5. Introduction to Gas Dynamics
(DW Chapter 6)

5.1 Is the ISM a fluid?
For fluid approximation,
- Divide the gas into small volumes with size << overall scale of the flow pattern. Each small volume must contain many atoms.
- Represent the contents of each of these volumes dV by density ρ, temperature T (hence pressure P), and a fluid velocity $\underline{v}$.
    - o Requires Maxwell distribution of random atomic velocities around mean $\underline{v}$.
    - o Size of volume $\delta x >> \Lambda$, the mean free path of the atoms. Overall condition $\Lambda <<< L$
    - o Atom velocity randomized (relative to flow) before it has traveled a significant distance.

See Problem 2.6(a).
Yes: the ISM is a good fluid.

5.2 Equations of Gas Dymanics
5.2.1 Conservation of Mass
Consider a stationary volume $dV =$ unit area $\times$ dx in the flow direction, with a flow passing through it.
On the first side that the flow encounters, there is P, ρ, u. On the far side, i.e. after distance dx, we have $P + dP, \rho + d\rho, u + du$. Note that $u = v_x$.
Rate of increase of mass in $dV =$ mass in flow – mass out flow.

$$\left.\frac{\partial}{\partial t}\right)_x (\rho dx) = \rho u - (\rho + d\rho)(u + du)$$

Neglect second order terms.

$$dx \left.\frac{\partial \rho}{\partial t}\right)_x = -\rho du - u d\rho$$

$$\left.\frac{\partial \rho}{\partial t}\right)_x + u \left.\frac{\partial \rho}{\partial x}\right)_t = -\rho \left.\frac{\partial u}{\partial x}\right)_x \quad \text{“Equation of Continuity”.}$$

NB: you can think of $\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)$ as an operator. It gives the change with time at a point moving with the flow.
“Lagrangian derivative” written $\frac{D}{Dt}$.
Hence continuity:

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u}{\partial x}$$

(in 3D: $\frac{D\rho}{Dt} = -p \nabla \cdot \underline{v}$, where $\nabla \cdot \underline{v}$ is the outflow from fluid volume.)

5.2.2 Conservation of Momentum
Consider inflow and outflow as before.

$$\frac{\partial}{\partial t}(\rho u dx) = u \cdot \underbrace{(\rho u)}_{momentum} - (u+du)(\rho+d\rho)(u+du) + \underbrace{P - (P+dP)}_{Net\ force}$$

$$\rightarrow \ \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} = -\frac{\partial P}{\partial x} - u^2\frac{\partial \rho}{\partial x} - 2\rho u\frac{\partial u}{\partial x}$$

$$\rho\frac{\partial u}{\partial t} + u\left(-u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} - u^2\frac{\partial \rho}{\partial x} - 2\rho u\frac{\partial u}{\partial x}$$

Cancel terms

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} - \rho u\frac{\partial u}{\partial x}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

In 3D: $-\nabla P$.

Euler Equation $\equiv$ Newton II for fluids.

We must add other forces if needed to the RHS.

e.g. gravity:

$\underline{f} = \rho\underline{g}$ where $\underline{g}$ is the local acceleration. $\underline{f} = -\rho\nabla\phi$, the gradient of gravitational potential.

e.g. viscosity:

$\underline{f} = \eta\nabla^2\underline{v}$ (see later)

$$\rightarrow \ \rho\frac{D\underline{v}}{Dt} = -\nabla P - \rho\nabla\phi + \eta\nabla^2\underline{v} \ \ etc.$$

5.2.3 Conservation of Energy

In general, complicated!

Consider steady 1D flow. How much energy flows through unit area in time dt?

To see, replace right-hand part of flow with piston.

At $t = 0$, we have $x = 0$, and u, P, ρ.

At time $t = dt$, we have $x = udt$, and u, P, ρ again.

Energy = (internal) + KE + PE $= \rho u dt\left(e + \frac{1}{2}u^2 + \phi\right)$

This is the energy in the gas that has flowed past $x = 0$ in time dt.

e = internal energy per unit mass = specific internal energy.

But: we've missed out part of the energy flow. We've also done work on the piston.

This is $PdV = Pudt$.

→ total energy flow per second $= \underbrace{\rho u}_{mass\ flux}\left(e + \frac{p}{\rho} + \frac{1}{2}u^2 + \phi\right)$

$e + \frac{P}{\rho} = h$ the "specific enthalpy".

Since ρu is conserved in steady 1D flow, $h + \frac{1}{2}u^2 + \phi = \xi$ is also conserved.

"Bernoulli Constant" or "stagnation enthalpy".

In steady 3D flow, ξ is conserved along streamlines.

