$$x \to x' = ax$$
$$\psi'(x') = S(a)\psi(x)$$
$$S^{-1}(a)\gamma^{\mu}S(a) = a^{\mu}{}_{\nu}\gamma^{\nu} \quad (5)$$
$$S^{-1}(a) = \gamma^{0}S(a)\gamma^{0} \quad (6)$$

Parity: $t \to t' = t$, $\underline{x} \to \underline{x}' = -\underline{x}$, i.e. $x'^{\mu} = a^{\mu}{}_{\nu}x^{\nu}$.

$$(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \ddots \end{pmatrix}$$

satisfies $a^{\mu}{}_{\nu}a^{\nu}{}_{\sigma} = g^{\mu}{}_{\sigma} = \delta_{\mu\sigma}$. Invariance requires $\psi'(x') = P\psi(x)$, where $P = P(a)$ satisfies (5) and (6), and we also require $P^2 = I$, i.e. $P^{-1} = P$. This is all satisfied with

$$P = \eta\gamma^{0}$$

where $n = \pm 1$ and is undetermined. Usually set $\eta = +1$ (convention).

Consider a particle at rest in Dirac representation, where:

$$\gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$
$$\psi_{+}(x) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} e^{-imt} \quad (E = m)$$
$$\psi_{-}(x) = \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} e^{imt} \quad (E = -m)$$

So:

$$P\psi_{+} = \eta\psi_{+}$$
$$P\psi_{-} = \eta\psi_{-}$$

A particle at rest is in an eigenstate of P with "intrinsic parity" η $(= +1$ by convention). An antiparticle at rest has intrinsic parity $-\eta$, i.e. particle and antiparticle have opposite parities. This is verified in $e^{+}e^{-} \to \gamma\gamma$, and is important in the quark model of mesons.

Consider a particle in motion with positive energy

$$\psi^{\dagger}{}_{p,s}(x) = N\sqrt{E(p)+m}\begin{pmatrix} \chi_s \\ \dfrac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_s \end{pmatrix} e^{-i(Et-\underline{p}\cdot\underline{x})}$$

$$\begin{aligned} P\psi^{\dagger}{}_{\underline{p},\underline{s}}(t,\underline{x}) &= P\psi(x') \\ &= \eta\gamma^{0}\psi^{\dagger}{}_{\underline{p},\underline{s}}(t,-\underline{x}) \\ &= \eta\sqrt{E(p)+m}\begin{pmatrix} \chi_s \\ -\dfrac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_s \end{pmatrix} e^{-i(Et-\underline{p}\cdot\underline{x})} \\ &= \psi^{\dagger}{}_{\underline{p},\underline{s}}(\underline{x}) \end{aligned}$$

i.e. under P transform, $\underline{p} \to -\underline{p}$, $s \to s$, phase change η.

3.6 Interactions with Fields

Consider an EM field,

$$A^\mu(x) = \left(A^0(x), \underline{A}(x)\right)$$

introduced by the minimal substitution

$$\partial_\mu \to \partial_\mu + iqA_\mu$$

where q is the charge. We can also consider the Lorentz scalar fields by substitution:

$$m \to m + S(x)$$

where m is the mass. Both obviously leave the Dirac equation Lorentz invalient, and give

$$\left[i\gamma^\mu\left(\partial_\mu + iqA_\mu - m - S\right)\right]\psi(x) = 0 \quad (1)$$

$$\text{or } i\frac{\partial\psi}{\partial t} = \left[\underline{\alpha}\cdot\left(-i\underline{\nabla} - q\underline{A}\right) + \beta(m+S) + qA^0\right]\psi(x) = 0 \quad (2)$$

3.6.1 The Dirac Magnetic Moment

Consider EM fields only $S(x) = 0$. Look for solutions of the form

$$\psi(x) = \begin{pmatrix}\phi(x)\\ \eta(x)\end{pmatrix} e^{-iEt} \quad (3)$$

in Dirac representation. Substitute in (2) → 2 coupled equations:

$$\underline{\sigma}\cdot\left(-i\underline{\nabla} - q\underline{A}\right)\eta + \left(qA^0 + m\right)\phi = E\phi \quad (4)$$

$$\underline{\sigma}\cdot\left(i\underline{\nabla} - q\underline{A}\right)\phi + \left(qA^0 - m\right)\eta = E\eta \quad (5)$$

