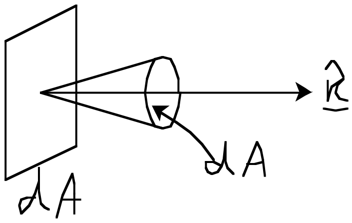
2. Radiative Transfer
DyW 2.2.3, B&M 8.1.4

2.1 Definitions
Frequency ν (Hertz)

Intensity $I(\nu)$ - a measure of power traveling along a ray.



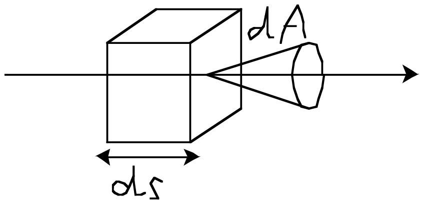
NB: area dA is perpendicular to the ray direction $\hat{k}$.

$$dE = I(\nu)\,d\nu\,dt\,d\Omega\,dA$$

Units of intensity are $W\,Hz^{-1}\,sr^{-1}\,m^{-2}$. ($sr$ is steradians)

Emission coefficient $j(\nu)$ - a measure of the power produced or radiated per unit volume.

$$dI(\nu) = j(\nu)\,ds$$



Absorption Coefficient $\kappa(\nu)$

Measure of the fraction of power lost per unit volume.

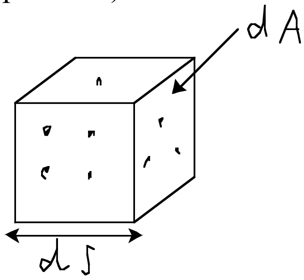
$dI(\nu) = -\kappa(\nu)\,I(\nu)\,ds$

Simple model for absorption:
Each particle has an absorption cross-section

$\sigma_{abs}(\nu) = Q_{abs}(\nu)\,\sigma_0$

where σ_0 is the true area, and $Q_{abs}(\nu)$ the absorption efficiency.

(not possible for electrons, as they have no true area, but good for things like dust particles).



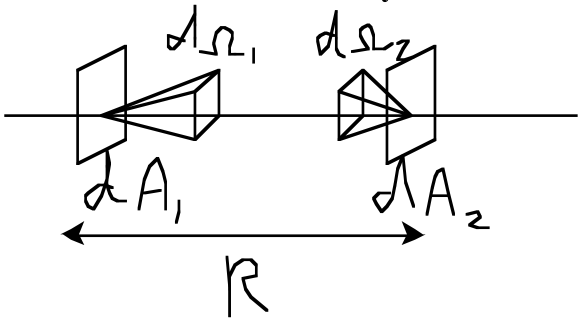
Number of particles $= n\,ds\,dA$.

→ total cross-section $= \sigma_{abs}(\nu)\, n\, ds\, dA$.

(assuming that the particles are not in front of each other, i.e. total cross-section is less than the total area)

→ absorption per unit area (dA): $\kappa(\nu) = n\sigma_{abs}(\nu) = nQ_{abs}(\nu)\sigma_0$

2.2 Conversation of Intensity



In free space (no absorption, emission or scattering) then the energy passing through a surface dA_1, heading for the second surface dA_2:

$$dE_1 = I_1(\nu)\, dA_1\, d\Omega_1\, d\nu\, dt$$

In a steady state, the energy arriving

$$dE_2 = I_2(\nu)\, dA_2\, d\Omega_2\, d\nu\, dt.$$

But from geometry (projecting the end of the cone all the way to the other area), $R^2\, d\Omega_2 = dA_1$ and $R^2\, d\Omega_1 = dA_2$.

So:

$$\begin{aligned} dE_1 &= I_1(\nu)\, R^2\, d\Omega_2\, d\Omega_1\, d\nu\, dt \\ &= I_2(\nu)\, R^2\, d\Omega_1\, d\Omega_2\, d\nu\, dt \end{aligned}$$

So $I_1(\nu) = I_2(\nu)$.