Full energy equation requires us to account for radiation, heat conduction, pressure fluctuations, etc.

Here we consider two special gases:
(a) Adiabatic Flow
Literally no heat flow, but in fluid dynamics means specific entropy s is constant for each fluid element.
Then $PV^\gamma = const.$ → $P = k(s)\rho^\gamma$
(γ is the ratio of specific heats).
NB for perfect gas, $e = \frac{1}{\gamma - 1}\frac{P}{\rho}$. Hence $h = \frac{\gamma}{\gamma - 1}\frac{P}{\rho}$
If all elements have same s, $k(s)$ is constant throughout flow. "isentropic flow".
$\gamma = 5/3$ for monatomic gases.

(b) Isothermal Flow
Often heat flow via emission and absorption of radiation is important, i.e. photoionized gas.
Then T is set by radiative heating and cooling.
For optically-emitting gas, forbidden-line thermostat keeps $T \approx const.$ throughout flow.

5.2.4 Equation of State
We use ideal gas law:

$$P = \left(\sum_i n_i\right)kT$$

I indexes particle species.
→ $P \approx n_H kT$ for neutral gas, $P = (n_p + n_e)kT = 2nkT$ for ionized gas.
(ignoring H_e.)
Adiabatic law $P = k\rho^\gamma$ is a second equation of state.
NB: for isothermal gas, $P \propto n = \frac{\rho}{m}$. Effectively $\gamma = 1$.

5.3 Sound Waves
Consider small fluctuations in the gas around equilibrium values P_0, ρ_0, $u = 0$.
Write:
$P = P_0 + P_1$
$\rho = \rho_0 + \rho_1$
$u = u_1$
$p_1 = dP = \frac{dP}{d\rho}d\rho = \gamma K \rho_0^{\gamma-1}\rho_1 = \frac{\gamma P_0}{\rho_0}\rho_1$
Linearize gas equations:
Continuity:
$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + u_1\frac{\partial}{\partial x}(\rho_0 + \rho_1) = -(\rho_0 + \rho_1)\frac{\partial u}{\partial x}$
$\frac{\partial \rho_1}{\partial t} = -\rho_0\frac{\partial u}{\partial x}$ (1)
Euler:

$$\left(\rho_0+\rho_1\right)\frac{\partial u_1}{\partial t}+u_1\frac{\partial}{\partial x}\left(u_1\right)=-\frac{\partial\left(P_0+P_1\right)}{\partial x}$$

→ $\rho_0\frac{\partial u_1}{\partial t}=-\frac{\partial P_1}{\partial x}=-\frac{\gamma P_0}{\rho_0}\frac{\partial\rho}{\partial x}$ (2)

$$\frac{\partial(1)}{\partial t}:\ \frac{\partial^2\rho_1}{\partial t^2}=-p_0\frac{\partial^2 u_1}{\partial t\partial x}$$

$$\frac{\partial(2)}{\partial x}:\ \rho_0\frac{\partial^2 u}{\partial x\partial t}=-\frac{\gamma P_0}{\rho_0}\frac{\partial^2 P}{\partial x^2}$$

→ $\frac{\partial^2\rho_1}{\partial t^2}=\left(\frac{\gamma P_0}{\rho_0}\right)\frac{\partial^2\rho_1}{\partial x^2}$

Wave equation, with wave speed (of sound):

$$c_3=\sqrt{\frac{\gamma P_0}{\rho_0}}=\sqrt{\frac{\gamma kT}{m}}$$

(For isothermal gas, put $\gamma=1$.

NB:

- Wave moves at plus or minus the sound speed $\pm c_s$ relative to the underlying fluid velocity u_0.
- $c_s\sim$ speed of molecules / atoms in gas
- If we convert all the gas' enthalpy into kinetic energy, we get
  $\frac{1}{2}u^2=\frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0}$
  → $u=\sqrt{\frac{2}{\gamma-1}}c_{s_0}$
  (of course this cools gas to $T=0$, $c_s=0$)
  i.e. pressure forces can create maximum speeds of order the original sound speed
- If flow changes slowly compared to the time it takes sound waes to cross a region, then the pressure has time to equalize across the whole region. → $P\approx const.$

5.4 Transport Properties

"Perfect fluid" has mean free path $\Lambda=0$.

Effects associated with finite Λ "transport properties".

- Diffusion
- Thermal and electrical conductivity
- Viscosity $\equiv$ internal friction in fluid.

Consider a shear flow. A set of layers with separation du, gradient $\frac{du}{dy}$. Take 3 layers: $u+du$, u and $u-du$.

