Derivatives

Consider a Lorentz scalar $\phi(x)$. $\phi'(x')=\phi(x)$. Through a 1st-order Taylor expansion,

$$\phi(x+\delta x)-\phi(x)=\delta\phi=\frac{\partial\phi(x)}{\partial x^\mu}\delta x^\mu$$

$\delta\phi$ is a Lorentz scalar, δx^μ is a contravarient 4-vector, so $\frac{\partial}{\partial x^\mu}$ is a covariant vector. Therefore

$$\partial_\mu\phi\equiv\frac{\partial\phi}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial\phi}{\partial t},\nabla\phi\right)$$

is a covariant 4-vector. Similarly,

$$\partial^\mu\phi=\frac{\partial\phi}{\partial x_\mu}=\left(\frac{1}{c}\frac{\partial\phi}{\partial t},-\nabla\phi\right)$$

is a contravarient 4-vector, and

$$\Box\equiv\partial_\mu\partial^\mu=\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)$$

is a scalar operator. For example, $\Box\phi=\partial_\mu\partial^\mu\phi=0$ is the wave equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi=0\,.$$

1.1.1 Equation of Continuity

Consider any charge density $\rho(x)$ such that the charge within a volume Ω is

$$Q_\Omega=\int_\Omega dv\rho(x)$$

is unchanged by a Lorentz transformation (i.e. it is a scalar). Then it can be shown that

$$J^\mu(x)=\left(c\rho,\underline{j}\right)$$

where $\underline{j}$ is the associated current, $\underline{j}=\rho\underline{v}$, is a 4-vector. i.e. $J^\mu\to J'^\mu=\Lambda^\mu{}_\nu J^\nu$ under a Lorentz transformation. The charge is conserved provided that the equation of continuity

$$\partial_\mu J^\mu(x)=0 \tag{1}$$

is satisfied. This is an explicitly covariant equation. To se this, rewrite (1):

$$\frac{\partial\rho}{\partial t}+\underline{\nabla}\cdot\underline{j}=0$$

Then

$$\begin{aligned}\frac{\partial Q_\Omega}{\partial t}&=\frac{\partial}{\partial t}\int_\Omega dv\rho\\&=-\int_\Omega\underline{\nabla}\cdot\underline{j}\,dv\\&=-\int\underline{j}\cdot\underline{ds}\end{aligned}$$

the last stage of which is using the Divergence Theorem. Hence the rate of increase of the charge enclosed $=-$ the rate of flow of the charge, i.e. the current / flux, through the surface. So no charge is created or destroyed. In particular, if ρ, $\underline{j}\to 0$ at ∞,

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}\int dv\rho(x) = 0 \ .$$

1.1.2 Electromagnetic Field

(Free space, $\varepsilon, \mu = 0$)

Maxwell's Equations (SI units):

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$$

$$\underline{\nabla} \cdot \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

In Rationalised Gaussian Units, $\mu_0 \varepsilon_0 = \frac{1}{c^2}$, so $\varepsilon_0 = 1$. Also redefine $\underline{B}$ as $\frac{B_{SI}}{c}$.

$$\underline{\nabla} \cdot \underline{E} = \rho \qquad (1)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad (2)$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \qquad (3)$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{c}\underline{j} + \frac{1}{c^2}\frac{\partial E}{\partial t} \qquad (4)$$

Also introduce the fine structure constant,

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137}$$

so to evaluate things with e, convert them to α.

Introduce the EM potentials ϕ and $\underline{A}$ which satisfy

$$\underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \ .$$

These guarantee that equations (2) and (3) are automatically satisfied.

If we write $A^\mu(x) = (\phi, \underline{A})$ (5), then (1) and (4) can be combined and written as

$$\Box A^\mu(x) - \partial^\mu\left(\partial_\nu A^\nu(x)\right) = \frac{1}{c} j^\mu(x)$$

so the Principle of Relativity requires $A^\mu(x)$ to be a 4-vector, i.e.

$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu X^\nu$, as implied but not proven by (5).