2.2 Conserved Current

$$\frac{\partial^2\phi}{\partial t^2} = \left(\nabla^2 - m^2\right)\phi \quad (1)$$

$$\rightarrow \frac{\partial^2\phi^*}{\partial t^2} = \left(\nabla^2 - m^2\right)\phi^* \quad (2)$$

ϕ^* x (1) – (2) x ϕ:

$$\rightarrow \frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{j} = 0$$

where $\rho = i\left[\phi^*\frac{\partial\phi}{\partial t} - \frac{\partial\phi^*}{\partial t}\phi\right]$ and $j = -i\left[\phi^*\underline{\nabla}\phi - \left(\underline{\nabla}\phi^*\right)\phi\right]$ (cf. Schrödinger case).

We now have that $\rho(x) \neq \left|\phi(\underline{x},t)\right|^2$, and it can be either positive (for positive energy solutions) or negative (for negative energy solutions).
ρ is not a probability density (charge density?)

Summary
We have covariant theory with a conserved current, but we have problems with negative energies and with the interpretation of ϕ.

Press on regardless with the positive energy solutions.

2.3 EM Fields and the "H atom"

If the potential $A^\mu \neq 0$, then $\partial_\mu \rightarrow \partial_\mu + iqA_\mu$ (which corresponds to $p_\mu \rightarrow p_\mu - qA_\mu$ in the classical case). This gives us the form for the Klein-Gordon equation,

$$\left(\partial_\mu + iqA_\mu\right)\left(\partial^\mu + iqA^\mu\right)\phi + m^2\phi = 0$$

Here consider an electrostatic field, so $\underline{A} = 0$ and $V(\underline{x}) = qA_0(\underline{x})$.

$$\left[\left(\frac{\partial}{\partial t} + iV\right)^2 - \underline{\nabla}^2 + m^2\right]\phi(\underline{x},t) = 0$$

For energy eigenvalues $\phi(x) = \psi(\underline{x})e^{-iEt}$ (time independent solution), we have

$$\left[-(E-V)^2 + \nabla^2 + m^2\right]\psi(\underline{x}) = 0$$

(cf the Schrödinger equation $(E-V)\psi = -\frac{\nabla^2}{2m}\psi$)

Now consider $V(\underline{x}) = -\frac{Z\alpha}{r}$ where $r = |\underline{x}|$ and $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$. What are the energy levels? Solve by comparing to the non-relativistic H-like atoms.
Time-Independent Schrödinger Equation (TISE):

$$\left[-\nabla^2 - 2m\frac{Z\alpha}{r}\right]\psi = 2m\varepsilon\psi$$

where $\varepsilon = E - mc^2$. Write $\psi(\underline{x}) = \frac{U(r)}{r}Y_{\ell m}(\theta,\phi)$, where ℓ, m are angular momentum eigenstates, and use $\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\hat{L}^2}{r^2}$. This gives us the radial Schrödinger Equation,

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}-\frac{2mZ\alpha}{r}\right]U(r)=2m\varepsilon U(r)$$

For the Klein-Gordon equation, using the same argument gives

$$\left[-\frac{d^2}{dr^2}+\frac{\left(\ell(\ell+1)-Z\alpha\right)^2}{r^2}-\frac{2EZ\alpha}{r}\right]U(r)=\left(E^2-m^2\right)U(r)$$

This is of the same form with $2m\to 2E$, $2m\varepsilon\to E^2-m^2$ and
$\ell\to\ell'=-\frac{1}{2}+\sqrt{\left(\ell+\frac{1}{2}\right)^2-(Z\alpha)^2}$.

Use the non-relativistic energy levels from the Schrödinger equation,

$$2m\varepsilon=-2\frac{m(Z\alpha)^2}{2n^2}m$$

where $n=n_r+\ell+1$. Making the substitutions from above, you get

$$E^2-m^2=-\frac{2E(Z\alpha)^2}{2n'^2}E$$

where $n'=n_r+\ell'+1$. Solving this for E gives

$$E=m\left[1+\frac{(Z\alpha)^2}{n'^2}\right]^{1/2}$$

There are 2 interesting cases;

1. $(Z\alpha)^2\ll 1$, a weak potential, i.e. $Z\ll 137$
   Expand in $Z\alpha$
   $\ell'=\ell-\frac{(Z\alpha)^2}{2\left(\ell+1/2\right)}+\ldots$
   $E=m\left[1-\frac{(Z\alpha)^2}{2n^2}-\frac{(Z\alpha)^4}{2n^4}\left(\frac{n}{\ell+1/2}-\frac{3}{4}\right)+\ldots\right]$
   Here the first part is the rest energy, the second part is the same as the non-relativistic result, and the third part is the fine structure – which is a relativistic correction because $\frac{v}{c}\sim Z\alpha$.
   Fine structure is wrong for H! Schrödinger abandoned the KG equation.
   But it is right for spin-0 particles in a Coulomb field, e.g. pionic atoms $\left(\pi^- p\right)$ etc.
   Scalar KG equation describes spin-0 particles!
2. $Z\alpha>\left(\ell+1/2\right)$
   $Z\geq 69$, so we have a strong potential.
   $\ell'=-\frac{1}{2}+\sqrt{\left(\ell+1/2\right)^2-(Z\alpha)^2}$ is complex – the energy is complex!
   Another disaster? To understand this, compare it with a simpler problem.

2.4 The Klein Paradox

Consider a particle incident on an electrostatic barrier in one dimension. Have a potential step of height V, which starts at $z=0$ and goes on in the $+z$ direction. Particle with kinetic energy $E-m$ traveling in the +z direction.

For $z<0$, time-independent equation

$$\left[-E^2-\frac{d^2}{dz^2}+m^2\right]\psi(z)=0$$

This has solutions $\psi(z)=Ae^{ipz}+Be^{-ipz}$, with $p^2=E^2-m^2$.

For $z>0$,

$$\left[-(E-V)^2-\frac{d^2}{dz^2}+m^2\right]\psi(z)=0$$

This has solutions $\psi=Ce^{ip'z}$ where $p'^2=(E-V)^2-m^2$. We have imposed that there are no incoming particles / waves from the right.

If $(E-m)>V$, $p'^2>0$ and propagation will occur on the right $(z>0)$.
If $(E-m)<V$, $p'^2<0$, so $p'=i|p'|$ and we will get exponential decay for $z>0$ ("evanescence").

Boundary conditions: ψ and $\dfrac{d\psi}{dz}$ are continuous at $z=0$.

$$\rightarrow\ B=\frac{p-p'}{p+p'}A,\ C=\frac{2p}{p+p'}A.$$

Consider a conserved current. $j_z=-i\left[\phi^*\nabla\phi-(\nabla\phi^*)\phi\right]$, $\phi=\psi(z)e^{-iEt}$.

$$\rightarrow\ -i\left[\psi^*\frac{\partial\psi}{\partial z}-\left(\frac{\partial\psi^*}{\partial z}\right)\psi\right]$$

For the incident wave Ae^{ipz} → $j_I=2p|A|^2$.

For the reflected wave $Be^{-ipz}\to j_R=-2p|B|^2$

For the transmitted wave $Ce^{ip'z}\to j_T=\begin{cases}2p'|C|^2 & p'^2>0,\ real\ p'\\ 0 & p'^2<0,\ imaginary\ p'\end{cases}$

Hence $j_I+j_R=j_T$ as expected, with propagation if $E-V^2>m^2$ and evanescence if $(E-V)^2<m^2$.