Fast particles from the upper layer diffuse into the middle layer, giving a forward force. But slow particles from the lower layer contribute equal and opposite retarding force. So there is no net viscous force when you have a uniform gradient of velocity.

→ viscous forces depend on the second derivative of the velocity:

$f_{visc} = \eta \nabla^2 \underline{v}$

where the coefficient of viscosity $\eta \approx 2\rho\bar{v}\Lambda$, where $\bar{v}$ Is the mean particle speed. (See D&W chapter 6.1.4)

$\eta \approx 2.5\rho c_s \Lambda$

5.5 Dimensionless Parameters

Flow pattern determined by dimensionless parameters of flow, including:

Mach Number $M = \frac{u}{c_s}$.

Reynold's Number $\mathrm{Re} = \frac{\rho v L}{\eta} \approx M \frac{L}{\Lambda}$

Where L is characteristic length scale of flow. The Reynold's number gives us the ratio of the pressure forces to the viscous forces.

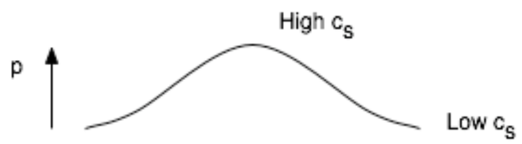
In astrophysical flows, $\mathrm{Re} >> 1$. This suggests that viscous forces should be relatively unimportant in astrophysics – we will see later that this is not necessarily so.

5.6 Shocks

Sound speed $= c_s = \sqrt{\frac{\gamma P}{\rho}}$.

For adiabatic flows, $P \propto \rho^\gamma$ → $c_s \propto \rho^{(\gamma-1)/2}$.

Consider a wave of finite amplitude.



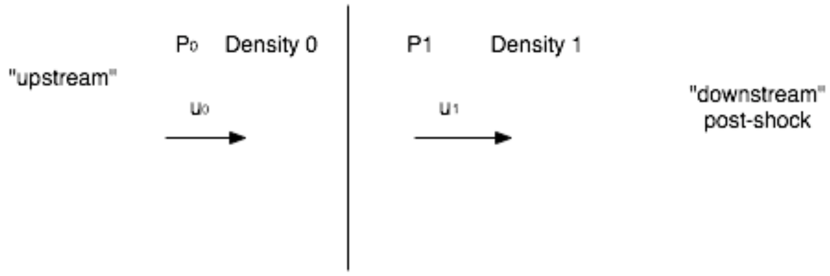
the peak tends to catch up with the trough.

Shock wave: sudden discontinuity (thickness $\sim \Lambda$) between fast-moving high pressure gas and slower gas ahead. The shock is supersonic with respect to c_s in the gas ahead.

Shocks are almost inevitable in supersonic flows, or equivalently in flows with large pressure gradients.

5.6.1 Jump conditions

In a frame moving with the shock:



Viscosity and thermal conduction at the shock heats and compresses the upstream gas, increasing entropy. But conservation laws still apply, with the special case of steady flow around the shock.
Mass flow:
$\rho_0 u_0 = \rho_1 u_1$ (1)
Momentum:
As in derivation of Euler equation, with $\frac{\partial}{\partial t} = 0$.
$\rho_0 u_0{}^2 - \rho_1 u_1{}^2 + P_0 - P_1 = 0$
i.e. $\rho_0 u_0{}^2 + P_0 = \rho_1 u_1{}^2 + P_1$ (2)
Energy:
$h_0 + \frac{u_0{}^2}{2} = h_1 + \frac{u_1{}^2}{2}$ (3)
"Shock Jump Conditions" aka "Rankine-Huyoniot Conditions"
From these equations and the condition that entropy is higher on the downstream side, we can solve for P_1, u_1, ρ_1 given P_0, u_0, ρ_0 (or vice versa).
→ but tedious in general. See DW 6.3.3.
NB: the conditions do not depend on details of viscosity etc. at the shock. The shock thickness automatically adjusts to satisfy the global conservation laws.

