2. Basic Physics and Equations of Stellar Structure

Stars are spherical → parameters are only functions of distance from the centre of the star r.

Newton's theorems for spherical mass distributions:

1. Gravitational force at distance R can be calculated by assuming that all mass at $r < R$ is located at the centre.
   $F(r) = \frac{GM(r)}{r}$
2. Mass located at $r > R$ does not contribute to the net gravitational force.
   $F(r) = 0$

NB: this is only true for symmetrical distributions of mass. If this is not the case, then this exact law does not apply.

2.1 Some timescales

2.1.1 Free Fall Time

Consider a sphere of cold gas, mass M and radius r, collapsing due to its self gravity.

Energy of a particle of mass m moving from infinity to r.

The astronomer's way (the rough approximation…)
Initial potential energy = kinetic energy.

$$\frac{1}{2}mv^2 = \frac{GMm}{r} \quad (4)$$

$$v^2 \approx \frac{GM}{r} \quad (5)$$

Assume that this velocity remains constant throughout the fall…

The free fall time, t_{ff}, is the time for the particle to reach the centre. Estimate $t_{ff} = \frac{r}{v}$

$$\frac{r^2}{t_{ff}^{\ 2}} \sim \frac{GM}{r} \quad (8)$$

Big errors: at start, it was stationary. It's actually going to accelerate… Big error at initial v (too big), big error at final v (too small). So hopefully the errors will cancel… :-)

$$t_{ff}^{\ 2} \sim \frac{r^3}{GM} \sim \frac{1}{G\rho} \quad (7)$$

where $M = \frac{4\pi}{3}\rho r^3$, and $\frac{4\pi}{3}$ as been ignored…

$$t_{ff} \sim \frac{1}{\sqrt{G\rho}} \quad (8)$$

For the sun, $t_{ff} \sim 1 hour$.

Exact calculations (e.g. use Kepler's law) give (The astrophysics way):

Start at r_0. At r, $\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{GM}{r} - \frac{GM}{r_0}$.

Then integrate along the line of sight.

$$t_{ff} = \int_{r_0}^{0} \frac{dt}{dr} dr = \int_{r_0}^{0} \left(\frac{2GM}{r} - \frac{2GM}{r_0} \right)^{\frac{1}{2}} dr$$

$$x = \frac{r}{r_0}.$$

$$= -\int_1^0 \left(\frac{r_0^{\ 2}}{2GM} \right)^{\frac{1}{2}} \left[\frac{1}{r_0 x} - \frac{1}{r_0} \right]^{\frac{1}{2}} dx = -\int_1^0 \left(\frac{r_0^{\ 3}}{2GM} \right)^{\frac{1}{2}} \left[\frac{x}{1-x} \right]^{\frac{1}{2}} dx$$

Now, let $x = \sin^2\theta$.

$$dx = 2\sin\theta\cos\theta d\theta$$

$$= \left(\frac{r_0^{\ 3}}{2GM} \right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} 2\sin^2\theta d\theta$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

$$= \left(\frac{r_0^{\ 3}}{2GM} \right)^{\frac{1}{2}} [\theta - 0]_0^{\frac{\pi}{2}} = \left(\frac{r_0^{\ 3}}{2GM} \right)^{\frac{1}{2}} \left(\frac{\pi}{2} \right)$$

$$t_{ff} = \sqrt{\frac{3\pi}{32G\rho}} \approx 0.5hr$$

(does the sun float...? i.e. what's the density of the sun?)

Use geometry…

$$P^2 = \frac{\pi^2}{GM} a^3$$

$$a = \frac{1}{2} r_0$$

$$t_{ff} = \frac{1}{2} P$$

So: if the internal heat of the sun goes, then the sun will collapse in 30 minutes.

2.1.2 Kelvin-Helmholtz time

The sun is emitting radiation. So how long will the sun survive?

Kelvin-Helmholtz time t_{KH} is the time to radiate away gravitational potential energy.

$$t_{KH} = \frac{|E_g|}{L} \quad (9)$$

where L is the luminosity, and

$$|E_g| \sim \frac{GM_\odot^{\ 2}}{R_\odot} \quad (10)$$

for the sun, and so $t_{KH} \sim 3.2 \times 10^7\, years$. The sun is stable on much longer timescales than this (the earth is older than this).

