3.7 Particles in a Spherical Potential

Compare with the TIDE.

$$\left[-i\underline{\alpha}\cdot\underline{\nabla}+\beta\left(m+S(r)\right)+V(r)\right]\psi=E\psi \quad (1)$$

where $S(r), V(r)=qA^0(r)$ depend on $r=|x|$ only.

Consider solutions with $j=\frac{1}{2}$, $P=+1$ - $S_{\frac{1}{2}}$ particles (or antiparticles in $P_{\frac{1}{2}}$). The wave function in Dirac representation have the form

$$\psi(r)=\begin{pmatrix} f(r)\chi_S \\ g(r)i\underline{\sigma}\cdot\hat{\underline{r}}\chi_S \end{pmatrix}$$

where the top one is large for positive energy at low-energies, and the bottom one is large otherwise. $f(r)$ and $g(r)$ are scalar functions, and $\underline{\sigma}\cdot\hat{\underline{r}}$ is an odd parity scalar function (a psudoscalar).

Substitute ψ into (1) in Pauli-Dirac representation.

$$\begin{pmatrix} m+S+V & -i\underline{\sigma}\cdot\underline{V} \\ -\underline{\sigma}\cdot\underline{V} & -m-S+V \end{pmatrix}\begin{pmatrix} f(r)\chi \\ g(r)i\underline{\sigma}\cdot\hat{\underline{r}}\chi \end{pmatrix}=E\psi \quad (2)$$

Use $\underline{\sigma}\cdot\underline{\nabla}f(r)=\underline{\sigma}.\hat{\underline{r}}\frac{df}{dr}$;

$$\underline{\sigma}\cdot\underline{\nabla}\left(\frac{\underline{\sigma}\cdot r}{r}g(v)\right)=\underbrace{\left(\underline{\sigma}\cdot\hat{\underline{r}}\right)^2}_{1}r\frac{d}{dr}\left(\frac{g}{r}\right)+\underbrace{\underline{\sigma}\cdot\underline{\sigma}}_{3}\frac{g}{r}=\frac{dg}{dr}+\frac{2g(r)}{r}$$

→ 2 coupled equations

$$\left(\frac{d}{dr}+\frac{2}{r}\right)g(r)+\left(m+S+V-E\right)f=0 \quad (3)$$

$$\frac{df}{dr}+\left(m+S-V+E\right)g(r)=0 \quad (4)$$

Note signs: for particles, the large components Sf and Vf, but for antiparticles the large components are Sg and $-Vg$.

3.7.1 M.I.T. Bag Model

This is a crude model of quarks confined inside hadrons. Consider a proton, uud, $m=940MeV/c^2$, $R_p\sim 1fm$. Consider the typical momenta. Say that $\lambda\sim 2R$.

$$p\sim\frac{h}{\lambda}(SI)=\frac{2\pi}{\lambda}(NU)=\frac{\pi}{R_p}\sim 600MeV/c$$

So we are obviously dealing with a relativistic situation.

We also know that quarks interact weakly at short distances $r<<1fm$, and interact strongly at distances $r\sim 1fm$ (because that's the size of the proton…).

A simple model is that the three quarks move independently inside a scalar potential well. Let $S(r)=0$ within $r<R$ $\left(R\sim R_p\right)$, and $S(r)=S_0$ at $r>R$. (5) Why a scalar potential S, not a vector V? Because a scalar confines both quarks and antiquarks, and we want the model to work for antiprotons too.

An addition is to set $m_q = 0$, which is good for m_u, m_d.
(Note that this is not the constituent mass = the mass of a bare quark + virtual gluons – the latter is modeled by S).

Consider the radial equations for f, g using (3) and (4).
$r < R$:

$$\left(\frac{d}{dr} + \frac{2}{r}\right) g(r) - Ef = 0$$

$$\frac{df}{dr} + Eg = 0$$

$$\rightarrow \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} + E^2 f = 0$$

$r > R$:

$$\left(\frac{d}{dr} + \frac{2}{r}\right) g + \left(S_0 - E\right) f = 0$$

$$\frac{df}{dr} + \left(S_0 + E\right) g = 0$$

$$\rightarrow \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \left(E^2 - S_0^{\ 2}\right) f = 0 \quad (6)$$

To solve these equations, set $f(r) = \frac{u(r)}{r}$ to get rid of the $\frac{2}{r}\frac{df}{dr}$ term, and solve for u. Note that the solutions are those for the Bessel functions.
Inside:

$$f(r) = \frac{A\sin(Er)}{r} \quad \text{(finite at } r = 0\text{ !)}$$

$$g(r) = A\left(\frac{\sin(Er)}{Er^2} - \frac{\cos(Er)}{r}\right)$$

Outside $S_0 > E$, $k = \sqrt{S_0^{\ 2} - E^2}$:

$$f(r) = \frac{Be^{-kr}}{r} \quad (\rightarrow 0 \text{ as } r \rightarrow \infty)$$

$$g(r) = \frac{B}{S_0 + E}\left(\frac{ke^{-kr}}{r} - \frac{e^{-kr}}{r^2}\right).$$

Assume S_0 is very large $\left(>> E, 1/R\right)$.

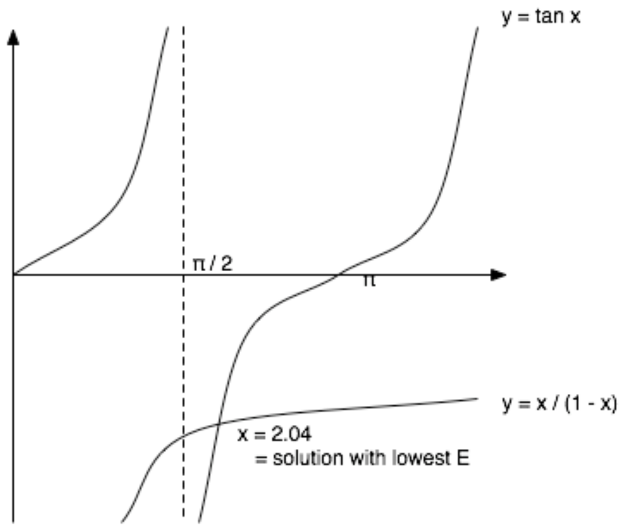
$$g(r) \approx \frac{Be^{-kr}}{r} = f(r) \quad (7)$$

At the surface, $f(r), g(r)$ continuous. So (7) implies that for $r \rightarrow R$ from below (i.e. inside),

$$f(R) = g(R)$$

i.e. $\frac{\sin ER}{R} = \frac{\sin ER}{ER^2} - \frac{\cos ER}{R}$, or $\tan ER = \frac{ER}{1 - ER}$.

This equation has a graphical solution:



$x = ER = 2.04$

→ $E_{S_{1/2}} = \dfrac{2.04}{R}$

and $M = 3E = \dfrac{6.12}{R}$ (three quarks)

Put $R = 1f$, $M \approx 1200 MeV$

Alternatively, $E = M = 940 MeV$, $R = 0.8 fm$.

This is not bad for a first attempt. Can elaborate e.g. by including single gluon exchange between the quarks, and can calculate more things.