5.6.2 Strong Shocks

The strength of the shock is fixed by the upstream Mach number.

$$M_0 = \frac{u_0}{c_{s_0}} = \frac{u_0}{\sqrt{\frac{\gamma P_0}{\rho_0}}}$$

$$M_0{}^2 = \frac{\rho_0 u_0{}^2}{\gamma P_0}$$

If $M_0 >> 1$, $P_0 << \rho_0 u_0{}^2$, $h_0 << \frac{u_0{}^2}{2}$.
Now, using (2), as well as the mass conservation:
$P_1 = \rho_0 u_0{}^2 - \rho_1 u_1{}^2 = \rho_0 u_0 (u_0 - u_1)$
From (3):
$h_1 = \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{1}{2}\left(u_0{}^2 - u_1{}^2\right)$
Eliminating the pressure P_1 between these two equations, and using (1):

$$\frac{\gamma}{\gamma-1}\frac{\rho_0 u_0 \cancel{(u_0-u_1)}}{\rho_1}=\frac{1}{2}(u_0+u_1)\cancel{(u_0-u_1)}$$

$$\frac{\gamma}{\gamma-1}\frac{\rho_0}{\rho_1}=\frac{(u_0+u_1)}{2u_0}=\frac{1}{2}+\frac{u_1}{2u_0}=\frac{1}{2}+\frac{\rho_0}{2\rho_1}$$

$$\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)\frac{\rho_0}{\rho_1}=\frac{1}{2}$$

$$\frac{\rho_0}{\rho_1}=\left(\frac{2\gamma}{\gamma-1}-1\right)^{-1}=\left(\frac{\gamma+1}{\gamma-1}\right)^{-1}=\frac{\gamma-1}{\gamma+1}$$

$$\boxed{\frac{u_0}{u_1}=\frac{\rho_1}{\rho_0}=\frac{\gamma-1}{\gamma+1}}$$

For monatomic gas $\gamma = 5/3$, $\frac{\rho_1}{\rho_0}=4$.

This is the condition for an adiabatic gas, i.e. there is no heat flow from one part of the gas to the other. This is broken in some cases in the ISM, hence is not always applicable.

Now $P_1=\rho_0 u_0^2\left(1-\frac{u_1}{u_0}\right)=\frac{3}{4}\rho_0 u_0^2$. The shock converts most of the incoming "ram pressure" ρu^2 to thermal pressure.

With $u_0^2=\frac{3}{5}\frac{\rho_0 u_0^2}{P_0}$:

$$\frac{P_1}{P_0}=\frac{5}{4}M_0^2$$

$$T_1=\frac{\rho_1}{n_1 k_B}=\frac{\bar{m}P_1}{\rho_1 k_B}=\frac{3}{16}\frac{\bar{m}u_0^2}{k_B}$$

$\bar{m}$ "mean molecular mass"

$=2m_H$ molecular H

$=m_H$ atomic H

$=\frac{1}{2}(m_p+m_e)=\frac{m_H}{2}$ ionized gas.

$$T_1=11k\times\left(\frac{u_0}{1km\,s^{-1}}\right)^2$$

Downstream Mach number:

$$M_1^2=\frac{u_1^2}{c_{s_1}^2}=\frac{3}{5}\frac{\rho_1 u_1^2}{P_1}=\frac{3}{5}\frac{\rho_0 u_0^2/4}{3/4\,\rho_0 u_0^2}=\frac{1}{5}$$

The flow behind the shock is subsonic in the frame in which the shock is not moving.

$M_1=\frac{1}{\sqrt{5}}$ for a strong shock.

5.6.3 Frames of Reference

Often better to think of the shock as moving through initially stationary gas. Then:
Speed of shock $V_s = -u_0$

$u_0' = u_0 + V_s = 0$

$u_1' = \frac{u_0}{4} + V_s = +\frac{3}{4}V_s$

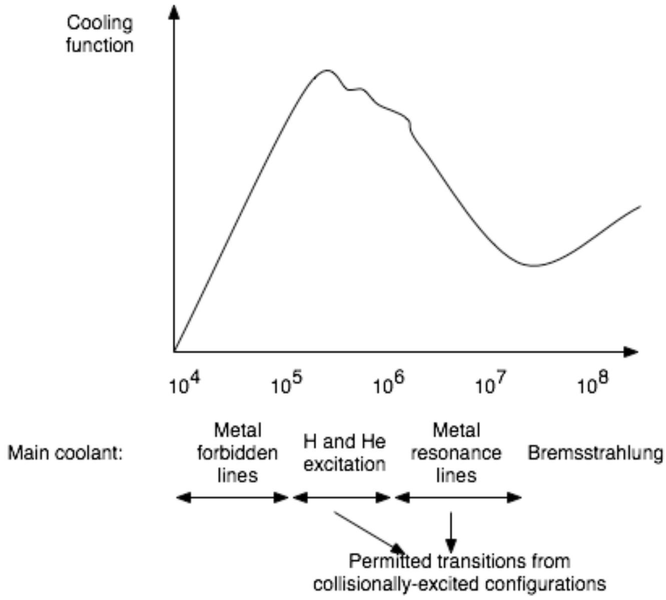
Shocked gas moves forward.