So:

1. Need a better source of energy
2. For main sequence stars, we can ignore time dependence in equations – look for equilibrium (hydrostatic equilibrium) solutions.

Enter fusion. This gives lifescales in the billions of years.

So: if the sun were unstable, then it would have already become so, and we'd have noticed… So we can use a stable sun, and ignore time dependence in equations.

2.2 Mass Equation

We now have to assume that the density of the star varies. So $\rho(r,\theta,t)$ but assume a spherical stable star, and ignore θ and t dependence.
If there is no mass flow, the mass dm in a spherical shell of thickness dr at radius r with density $\rho(r)$ is:

$$dm = 4\pi\rho(r)r^2dr$$

$$\frac{dm}{dr} = 4\pi r^2\rho(r) \quad (11)$$

$$\frac{dm}{dt} = 0 \quad (12)$$

So the total mass of a star is (NB: $R_\odot \equiv r_0$):

$$M_\odot = \int_0^{R_\odot} dm \quad (13)$$

$$M_\odot = \int_0^{R_\odot} 4\pi\rho(r)r^2dr \quad (14)$$

Only if ρ is a constant, independent of r, is $M_\odot = \frac{4\pi}{3}r^3\rho$.

2.3 Hydrostatic Equilibrium

As gravitational force changes as you vary r, so will the force due to the pressure (as the two must balance…)
Force due to pressure gradient $= F_p$.

$$F_p = dA\left[P(r) + \frac{dP}{dr}dr - P(r)\right]$$

$$F_p = \frac{dP}{dr}drdA \quad (11)$$

Inward gravitational force $= F_g$

$$F_g = gdm = g\rho(r)drdA \quad (12)$$

where $g = \frac{Gm(r)}{r^2}$.

$$m(r) = \int_0^r dm = \int_0^r 4\pi r^2\rho(r)dr \quad (13)$$

In equilibrium, $\Delta F = F_p + F_g = 0$, i.e. $F_p = -F_g$.

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \quad (14)$$

Which is the equation of HydroStatic Equilibrium (HSE).

Aside: coordinate systems
What it the independent coordinate when modeling a star?

- Radius r - each variable is labeled by the radius – Eulerian description
- Mass m - each variable is labeled by a mass element and so follows the mass in a system. There is a one to one mapping with radius. Very useful in what follows – Lagrangian description.

$$\frac{\partial}{\partial m} = \frac{\partial r}{\partial m}\frac{\partial}{\partial r}$$
$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho(r)}$$
$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho(r)}\frac{\partial}{\partial r}$$

Therefore in Lagrangian coordinates the equation of hydrostatic equilibrium (hse) (Equation. 14) becomes:

$$\frac{\partial P}{\partial m} = -\frac{Gm(r)}{4\pi r^4} \quad (15)$$

Multiply equation of HSE.

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

by $4\pi r^3$ and integrate from $r = 0$ to $r = R_\odot$.

$$\int_0^{R_\odot} \frac{dP}{dr} 4\pi r^3 dr = -\int_0^{R_\odot} \rho(r)\frac{Gm(r)}{r} 4\pi r^2 dr \quad (16)$$
$$RHS = -\int_0^{M_\odot} \frac{Gm(r)}{r} dm \quad (17)$$
$$RHS = W \quad (18)$$

this is the gravitational potential energy of the star. So the other side has to be an energy too.

$$\text{LHS} = \left[P 4\pi r^3\right]_0^{R_\odot} - \int_0^{R_\odot} 3P(r) 4\pi r^2 dr \quad (19)$$

For the LHS, the first term is 0 (the pressure at the edge of the star must be 0). For an ideal gas (we can assume this…), $P = nkT$ and the thermal energy (per unit volume) is $E_{th} = \frac{3kTn}{2}$ and so:

$$P = \frac{2}{3}E_{th} \quad (20)$$

This is a standard result for a non-relativistic gas. For a relativistic one, the $2/3$ would become $1/3$ (or so).

