3.7.2 H-like atoms

Now have $S(r)=0$, and $V(r)=-\frac{Z\alpha}{r}$ is the Coulomb potential, where we're assuming $q\equiv -e$. We can solve this – see textbooks. We get:

$$E_n = m\left[1+\left(\frac{Z\alpha}{n-\left(j+\frac{1}{2}\right)+\sqrt{\left(j+\frac{1}{2}\right)^2-Z\alpha^2}}\right)^2\right]^{\frac{1}{2}}$$

Note that this doesn't depend on ℓ.

There are 2 interesting limits.

1. Strong potentials

$$Z\alpha > j+\frac{1}{2}$$
$$j=\frac{1}{2} \rightarrow Z>137$$

Here, E is imaginary (cf. KG equation)

In principle, we have the same as for the KG equation for $Z>69(?)$. This tells us that pair production can't be ignored. In practice, there are no atoms with $Z>137$.

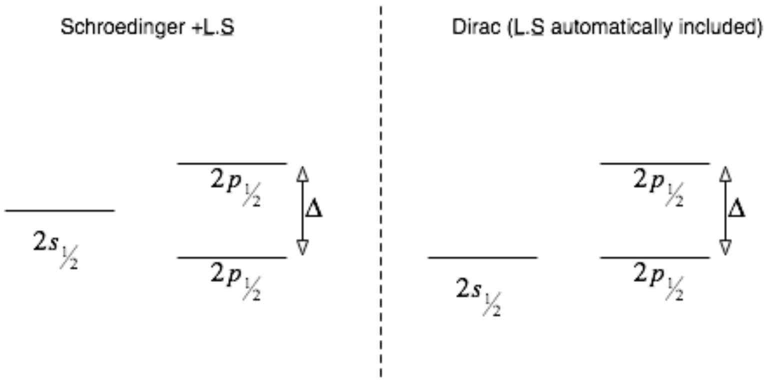
2. Hydrogen Atom

$Z=1$, and expand in terms of $\alpha=\frac{1}{137}$.

$$E_n = E_n - m$$
$$= -\frac{m\alpha^2}{2n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{1}{j+\frac{1}{2}}-\frac{3}{4n}\right)+O\left(\alpha^4\right)\right]$$

The 1 is the non-relativistic result; the second part is the relativistic corrections, including the spin-orbit terms, which are typically of order 10^{-5}.

Consider the $n=2$ levels.

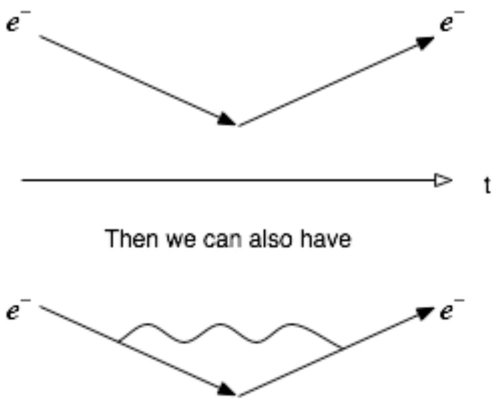


Experimentally:

- Schrödinger equation + $\underline{L}\cdot\underline{S}$ is incorrect
- Dirac almost correct

But experimentally, $2S_{1/2}$ level shifted upwards by the "Lamb shift", $\delta \sim O(\alpha^3)$ relative to the other relativistic corrections. This was a great discovery (1947) → Nobel prize because is goes beyond the Dirac theory + hole theory. It is mainly due to quantum fluctuations of the EM field.

For comparison, if we represent the interaction with a classical field $-\frac{Z\alpha}{r}$ by



The same effect causes $g = 2$ (Dirac) → $g = 2\left(1 + \frac{\alpha}{2\pi} + ...\right)$. Both results have been confirmed experimentally to extraordinary precision. $g - 2$ has been measured and calculated to an accuracy of 1 in 10^9, in perfect agreement. To understand this, we need QFT.