$M_s = \frac{V_s}{c_{s_0}} \equiv M_0$ (unsigned – we don't have negative mach numbers)

5.7 Shocked Gas

Post-shock H is collisionally ionized if $T \geq 4\times10^4 k$. i.e. $V_s \geq 50 km\, s^{-1}$.
Faster shocks give highly ionized metals, e.g. O*III* → O*IV* → O*VI* etc. as T increases.
Above $10^8 k$ $\left(V_s \geq 3000 km\, s^{-1}\right)$, all the atoms are fully ionized.

Shock-heated gas cools by radiation.
Emitted power per unit volume = $\int_0^\infty 4\pi j(\nu)d\nu = n^2\Lambda(T_e)$
This Λ is called the cooling function, and is nothing to do with the mean free path.

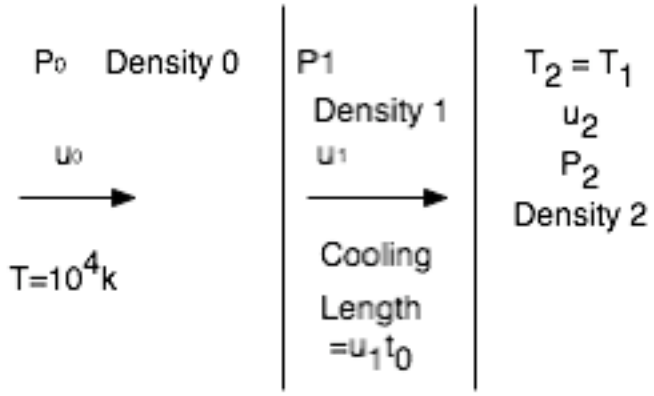


The cooling function has $\Lambda_{\max} \sim 3\times10^{-33} Wm^3$.
Cooling most effective at $\sim 10^5 k$.
Gas cools at ~ constant pressure → $n\uparrow$ as $T\downarrow$. Hence $n^2\Lambda$ increases as the gas cools.
Hence cooling very effective for $10 \leq V_s \leq 400 km\, s^{-1}$.

Cooling time $t_c \leq \frac{heat\ content\ per\ unit\ volume}{{n_1}^2\Lambda} = \frac{\rho h}{{n_1}^2\Lambda}$.

(Enthalpy not internal energy as constant pressure not volume.)
If cooling length $= u_1 t_c$ << scale of the flow, then the gas will rapidly cool until around $\sim 10^4 k$.

5.8 "Isothermal Shocks"
Shock with effective cooling.



The shock is adiabatic.
If cooling zone is thin enough, count as part of the shock transition.
New jump conditions:

$\rho_0 u_0 = \rho_2 u_2$ (1)

$\rho_0 u_0^2 + P_0 = \rho_2 u_2^2 + P_2$ (2)

$T_0 = T_2$ (3)

For isothermal gas;

$$c_{iso}^2 = \frac{P}{\rho} = const.$$

$$\rho_0\left(u_0^2 + c_{iso}^2\right) = \rho_2\left(u_2^2 + c_{iso}^2\right)$$

For strong shock, $M_0 = \frac{u_0}{c_{iso}} >> 1$.

$$\underbrace{\left(\frac{u_2}{u_0}\right)}_{from(1)} u_0^2 = u_2^2 + c_{iso}^2$$

Or $u_2^2 - u_0 u_2 + c_{iso}^2 = 0$.

$$u_2 = \frac{u_0 \pm \sqrt{u_0^2 - 4c_{iso}^2}}{2} \approx \frac{u_0}{2}\left(1 \pm \left(1 - 2\left(\frac{c_{iso}}{u_0}\right)^2\right)\right)$$

as $u_0 >> c_{iso}$.

Positive solution: $u_2 \approx u_0$ i.e. no shock.

Negative solution: $u_2 = \frac{c_{iso}^2}{u_0} = \frac{u_0}{M_0^2}$.

→ $\frac{\rho_2}{\rho_0} = \frac{u_0}{u_1} = M_0^2$.

This will no longer peak at 4 as it did in adiabatic shocks, but will keep on increasing without limit.
In rest frame of the upstream gas:

$$u_2` = u_2 + V_s = \left(-\frac{1}{M^2} + 1\right)V_s$$

i.e. $u_2` \approx V_s$. "snowplow" effect.

i.e. the shocked gas moves along with shock as a thin layer of high density "swept up" gas.

NB: $P_2 = \rho_2 c_{iso}{}^2 = \rho_0 M^2 c_{iso}{}^2 = \rho_0 u_0{}^2$.

We have 100% conversion of ram pressure to thermal pressure (unlike adiabatic, where we had ¾).