2.3 Equation of Transfer

$$\boxed{\frac{dI(\nu)}{ds} = j(\nu) - \kappa(\nu) I(\nu) + scattering}$$

s is distance along the ray. Sign convention is s increases from source towards the observer.

Scattering: complicated! Light can be scattered both into and out of the ray. See later, ignore for now.

Optical Depth: $\tau_\nu = \int_{s_0}^{s_1} \kappa(\nu)\, ds'$ where s_0 is the start point, and s_1 is the observer.

$$d\tau_\nu = \kappa(\nu)\, ds$$

So divide the radiative transfer equation by $\kappa(\nu)$.

$$\frac{dI(\nu)}{d\tau_\nu} = \frac{j(\nu)}{\kappa(\nu)} - I(\nu) = S(\nu) - I(\nu)$$

where $S(\nu)$ is called the Source Function.

2.4 Solutions
(See Examples 1)

(1) No absorption, just emission (common in radio and infrared)
$I(\nu) = \int_{-\infty}^{\infty} j(\nu)\,ds'$
(2) No emission in region considered, just absorption.
Start at $s = s_0$ with $I = I_0$.
$I(\nu) = I_o(\nu)e^{-\tau_\nu}$
(3) Both emission and absorption. But $S(\nu)$ is a constant along the line of sight.
$I(\nu) = I_0(\nu)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$
Tends to $I_0(\nu)$ as $\tau \to 0$.
Tends to $S(\nu)$ as $\tau >> 1$.

2.5 Kirchoff's Law
If the region is in Local Thermodynamic Equilibrium (LTE), i.e. within a small volume all the energy levels are populated according to the Boltzmann distribution with temperature T, then:

$$S(\nu) = B(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}$$

the Planck "Black Body" spectrum.
Peak of $B(\nu, T)$ at $\nu = 2.8\,{kT}/{h}$ (Wein displacement law).

At low frequency / long wavelength $B(\nu, T) = 2kT\frac{\nu^2}{c^2} = \frac{2kT}{\lambda^2}$. (Rayleigh-Jeans approximation [RJ])

2.6 Brightness Temperature
Radio astronomers often use the RJ approximation to measure the intensity in terms of "brightness temperature".

$$T_b(\nu) \equiv \frac{I(\nu)\lambda^2}{2k}$$

Caution: this is just a way of measuring the intensity $I(\nu)$ if the source radiation is in LTE then $T_b \le T$, and then if $\tau_\nu >> 1$ $\left(T_b(\nu) \approx T\right)$. Otherwise there is no definite connection to source temperature, especially if :
- Scattering
- Stimulated emission, e.g. lasers or masers.

Typo's:
Ex. 5: add $R_0 = 8.5 kpc$

Ex. 6: in equation, $\rho_g \to \rho_s$.

For an emitter / absorber in thermal equilibrium,

$$I(\nu) = I_0(\nu)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$$

Can be written as:

$T_b(\nu) = T_{b0}e^{-\tau_\nu} + T\left(1 - e^{-\tau_\nu}\right)$

→ $T_b(\nu) - T_{b0}(\nu) = \left[T - T_{b0}(\nu)\right]\left(1 - e^{-\tau_\nu}\right)$

If $T > T_{b0}$ we see emission lines.

If $T < T_{b0}$ then we get absorption lines.

2.7 Flux Density

Total apparent brightness of an object ("flux density" F_ν) is:

$$F_\nu = \int I\left(\nu, \underline{\hat{k}}\right) d\Omega$$

integral over the region of the sky covered by the light from the object.
Apparent magnitude is a measure of this flux density.

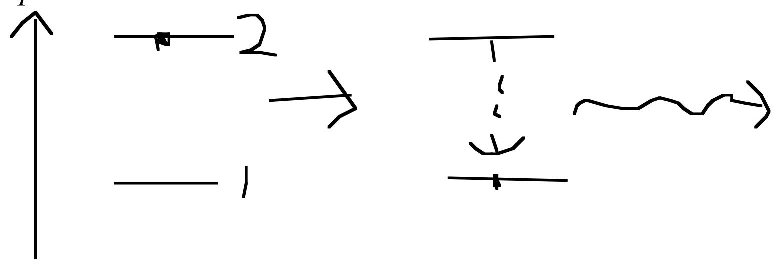
NB: the inverse square law applies to F_ν since $\Omega = \frac{A}{D^2}$. Hence $F_\nu \sim I(\nu)\Omega = \frac{I(\nu)}{D^2}$.

