^\mu{}_{\alpha\beta}(0) + \dots$$

→

$$\Delta A^\mu = -\int \left[\Gamma^\mu{}_{\alpha\beta}(0) + x^\rho \partial_\rho \Gamma^\mu{}_{\alpha\beta}(0) + \dots\right]\left[A^\alpha(0) - x^\rho \Gamma^\alpha{}_{\gamma\rho} A^\gamma(0) + \dots\right] \frac{dx^\beta}{du} du$$

$$= -\Gamma^\mu{}_{\alpha\beta}(0) A^\alpha(0) \oint \frac{dx^\beta}{du} du - A^\alpha(0)\left[\partial_\rho \Gamma^\mu{}_{\alpha\beta}(0) - \Gamma^\mu{}_{\gamma\beta}(0)\Gamma^\gamma{}_{\alpha\rho}(0)\right] \oint x^\rho \frac{dx^\beta}{du} du + O(2)$$

For a closed curve, the integral $\oint \frac{dx^\beta}{du} du = 0$ and

$$\oint x^\rho \frac{dx^\beta}{du} du = \oint \left[\frac{d}{du}\left(x^\rho x^\beta\right) - x^\beta \frac{\partial x^\rho}{\partial u}\right] = -\oint x^\beta \frac{dx^\rho}{du} du$$

→

$$\Delta A^\mu = -\frac{1}{2}\left(\partial_\rho \Gamma^\mu{}_{\alpha\beta}(0) - \partial_\beta \Gamma^\mu{}_{\alpha\rho}(0) + \Gamma^\mu{}_{\alpha\rho}(0)\Gamma^\alpha{}_{\gamma\beta}(0) - \Gamma^\mu{}_{g\beta}(0)\Gamma^\gamma{}_{\alpha\rho}(0)\right) A^\gamma(0) \oint x^\beta \frac{dx^\rho}{du} du$$

$$= -\frac{1}{2} R^\mu{}_{\alpha\rho\beta}(0)\left[\oint x^\beta \frac{dx^\rho}{du} du\right] A^\gamma(0)$$

→ $\Delta A^\mu = 0$ if $R^\mu{}_{\alpha\rho\beta} = 0$ i.e. there is no curvature.

i.e. $\Delta A^\mu = 0$ if space is flat.