4. Quantum Fields

4.1 Quantum Mechanics of a String

Compare with a taut string. Take the dynamical variable as the transverse displacement $\phi(z,t)$. This satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial z^2}$$

where $c = \sqrt{\frac{T}{\mu}}$, where T is the tension and μ is the mass per unit length. The energy is

$$E = \int_0^L dz \left(\frac{1}{2}\rho\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\phi}{\partial z}\right)^2 \right)$$

where the first term is the kinetic energy, and the second term is the potential energy. For simplicity, set $T = 1$, $\mu = 1$ → $c = 1$.

$$\rightarrow \frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial z^2} \quad (1)$$

like the KG equation with $m = 0$ but $c = 1 =$ the speed of elastic waves. The energy

$$E = \int_0^L dz \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 \right]. \quad (2)$$

To describe traveling waves, impose periodic boundary conditions

$$\phi(t,z) = \phi(t,z+L) . \ (3)$$

(we can take $L \to \infty$ at the end)

The normal modes are

$$\phi_n(z,t) = e^{\pm i(\omega_n t - k_n z)}$$

where $k_n = \frac{2\pi n}{L}$ with n an integer (from boundary condition), and $\omega_n = |k_n|$ to satisfy the wave equation. For a real string, take the real part of this.

Note that at fixed z, each bit of the string performs simple harmonic motion, i.e. we have an infinite number of simple harmonic oscillators.

Expand an arbitrary ϕ in terms of ϕ_n, where we want ϕ to be real;

$$\phi(t,z) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\omega_n L}} \left[a_n e^{-i(\omega_n t - k_n z)} + a_n{}^* e^{i(\omega_n t - k_n z)} \right]$$

The $\frac{1}{\sqrt{2\omega_n L}}$ is there for later convenience. The first part is like positive energy, while the second part is like negative energy.

Calculate the energy in terms of new variables $a_n, a_n{}^*$. The "field velocity" is

$$\frac{\partial\phi}{\partial t} = -i \sum_n \sqrt{\frac{\omega_n}{2L}} \left(a_n e^{-i(\omega_n t - k_n z)} - a_n{}^* e^{-i(\omega_n t - k_n z)} \right)$$

The kinetic energy (at $t = 0$) is

$$\frac{1}{2}\int \left(\frac{\partial\phi}{\partial t}\right)^2 dz = -\sum_{n,m} \sqrt{\frac{\omega_n \omega_m}{4L}} \begin{pmatrix} a_n a_m e^{-i(k_n + k_m)z} - a_n a_m{}^* e^{i(k_n - k_m)z} \\ -a_n{}^* a_m e^{i(k_m - k_n)z} + a_n{}^* a_m{}^* e^{-i(k_n + k_m)z} \end{pmatrix}$$

(Note that everything in the brackets should be on the same line).
Using orthogonality,

$$\int_0^L e^{i(k_n - k_m)z} dz = L\delta_{mn}$$

$$\rightarrow \frac{1}{2}\int_0^L \left(\frac{\partial\phi}{\partial t}\right)^2 dz = -\sum_n \frac{\omega_n}{4} \left(a_n a_{-n} - 2a_n{}^* a_n + a_n{}^* a_{-n}{}^* \right)$$

Similarly for the potential energy,

$$\frac{1}{2}\int dz \left(\frac{\partial\phi}{\partial z}\right)^2 = \sum_n \frac{\omega_n}{4} \left(a_n a_{-n} + 2a_n{}^* a_n + a_n{}^* a_{-n}{}^* \right)$$

So the total energy (the Hamiltonian) is

$$H = \sum_n \omega_n a_n{}^* a_n .$$

Alternatively, introduce real variables

$$q_n = \frac{a_n + a_n{}^*}{\sqrt{2\omega_n}}, \quad p_n = -i\sqrt{\frac{\omega_n}{2}} \left(a_n - a_n{}^* \right)$$

$$\rightarrow H = \sum_n \frac{1}{2} \left(p_n{}^2 + \omega_n{}^2 q_n{}^2 \right)$$

Consider the SHO in one dimension;

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 x^2 .$$

We have an infinite sum of simple harmonic oscillators, which we can quantize.