2.8 The Einstein Coefficients

(B&M 8.1.4)

Any time a photon is emitted or absorbed, the photon changes the energy level in the "atom" (molecules, crystals, light bulbs…). Consider just two energy levels (as just one photon is being considered). There are three possibilies:

1) *Spontaneous Emission*

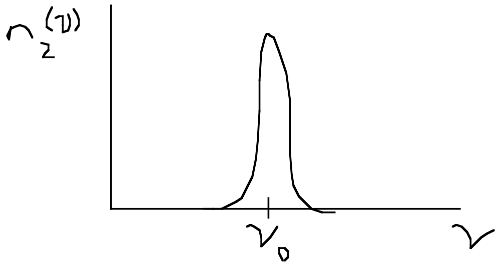


Photon has energy $h\nu = E_2 - E_1$.

Probability per atom in state 2 per second = A_{21}.

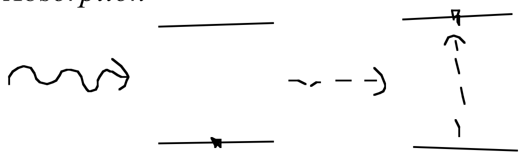
In reality there is always a small range of possible energies.

Write $n_2^{(\nu)}d\nu =$ the number of atoms in state 2 which can emit a photon with frequency between ν and $\nu + d\nu$.



$$\nu_0 = \frac{E_2 - E_1}{h}$$

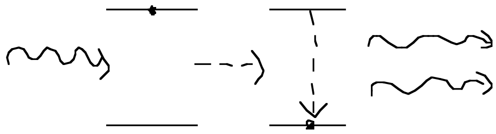
2) *Absorption*



Probability per atom in state 1 receptive to $\nu \to \nu + d\nu$ is $B_{12}u(\nu)$

where the photon energy density $u(\nu) = \frac{1}{c}\int_{4\pi} I(\nu, \hat{\underline{k}})\,d\Omega$.

3) *Stimulated emission*



Incoming photon energy $\approx h\nu_0$. Outgoing ones $= h\nu_0$ also.

Probability per atom receptive $\nu \to \nu + d\nu = B_{21}u(\nu)$.

To avoid perpetual motion machine of 2nd kind, we need the *Principle of Detailed Balance* – "In thermal equilibrium, every elementary process is statistically balanced by its exact reverse."

Here $N(2 \to 1 + \gamma) = N(1 + \gamma \to 2)$.

→ in LTE, $n_1^{(\nu)} B_{12} u(\nu) = n_2^{(\nu)} A_{21} + n_2^{(\nu)} B_{21} u(\nu)$.

→ $u(\nu) = \dfrac{n_2^{(\nu)} A_{21}}{n_1^{(\nu)} B_{12} - n_2^{(\nu)} B_{21}}$.

Boltzmann Distribution specifies that $\dfrac{n_1^{(\nu)}}{n_2^{(\nu)}} = \dfrac{g_1 e^{-E_1/kT}}{g_2 e^{-E_2/kT}} = \dfrac{g_1}{g_2} e^{h\nu/kT}$.

So $u(\nu) = \dfrac{A_{21}/B_{21}}{\dfrac{g_1}{g_2}\dfrac{B_{12}}{B_{21}} e^{h\nu/kT} - 1}$

In thermal equilibrium $u(\nu) = \dfrac{4\pi}{c} B(\nu, T)$.