Therefore we have:

$$-\int_0^{R_\odot} 2E_{th} 4\pi r^2 dr = -2U \quad (21)$$

where U is the total thermal energy. We therefore arrive at the *virial theorem* for self-gravitating systems: NB: it is only valid when you are in LTE. It is also valid for the Earth, where $E_k = \frac{1}{2}mv^2$, $W = -\frac{GMm}{r}$. Balance the forces: $\frac{GMm}{r} = \frac{mv^2}{r}$.

$$2U + W = 0 \quad (22)$$

The total energy of a star is:

$$E_{total} = U + W \quad (23)$$

$$E_{total} = -U = \frac{W}{2} \quad (24)$$

As stars contract, $|W|$ (the gravitational energy) increases, W becomes more negative. Therefore since $2U + W = 0$, U becomes more positive. The thermal energy increases.

Luminosity $L =$ rate of change of total energy,

$$L = \frac{dE}{dt} = \frac{\dot{W}}{2} \quad (25)$$

Half the gravitational energy released is radiated away. The other half heats the gas. So the heat capacity of the star is negative. (NB; it will also shrink…)

During the collapse of a protostar, it is very luminous to get rid of potential energy (this is before nuclear reactions start).

Change in gravitational energy moving a mass element (shell) of mass Δm from distance ∞ to r from the centre of the star $= \Delta E_g$. Let s be the distance, to avoid confusion with r.

$$\Delta E_g = \int_\infty^r F ds$$

$$F = \frac{GM}{s^2}\Delta m$$

$$\Delta m = 4\pi r^2 \rho dr$$

$$\Delta E_g = \int_\infty^r \frac{GM}{s^2}\Delta m ds = -\frac{GM}{r}4\pi r^2 \rho dr$$

The total gravitational energy is E_g.

$$E_g = -\int_0^{R_\odot} \Delta E_g dr = \int_0^{R_\odot} \frac{GM(r)}{r} 4\pi r^2 \rho(r) dr$$

We can't solve this until we know $\rho(r)$.

(So out comes the thermal energy of the star…)

2.4 Energy Conservation

In equilibrium, the change in energy transversing a shell per second = energy generated in the shell. Therefore the change in luminosity (energy per second) dl in a shell is:

$$dl(r) = 4\pi r^2 \rho(r)\varepsilon dr \quad (26)$$

$$\frac{dl(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon \quad (27)$$

where ε is the energy generation rate per unit mass.

For non-stationary system,

$$\frac{dl}{dr} = 4\pi r^2 \rho(r)\left[\varepsilon - \frac{\partial u}{\partial t} - P\frac{\partial V}{\partial t}\right] \quad (28)$$

u : internal energy km^{-3}

PdV : work done. (if the gas is moving up and down; pressure will do work).

We won't be doing too much with this. We'll mainly stick with the stationary version.

Integrate over the radii of the star, and out comes the luminosity…

2.5 Energy Transport

2.5.1 Radiation

Energy transported by the diffusion of photons.

Mean Free Path

Consider a moving atom with velocity v and radius a (This is easier to visualize than a photon). In a time t, the atom will collide with any other atom inside a cylinder of length vt and area πa^2. For n atoms per unit volume, there will be $n\pi a^2 vt$ collisions.

Therefore the mean distance between collisions, the mean free path s, is:

$$s = \frac{1}{n\pi a^2}.$$

Define the collisional cross-section per atom, σ:

$$s = \frac{1}{n\sigma}$$

Define κ as the collisional cross-section per unit mass:

$$s = \frac{1}{\rho\kappa}$$

For the sun:

$\rho = 1400 kgm^{-3}$

$\kappa = 0.1 - 0.04 m^2 kg^{-1}$

→ $s_{ph} \sim 0.017 - 0.7m \sim 5cm$!!

s_{ph} is the mean free path of a photon.

Energy transport by radiation is slow.

Local radiation field is a black body with flux $F(r)$ and momentum $\frac{F(r)}{c}$.

