Mass generation in the Standard Model
(NB: throughout the whole of this document, 'massive' just means that it has mass)
<u>Helicity</u>
For a spin ½ particle, helicity is the projection of a particles' spin (Σ) along its' direction of motion $\underline{p}$.
→ Helicity operator $\Sigma \cdot \underline{p}$
There are two states: one is when the spin is aligned with the direction of motion ('positive' or 'right handed'), the other is when it is in the opposite direction ('negative' or 'left handed').

If the particle is massive, this is frame dependent (can 'boost' past it, and reverse the sign of $\underline{p}$).

Handedness, or alternatively chirality, is the Lorentz invariant (i.e. frame independent) analogue of helicity for massive and massless particles. Either 'Left handed' (LH) or 'right handed' (RH).

A massless fermion is either LH or RH.
A helicity eigenstate for a massive particle is a linear combination of LH and RH.

$$\psi = \psi_L + \psi_R$$

The weak interaction has a preferred handedness (unlike Strong and EM). Neutrinos are LH, and antineutrinos are RH (experimental fact).

$$\gamma^5 \psi_{L,R} = \mp \psi_{L,R}$$

where γ^5 essentially represents the helicity operator, such that this is a standard eigenfunction / eigenvalue equation (eigenvalues: ∓ 1). Note that $\gamma^5 \neq (\gamma)^5$; it is just the name of the operator matrix.

$$\psi_{L,R} = \frac{1}{2}\left(1 \mp \gamma^5\right)\psi$$

So every weak vertex has a factor of $\frac{\left(1-\gamma^5\right)}{2}$ to project out the LH state.

Therefore RH neutrinos could exist, but would be 'sterile', i.e. they could not interact via SM interactions.

The Higgs Mechanism generates masses through terms like, for the example of the electron,

$$m_e\left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L\right)$$

where the bar represents the complex conjugate.

$$m_e = \frac{g_e v}{\sqrt{2}}$$

where g_e is the coupling constant of e to the Higgs field. v is the Higgs vev (vacuum expectation value), which we know as it is related to the masses of the W and Z particles.

i.e. for quarks in the SM (denoting $i = 1,2,3$ as the generation number, such that the following represents all three quark generations):

$$\begin{pmatrix} u_i \\ d_i \end{pmatrix}_L, \; u_{iR}, \; d_{iR}.$$

For leptons in the SM,

$$\begin{pmatrix} \ell_i \\ \nu_i \end{pmatrix}, \; \ell_{iR}$$

So the Higgs mechanism doesn't give us a mass for the neutrino as there is no right-handed state for it.

We can do the same as the Higgs mass generation for the ν by introducing the (sterile) RH ν, then

$$m_D = \frac{g_\nu v}{\sqrt{2}}$$

But this implies that $g_e > 5 \times 10^4 g_\nu$ - very ad-hoc. (It would work, but it wouldn't give a reason for the masses of the neutrinos to be very small – not very satisfactory).

Alternative: the "See-saw mechanism".
This is one example of why ν mass could be a window onto new (and profound) physics. We can write down the usual "Dirac" mass term in the Lagrangian L;

$$L_D = m_D\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right) \text{ (1) } \left(= m\psi\bar{\psi}\right)$$

In the bracketed expression, these are mass eigenstates, where $\psi = \psi_L + \psi_R$, and is just another way of writing the first expression.
(Note that putting this into the Euler-Lagrange equations would give $p^2 + m^2 = E^2$).

Can also write down two other terms that are consistent with Lorentz invariance, the Majorana mass terms:

$$L_{M_L} = m_L\left(\bar{\psi}_L^c\psi_L + \bar{\psi}_L\psi_L^c\right) = m_L\bar{\chi}\chi$$

$$L_{M_R} = m_R\left(\bar{\psi}_R^c\psi_R + \bar{\psi}_R\psi_R^c\right) = m_R\bar{\omega}\omega \text{ (2)}$$

where here the superscript c means the charge conjugate (electric, weak, strong, …). The mass eigenstates are the self conjugate fields $\chi = \psi_L + \psi_L^c \; \left(\chi^c = \chi\right)$, $\omega = \psi_R + \psi_R^c \; \left(\omega^c = \omega\right)$. i.e. the Majorana neutrino is its own antiparticle.

Gauge invariance:

$$\psi = e^{i\alpha(x)Q}\psi$$

$$\bar{\psi} = e^{-i\alpha(x)Q}\bar{\psi}$$

where $\alpha(x)$ is an arbitrary function of x. Gauge invariance says that it should be possible to do this, and the theory should be completely independent of the transformations. This is a $U(1)$ transformation. It can easily be shown that

$$\bar{\psi}\psi \to \bar{\psi}\psi$$

i.e. this is gauge invariant. But

$$\bar{\psi}^c\psi \to e^{2i\alpha(x)Q}\bar{\psi}^c\psi$$

is not gauge invariant, i.e. $\bar{\psi}^c\psi$ is only $U(1)$ gauge invariant if $Q = 0$.

Note that this is not the only gauge transformation that it has to satisfy – it also has to satisfy the $SU(2)$ (weak force) and $SU(3)$ (strong force) gauge transformations.

Notice that

$$\psi_L = \frac{1}{2}(1-\gamma^5)\chi\ ;\ \psi_L^c = \frac{1}{2}(1+\gamma^5)\chi$$
$$\psi_R = \frac{1}{2}(1+\gamma^5)\omega\ ;\ \psi_R^c = \frac{1}{2}(1-\gamma^5)\omega$$

Actually, the $m_L\,\bar{\chi}\chi$ term is only possible if there is a more complicated Higgs sector (at least 5 Higgs particles), but the $m_R\,\omega\bar{\omega}$ term can still survive because the RH ν is sterile (It has no charge whatsoever – be it electric, weak or strong) (and therefore this term must always be invariant under the SM gauge transformations).

We can write the sum of (1) and (2) in matrix form,

$$(\bar{\chi},\bar{\omega})\begin{pmatrix} m_R & \frac{1}{2}m_D \\ \frac{1}{2}m_D & 0 \end{pmatrix}\begin{pmatrix}\chi \\ \omega\end{pmatrix} + \text{charge conjugated terms}$$

This should be diagonalised to get 2 mass eigenvalues.

$$m_{1,2} = \frac{1}{2}\left\{m_R \pm \left(m_R^{\ 2} + m_D^{\ 2}\right)^{1/2}\right\} = \frac{1}{2}\left\{m_R \pm m_R\left(1+\frac{m_D^{\ 2}}{m_R^{\ 2}}\right)^{1/2}\right\}$$

such that we can set $m_1 = m_R$, $m_2 = \frac{m_D^{\ 2}}{m_R}$.

→ Majorana mass eigenstates.

$$\eta_1 = \cos\theta\chi - \sin\theta\omega$$
$$\eta_2 = \sin\theta\chi + \cos\theta\omega$$

where

$$\tan 2\theta = \frac{m_D}{m_R}.$$

Note that m_D should naturally be the same as m_e or m_q. since it came from the Higgs mechanism. The ν masses we see are the Dirac masses (squared) divided by m_R.

Take m_D to be a few hundred GeV (i.e. the top quark mass), and we want $m_\nu \sim 0.05eV$ (from $\left(\Delta m_{atm}^{\ 2}\right)^{1/2}$). Then $m_R \sim 10^{15}\,GeV$.