So:

$g_1 B_{12} = g_2 B_{21}$

$$A_{21} = \frac{8\pi}{c^3}\nu^3 B_{21}$$

These are called the Einstein Relations. They are always true as A_{21}, B_{12}, B_{21} are micro properties which are independent of the incident radiation field.
Hence:

$$j(\nu) = \frac{1}{4\pi} h\nu n_2^{(\nu)} A_{21}$$

where $\frac{1}{4\pi}$ gets back to emission per steradian, $h\nu$ is the energy per transition, and $n_2^{(\nu)} A_{21}$ is the number of transitions per second per unit volume per unit frequency.

$$\kappa(\nu) = \frac{h\nu}{c}\left(n_1^{(\nu)} B_{12} - n_2^{(\nu)} B_{21}\right)$$

NB: stimulated emission included in absorption coefficient (negative absorption) as rate is proportional to $I(\nu)$.
Using the Einstein relations:

$$\kappa(\nu) = \frac{h\nu}{c} B_{12} n_1^{(\nu)}\left(1 - \frac{g_1}{g_2}\frac{n_1^{(\nu)}}{n_2^{(\nu)}}\right)$$

2.9 Masers
Boltzmann says

$$\frac{g_1 n_2^{(\nu)}}{g_2 n_1^{(\nu)}} = e^{-h\nu/kT} < 1.$$

→ $\kappa(\nu)$ is always positive provided we're in local thermal equilibrium. Otherwise, we can (and do) get "inverted populations":

$$\frac{n_2}{g_2} > \frac{n_1}{g_1}.$$

Hence, $\kappa(\nu)$ is negative and so is τ_ν.

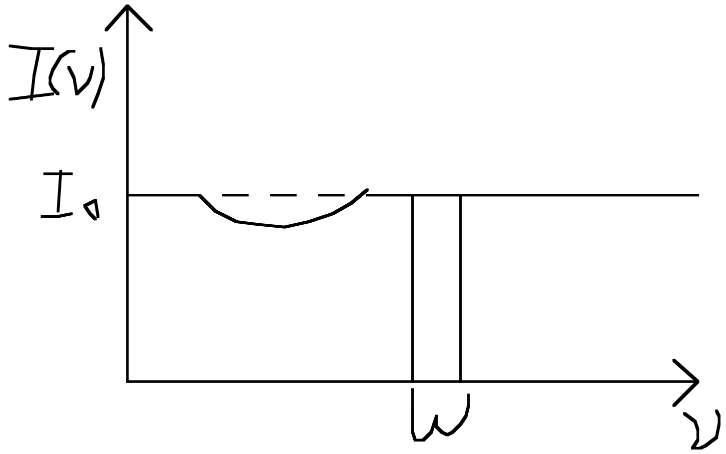
→ "Attenuation factor" $e^{-\tau_\nu}$ becomes an amplification factor.
Microwave Amplification by Stimulated Emission of Radiation (MASER).
Masers (even IR lasers) occur naturally in various astrophysical settings.

2.10 Spectral lines vs. Continuum
Isolated atoms (very simple molecules): levels are well separated. → photons have well-defined energy "line emission".
→ energy of the photon (or λ or ν) is a diagnostic of the emitting atom.

Molecules: energy levels crowd together → "band" (a bunch of close lines, sometimes overlapping).

Solids: lines / bands get broader and broader, "resonance". Can't tell you which molecules are involved exactly, but is characteristic of individual chemical bond vibrations e.g. C-H bonding.
Bulk solid / hot solid: resonances merge together → continual emission at all wavelengths over a very broad range of λ. This is a "continuum".

2.11 Equivalent width
We often observe spectral lines in absorption against a continuum emitter, such as a star or a quasar.
The point of a continuum is that, because it is so smooth, $I_0(\nu) \approx$ independent of ν.
τ_ν depends strongly on ν → large only near line centre ν_0.
$I(\nu) = I_0 e^{-\tau_\nu}$.