Momentum added to gas by photon collisions is:

$$\frac{F(r)}{c}\frac{1}{s}m^{-1}$$

This is a radiative pressure.

$$\frac{F(r)}{cs} = -\frac{dP_{rad}(r)}{dr}$$

$$\frac{F(k)\kappa\rho}{c} = -\frac{dP_{rad}(r)}{dr}$$

For radiation,

$$P_{rad} = \frac{1}{3}E_{rad} = \frac{1}{3}aT^4$$

where a is the radiation constant. (NB; this is now relativistic…)
As there are a lot of collisions, this normally small pressure becomes quite important.

$$\frac{dP_{rad}(r)}{dr} = \frac{4}{3}aT^3\frac{dT(r)}{dr}$$

$$\frac{dT(r)}{dr} = \frac{3\kappa\rho}{4acT^3}F$$

In terms of σ the Stefan-Boltzmann constant, $a = \frac{4\sigma}{c}$, we arrive at:

The equation of radiation transport

$$\frac{dT(r)}{dr} = -\frac{3\kappa\rho(r)l(r)}{64\pi\sigma r^2 T^3}$$

where the flux $F(r)$ is related to the luminosity $l(r)$ by:

$$F(r) = \frac{l(r)}{4\pi r^2}$$

There is another way of transferring energy: convection. In some cases, this can be more efficient then radiation.

2.5.2 Conduction

The equations are very similar to those for radiation transport. The conductive flux, F_{cd}, is given by:

$$F_{cd} = -k_{cd}\frac{dT}{dr} \quad (34)$$

$$k_{cd} = \frac{16\sigma}{3}\frac{T^3}{\kappa_{cd}\rho} \quad (35)$$

where κ_{cd} is the conductive opacity.
If the total flux carried by the radiation and conduction is:

$$F = F_r + F_{cd},$$

then

$$F \propto \frac{1}{\kappa_{cd}} + \frac{1}{\kappa_v}$$

Transport mechanism with the smallest opacity carries the greatest energy. cf. resistors in electricity.

There are three methods of transport: radiation F_v, conduction F_{cd}, and convection.

2.5.3 Convection
Energy transport by mass motions.
Assume a bubble of gas is at a lower density then its' surroundings, and is forced upwards. During rise, density, pressure and temperature drop in both bubble and surroundings.
δ variables: changes in bubble parameters.
Δ variables; changes in surroundings.
$T \to T_0 + \delta T$
$P \to P_0 + \delta P$
$\rho \to \rho_0 + \delta\rho$
ΔT, ΔP, $\Delta\rho$.
Convection occurs if $\delta\rho < \Delta\rho$, i.e. the density of the bubble is smaller than the surrounding density.
Assume:

- During rising, $\delta P = \Delta P$: rapid pressure equation.
- Ideal gas with:
    - $\gamma = \frac{c_p}{c_v} = \frac{5}{3}$ (monatomic H; not always true)
    - $P \propto \rho^\gamma$ (37) (Adiabatic; $PV^\gamma = const.$; $V \propto \rho^{-1}$)
    - $\frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\delta P}{P}$ (38) (From last equation, looking at $P + \delta P$ vs. P, plus binomial expansion of $(\rho + \delta\rho)^\gamma \approx \rho^\gamma - \gamma\rho^{\gamma-1}d\rho$)

For an ideal gas, $\rho \propto \frac{P}{T}$. Using $\delta P = \Delta P$, and the convection requirement $\delta\rho < \Delta\rho$;

$$\frac{1}{\gamma}\frac{\delta P}{P} < \frac{\Delta\rho}{\rho}$$

$$= \frac{\Delta P}{P} - \frac{\Delta T}{T} \quad (39)$$

$$\frac{\Delta T}{T} < \frac{\gamma - 1}{\gamma}\frac{\Delta P}{P} \quad (40)$$

$$\frac{dT}{dr} < \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr} \quad (41)$$

Note that $\frac{dT}{dr} < 0$, and $\frac{dP}{dr} < 0$.

Now find an expression for the adiabatic change (no energy in or out). Take μ to be mean mass of a particle in the gas.

$$P = A\rho^\gamma, \quad (42)$$

from (37), where A is a constant. Through $P = nkT$, where n can be represented by the density divided by the average particle mass,

$$P = \frac{\rho}{\mu} kT \quad (43)$$

Differentiating (43) with respect to the radius, remembering to use the chain rule, gives

$$\frac{dP}{dr} = \frac{k}{\mu}\left(T\frac{d\rho}{dr} + \rho\frac{dT}{dr}\right). \quad (44)$$

Differentiating (42), and using $\frac{d\rho^{\gamma}}{dr} = \gamma\rho^{\gamma-1}\frac{d\rho}{dr}$, i.e. the chain rule, gives

$$\frac{dP}{dr} = \gamma\rho^{\gamma-1}A\frac{d\rho}{dr}$$

$$= \gamma\frac{P}{\rho}\frac{d\rho}{dr} \quad (45)$$

Substituting (45) into (44) gives

$$\left(\frac{dP}{dr}\right)_{ad} = \frac{k}{\mu}\left[\frac{T\rho}{\gamma P}\left(\frac{dP}{dr}\right)_{ad} + \rho\left(\frac{dT}{dr}\right)_{ad}\right] \quad (46)$$

Useful, however these equations make too many assumptions to be valid.