Consider non-relativistic approximations for positive energies:

$$\varepsilon = E - m << m, \; \left|qA^\mu\right| << m$$

(5) becomes

$$\eta = \frac{1}{2m}\underline{\sigma}\cdot\left(-i\underline{\nabla} - q\underline{A}\right)\phi \to \eta << \phi$$

Substitute this into (4);

$$\frac{1}{2m}\left[\underline{\sigma}\cdot\left(-i\underline{\nabla} - q\underline{A}\right)\right]^2\phi + qA^0\phi = \varepsilon\phi \quad (6)$$

Tidy up the first term using

$$\left(\underline{\sigma}\cdot\underline{a}\right)\left(\underline{\sigma}\cdot\underline{b}\right) = \underline{a}\cdot\underline{b} + i\underline{\sigma}\left(\underline{a}\times\underline{b}\right) \quad \text{(cf. Ex. 1)}$$

$$\to \left[\;\right]^2\phi = \left(-i\underline{\nabla} - q\underline{A}\right)\cdot\left(-i\underline{\nabla} - q\underline{A}\right)\phi + i\underline{\sigma}\left(-i\underline{\nabla} - q\underline{A}\right)\phi = (A) + (B)$$

$$(B) = -q\underline{\sigma}\cdot\left[\underline{\nabla}\times(A\phi) + \underline{A}\times\underline{\nabla}\phi\right]$$

$$= -q\underline{\sigma}\left(\underline{\nabla}\times\underline{A}\right)\phi$$

using $\underline{\nabla}\times\left(\underline{A}\phi\right) = \phi\left(\underline{\nabla}\times\underline{A}\right) - \underline{A}\times\left(\underline{\nabla}\phi\right)$.

$$\underline{S} = \frac{1}{2}\underline{\sigma}, \; \underline{B} = \underline{\nabla}\times\underline{A}$$

$$(A) = \left(-i\underline{\nabla} - q\underline{A}\right)\cdot\left(-i\underline{\nabla} - q\underline{A}\right)\phi$$

$$= -\nabla^2\phi + iq\underline{\nabla}\cdot\left(\underline{A}\phi\right) + iq\underline{A}\cdot\underline{\nabla}\phi + q^2\underline{A}^2\phi$$

Exploit gauge invariance, i.e. $\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$, $\underline{B} = \underline{\nabla} \times \underline{A}$, and physics in general is unchanged by $\phi \to \phi + \frac{\partial f}{\partial t}$, $\underline{A} \to \underline{A} - \underline{\nabla} f$, i.e. $\phi, \underline{A}$ are not uniquely defined by ∇.
This is a popular, but not unique, choice:
Coulomb gauge: $\underline{\nabla} \cdot \underline{A} = 0$
Then $\underline{\nabla} \cdot (\underline{A}\phi) = \underbrace{(\underline{\nabla} \cdot \underline{A})}_{=0}\phi + \underline{A} \cdot (\underline{\nabla}\phi)$

$$\rightarrow (A) = \left(-\nabla^2 + 2iq\underline{A} \cdot \underline{\nabla} + q\underline{A}^2\right)\phi$$

Specialize to uniform magnetic field $\underline{B}$:

$$\underline{A} = \frac{1}{2}(\underline{B} \cdot \underline{x}) \text{ (Ex. 1)}$$

$$\rightarrow \begin{aligned} 2iq\underline{A} \cdot \underline{\nabla} &= iq(\underline{B} \times \underline{x}) \cdot \underline{\nabla} \\ &= iq(\underline{x} \times \underline{\nabla})\underline{B} = -q\underline{B} \cdot \underline{L} \end{aligned}$$

since $\underline{L} = \underline{x} \times \underline{p} = -i(\underline{x} \times \underline{\nabla})$. Neglecting the $i\hbar$ factor, $\underline{x} \cdot \underline{\nabla} =$ angular momentum operator.

Put it all together: Dirac equation (6) in the non-relativistic limit with $E > 0$ and constant magnetic fields becomes

$$\left[-\frac{1}{2m}\nabla^2 + qA^0 - \underline{\mu} \cdot \underline{B} + \frac{q^2\underline{A}^2}{2m}\right]\phi = \varepsilon\phi$$

this is the Schrödinger equation (i.e. Pauli equation) with magnetic moment

$$\mu = \frac{q}{2m}(\underline{L} + 2\underline{S})$$

i.e. predict hitherto mysterious factor

$$g = 2$$

for pure spin. Another triumph for Dirac equation.