The solid line is the line as observed. The straight line at I_0 is the unaltered continuum emitter.
Define the "equivalent width" W which is the width of the continuum with equal area to that between the dotted line, and the actual observed line.

$$W = \int \left(1 - \frac{I(\nu)}{I_0}\right) d\nu = \int 1 - e^{-\tau_\nu} d\nu$$

For low τ_ν ("optically thin"), write $e^{-\tau_\nu} = 1 - \tau_\nu$.
$W = \int \tau_\nu d\nu$.
NB: W is a measure of line strength not the width.

2.12 Line Shape
(DW 2.2.2, B&M 8.1.1)
If the atoms are isolated and stationary, then the shape of the line obtained is the "natural line shape". This is determined by quantum mechanics, essentially:

$$\Delta E \Delta t = h \Delta \nu \Delta t \geq \frac{h}{4\pi}$$

Δt is the mean time per photon emission $= \frac{1}{A}$ (the Einstein coefficient for this particular transition).
In detail, the shape is a Lorentzian.

$$\phi(\nu) \propto \frac{1}{(\nu-\nu_0)^2 + \left(\gamma/4\pi\right)^2}$$

$$\gamma = \frac{A}{2}$$

If the atoms are moving according to Maxwell distribution, line is Doppler broadened.
Line of sight velocity

$$v_{los} = -\underline{v} \cdot \hat{\underline{k}}$$

(sign convention is that a positive velocity is moving away from the observer.)

$$\nu = \nu_0\left(1 - \frac{v_{los}}{c}\right).$$

$$P(v_{los}) \propto e^{-mv_{los}/2kT}$$

$$P(\nu) \propto \phi(\nu) \propto e^{-(\nu-\nu_0)^2/2\delta^2} \text{ Gaussian}$$

with $\delta^2 = \dfrac{\nu_0{}^2 kT}{mc^2}$.

Almost always $\delta >> A$ → line shape is Gaussian.
But this is only true at the line's centre. Lorentzian "winds" dominate at large $(\nu-\nu_0)^2$.

Bulk motions of the gas give further Doppler shifts and broadening of the line.

2.13 Notation

Line from a neutral atom of element x → denoted as x I e.g. H I.
Line from a singly-ionized ion x^+ denoted by x II
And so on.
e.g.: c^{3+} "C IV".

'Forbidden' lines (with very low A coefficient are:
$[OIII]$
or $[OIII]5007$ where the number is λ in Angstrom. One Angstrom $=10^{-10}m$.

Special case: Hydrogen HI.

Quantum Number n_2 n_1	$\lambda/\overset{o}{A}$	
2 1	1216	$Ly\alpha$
3 1	1026	$Ly\beta$
4 1		$Ly\gamma$
3 2	6563	$H\alpha$
4 2	4861	$H\beta$
		$H\gamma$
4 3	$1.875\mu m$	$Pa\alpha$

5 3		$Pa\beta$

Ly = Lyman series (UV)
H = Balmer series (optical)
Pa = IR.

2.14 Scattering and Extinction
(D&W 4.1.2)
Light removed from ray by both absorption (photon with $h\nu$, particle with E ending with particle with energy $E+h\nu$) and scattering (photon and particle, ending with particle and photon going off in another direction).
Pure dielectrics (real refractive index) scatter, don't absorb.
Real materials all absorb to some extent.
Combination of absorption and scattering = extinction.
We saw $\kappa(\nu) = nQ_{abs}(\nu)\sigma_0$.
Now define $\alpha(\nu) = n\left(Q_{abs}(\nu) + Q_{scatt}(\nu)\right)\sigma_0 = nQ_{ext}(\nu)\sigma_0$.
Then $I(\nu) = I_0(\nu)\exp\left[-\int_{s_1}^{s_0}\alpha(\nu)\,ds\right] = I_0 e^{-\tau_\nu}$.
On line of sight towards some strong emitter, e.g. a star.
In terms of magnitudes, the extinction coefficient

$$A(\lambda) = \Delta m = -2.5\log_{10}\left({I(\lambda)} \big/ {I_0(\lambda)} \right) = 1.086\tau_\nu$$

Usually quoted for a broad 'band'
Johnson U (UV), B (Blue), V (Visual - greenish), etc.
Hence A_V, A_B, …