Rearranging (46) gives

$$\left(\frac{dP}{dr}\right)_{ad}\left[1 - \frac{k}{\mu}\frac{T\rho}{\gamma P}\right] = \frac{P}{T}\left(\frac{dT}{dr}\right)_{ad},$$

hence as (43) gives $\frac{kT\rho}{\mu P} = 1$,

$$\left(\frac{dT}{dr}\right)_{ad} = \frac{T}{P}\left(\frac{dP}{dr}\right)_{ad}\frac{\gamma - 1}{\gamma}. \quad (47)$$

Schwarzschild criterion for convective instability:

$$\frac{dT}{dr} < \left(\frac{dT}{dr}\right)_{ad} \quad (48)$$

Convection occurs if the temperature gradient is steeper than the adiabatic gradient. Taking logs,

$$\left(\frac{d(\log T)}{d(\log P)}\right)_{ad} \le \frac{d(\log T)}{d(\log P)} \quad (49)$$

Where does convection take place?

When the adiabatic gradient is small, $\gamma \to 1$. i.e. c_v becomes large. $\left(\gamma = \frac{c_p}{c_v} = 1 + \frac{R}{c_v}\right)$

(Remember $c_p - c_v = R$ the molar gas constant).

$c_v = \left(\frac{dQ}{dT}\right)_v$ large → large amounts of energy input gives only a small temperature rise.

1. Regions in a star where H and He make the transition from ionized to neutral.
   $T \sim 6000 \to 7000k$, $H \to p + e^-$.
   $T \sim 20{,}000 \to 50{,}000k$, $He \to He^+ + e^-; He^{2+} + 2e^-$
2. Regions with large energy flux – cores of hot stars (no cycles.)
   $L_r = 4\pi r^2 F(r)$
   $F \sim \frac{1}{r^2}$
   Though if it all occurred in a small bit near the core, bad things happen.
   $T_{eff} > 9000k$, AO core convection.
   $T_{eff} < 7600k$, late A, convection zone below thin radiative surface layers.
   Convective layer thin for $T_{eff} > 7000k$.
   $T_{eff} < 7000k$, outer convection zones and radiative inner regions.
   Very cool, convective zone extends to core.

2.5.4 Convective Energy Transport
At radius r, the total flux is:

$$F = F_r + F_c = \sigma T_{eff}^{\ 4} \frac{R_\odot^{\ 2}}{r^2} \quad (54)$$

Look at the convective part of this.

Material moved per second $= \rho v \sigma$, which has heat content $e = c_p T$. With $\sigma = \sigma_u + \sigma_d$ and $\sigma_d = \sigma_u$, net energy transport through area σ is:

$$2F_c = \rho_u v_u \left(c_p T_u + \frac{1}{2} v_u^{\ 2} \right) - \rho_d v_d \left(c_p T_d + \frac{1}{2} v_d^{\ 2} \right) Wm^{-2} \quad (55)$$

[In essence, this is the heat content + the kinetic energy, in both directions]
But there must be zero net mass transfer, so

$$\rho_u v_u = \rho_d v_d \quad (56)$$
$$\rho_u \approx \rho_d = \rho$$
$$v_u = v_d = v$$
$$F_c = \frac{1}{2} \rho v c_p \left(T_u - T_d \right) \quad (57)$$
$$F_c = \rho v c_p \Delta T \quad (58)$$

where v is the convective velocity.
But what are ΔT and v?

