7. The Complete Problem

To model a star, we have four differential equations:

$$\frac{dm}{dr} \quad \frac{dP}{dr}$$
$$\frac{d\ell}{dr} \quad \frac{dT}{dr}$$

and three other functions:

$$\varepsilon(P,T) \quad \rho(P,T)$$
$$\kappa(P,T)$$

What are the boundary conditions?

At centre, $r = 0$, $m = 0$, $\ell = 0$.
At surface, $r = R_*$, $m = M_*$, $P \to 0$

But an atmosphere has no sharp edge.

Define the outer edge as optical depth $\tau = 2/3$ $(\tau = \kappa\ell)$:

$$\tau = \int_{R_*}^{\infty} \kappa\rho dr$$
$$= \bar{\kappa}\int_{R_*}^{\infty} \rho dr = \frac{2}{3} \quad (171)$$

Here $T = T_{eff}$.
As $r \to R_*$

$$\frac{Gm(r)}{r^2} \to \frac{GM_*}{R_*^{\,2}} = g_0$$

which is a constant. Integrate the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho dr$$

from infinity to R_*.

$$P(r = R_*) \approx g_0 \int_{R_*}^{\infty} \rho dr = g_0 \frac{2}{3\bar{\kappa}}$$
$$P(r = R_*) = \frac{GM_*}{R_*^{\,2}} \frac{2}{3} \frac{1}{\bar{\kappa}}$$

Other outer boundary conditions at $r = R_*$:

$$T = T_{eff}$$
$$L = 4\pi R_*^{\,2}\sigma T_{eff}^{\,4}$$

7.1 Solving the Problem

Boundary conditions are split between the inner and outer boundaries. Two methods to find a solution:

1. *Shooting*: start at $r = 0$, guess P and T, integrate outwards, miss outer boundary conditions. Tweak P and T at $r = 0$ and try again. Difficult due to T^4 and r^{-4} in equations.
2. *Henyey method (relaxation):*
   a. Estimate solution at all radii
   b. Calculate by how much the estimate misses the solution.
   c. Correct values at all points using these errors.
   d. Iterate.

7.2 Basic Set

$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}$$
$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{\ell}{4\pi r^2}$$
$$\frac{d\ell}{dr} = 4\pi r^2 \rho\varepsilon$$
$$P = nkT + P_e + \frac{1}{3}aT^4$$
$$\kappa = \kappa_0 \rho^a T^b$$
$$\varepsilon = q_0 \rho^m T^n \quad (172)$$

Equations are strongly non-linear. Equations 1 and 2 depend on 3 and 4 only if P is a function of T.

7.3 Polytropic Equations of State

If P does not depend on T: can solve separately for hydrostatic structure (equations 1 and 2), and heat flow (2 and 3).

$$\frac{dm}{dr} = 4\pi r^2 \rho$$
$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho$$
$$m(r) = -\frac{r^2}{\rho G}\frac{dP}{dr}$$

Insert into mass equation to find

$$\frac{d}{dr}\left(-\frac{r^2}{\rho G}\frac{dP}{dr}\right) = 4\pi r^2 \rho \quad (173)$$

If $P = K\rho^\gamma$

$$\frac{K}{r^2}\frac{d}{dr}\left(r^2 \gamma \rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho \quad (174)$$

This is a simple second order differential equation: integrate to find the density profile.

Define

$$\gamma = 1 + \frac{1}{n} \quad (175)$$

where n is the polytropic index.

$$\frac{(n+1)K}{nr^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{(n-1)/n}}\frac{d\rho}{dr}\right) = -4\pi G\rho \quad (176)$$

with boundary condition $\rho = 0$ at $R = R_*$ and $d\rho / dr = 0$ at $r = 0$.

(At $r = 0 \to m(r) = 0 \to \frac{dP}{dr} = 0$. $P = \kappa\rho^\gamma$. $\frac{dP}{dr} = \kappa\,\gamma\,\rho^{\gamma-1}\frac{d\rho}{dr} = 0$. The only thing here that can be 0 is $d\rho / dr$.)

Examples of polytropes: degenerate stars.

- Relativistic: $n = 3$
- Non-relativistic: $n = 1.5$

A polytrope is fully determined by K, n, R_*.

(Traditionally, things were defined as:

$\rho = \rho_c w^n$

$r = \alpha z$

$\alpha^2 = \frac{n(n+1)K}{4\pi G\rho_c^{(n-1)/n}}$)

($P = \kappa\rho^\gamma$

$n = \frac{1}{\gamma - 1}$

$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{d\omega}{dz}\right) + \omega^n = 0$ the Lane-Emden Equation.)

(Chandrasekhar)

7.4 A Gaussian Model

Another way to simply model a star is to adopt some model for the pressure within the star, to solve for (1,2) and (3,4) separately.

The equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

At the centre of the star $(r = 0)$,

$$\frac{dP}{dr} = 0 \quad \frac{d\rho}{dr} = 0$$

The enclosed mass can be approximated as

$$m(r) \approx \frac{4\pi}{3} r^3 \rho_c \quad (177)$$

where ρ_c is the central density. Therefore

$$\frac{dP}{dr} = -\frac{4\pi}{3} G \rho_c^{\ 2} r \quad (178)$$

As $r \to 0$, dP/dr varies linearly with radius.

Near the surface of the star, $\rho \to 0$ and $m \to M_*$:

$$\frac{dP}{dr} = -\frac{GM_* \rho(r)}{r^2} \to 0 \quad (179)$$

Based on these arguments, Clayton (1986?) suggested a model with adopts a pressure gradient of

$$\frac{dP}{dr} = -\frac{4\pi}{3} G \rho_c^{\ 2} r e^{-\frac{r^2}{a^2}} \quad (180)$$

where a is some length parameter.

At the outer radius of the star R, take $P = 0$. Integrate equation 180 to find

$$P(r) = \frac{2\pi}{3} G \rho_c^{\ 2} a^2 \left[e^{-\frac{r^2}{a^2}} - e^{-\frac{R^2}{a^2}} \right] \quad (181)$$

To find the other parameters:
Mass: Integrate $-4\pi r^3 dP = Gm(r)dm$
Density: integrate mass conservation equation
Temperature: the ideal gas equation.

For $a/R << 1$ (true for the sun: $a = R_\odot / 5.4$)

$$M = m(R) = \frac{4\pi \rho_c a^3 \sqrt{6}}{3} \quad (182)$$

$$P_c = \frac{2\pi}{3} G \rho_c^{\ 2} a^2 \quad (183)$$

$$P_c = \left(\frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho_c^{\ 4/3} \quad (184)$$

7.5 Other Models

Eddington

$$\frac{\ell}{m} = \eta \frac{L}{M} \quad (185)$$

Assume energy generation in the core only: η increases inward. κ increases outward. Assume

$$\kappa \eta = const = \kappa_{surface} \quad (186)$$

This gives a polytrope of index $n = 3$.

Point source model
All energy generation at $r = 0$. Still leads to complicated equations.

Such models are no longer required now that computer models can be designed.

7.6 Real Models
See diagrams in handouts.