This is a research area, so this description could not be valid – bubbles are too ordered, it's likely to be more chaotic, for example. We don't really know the numbers, either…

Consider a convective shell with mass m_c, velocity v_c, thickness r_c, located at r.
Energy ℓ_c transported by heat excess $\frac{\delta T}{T}$.
Since $\delta P = 0$ (we are assuming equilibrium; differing pressure in the shell would quickly even out):

$$\left|\frac{\delta\rho}{\rho}\right| = \left|\frac{\delta T}{T}\right|$$

$$\frac{\delta T}{T} \sim \frac{\ell_c t}{u m_c} \sim \frac{\ell_c \left(r_c / v_c \right)}{u m_c} \quad (55)$$

Buoyancy acceleration

$$g' = g\left|\frac{\delta\rho}{\rho}\right|$$

Convective velocity

$$v_c \sim \sqrt{g' r_c}$$

Substituting into this, and remembering that $g = \frac{Gm(r)}{r^2}$,

$$v_c \sim \sqrt{\frac{Gm(r)}{r^2}\left|\frac{\delta\rho}{\rho}\right| r_c} \sim \sqrt{\frac{GM(r)}{r^2}\left|\frac{\delta T}{T}\right| r_c} \quad (56)$$

Substituting into (55), and rearranging,

$$\left(\frac{\delta T}{T}\right)^{3/2} \sim \frac{\ell_c r}{u m_c \sqrt{\frac{Gm(r)}{r_c}}} \quad (57)$$

To get order of magnitude estimate, consider whole star as convective.
$\ell_c \to L$
$m, m_c \to M$
$uM \to U$
$r, r_c \to R$

$$\left(\frac{\delta T}{T}\right)^{3/2} \sim \frac{L}{U}\frac{1}{\sqrt{\frac{GM}{R^3}}}$$

$$\frac{\delta T}{T} \sim \left(\frac{t_{ff}}{t_{KH}}\right)^{3/2} \quad (58)$$

$\frac{\delta T}{T} \sim 10^{-8}$
$v_c \sim 45 ms^{-1}$ (59)
Temperature gradient is adiabatic.

“All you need is a roof, and you’ve got a greenhouse.”

Pressure Scale Height
For a perfect gas in HSE we have:

$$P = \frac{\rho}{\mu} kT$$

$$\frac{dP}{dr} = -\rho g$$

$$\frac{1}{P}\frac{dP}{dr} = \frac{d(\ln P)}{dr} = -\frac{g\mu}{kT}$$

$$d(\ln P) = -\frac{g\mu}{kT} dr$$

Assuming that the gas is isothermal, so that at $T \neq T(r)$, this can be directly integrated to give

$$P_1 = P_0 e^{-\left(\frac{r_1 - r_0}{\lambda_p}\right)} \quad (60)$$

where λ_p is the isothermal pressure scale height

$$\lambda_p = \frac{kT}{\mu g} \quad (61)$$

More generally,

$$\lambda_p = -\left(\frac{d(\ln P)}{dr}\right)^{-1} = \frac{P}{g\rho} \quad (62)$$

The Mixing Length
If rising column and falling column have the same cross-section, after moving up one scale height the pressure has decreased by a factor e. Therefore the area of the rising column has increased by a factor e and there is no room for the falling column. Therefore the mixing length ℓ is less than or equal to a pressure scale height. Typically, $\lambda = 0.1\lambda_p$ - fit from observations.

We know that this is wrong, but we can't do much about it – the proper answer also describes nuclear bombs, so is classified.

2.6 Equations of Stellar Structure

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \quad (63)$$

[Lagrangian description; Aside in 2.3]

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \quad (64)$$

[Equation of HydroStatic Equilibrium; Aside in 2.3]

$$\frac{dl}{dm} = \varepsilon \quad (65)$$

[Lagrangian version of (27); 2.4]

$$\frac{dT}{dm} = -\frac{GmT}{4\pi r^4 P}\nabla \quad (66)$$

where

$$\nabla = \frac{d\ln T}{d\ln P} \quad (67)$$

[Probably something to do with equation (47) in Lagrangians; 2.5.3]
and for radiative transport we have

$$\nabla = \nabla_r = \frac{3}{16\pi acG}\frac{\kappa \ell P}{mT^4} \quad (68)$$

[Radiative transport equation in Lagrandians; 2.5.1]
while for convection

$$\nabla = \nabla_c \approx \nabla_{ad} \quad (69)$$

- Lagrangian coordinates, in equilibrium – no time dependencies.

But also need

- Equation of state $\rho(P,T)$
- Energy generation rate, ε
- Opacity κ