(Lectures 7-8)
Liddle, Chapter 5
Simple cosmological models
(i) Hubble's "Law" – revisited.
Self-similar stretch of the universe. All universe models have this characteristic.
$v \propto r$; $v = Hr$.
since only this conserves homogeneity and isotropy.

Recall
$\underline{r}(t) = a(t)\underline{x}$
where $\underline{r}$ s the proper (physical) distance, $a(t)$ is the stretch factor, and $\underline{x}$ the commoving distance.
$\dot{r}(t) = \dot{a}(t)\underline{x}$ (since $\dot{x} = 0$ by definition).

$$\dot{r}(t) = \frac{\dot{a}(t)}{a(t)} \underbrace{a(t)x}_{r(t)}$$

where $\dot{r}(t)$ is the velocity, $a(t)x$ the distance, and hence $\frac{\dot{a}(t)}{a(t)}$ is Hubble's "constant".
"Velocity distance law" is a snapshot of the universe at a given time. However, note that as we look out in distance, we look back in time ($c \neq \infty$). Hence we would expect the Hubble 'law' to change at large distances (earlier times), when things were moving faster, and the universe was expanding at a different rate. *(Picture 26)*. We will return to this later.

Redshift and expansion of the universe (revisited)
Redshift $\equiv z = \dfrac{\lambda_{obs} - \lambda_{emitted}}{\lambda_{emitted}}$.
This implies that $1 + z = \dfrac{\lambda_{obs}}{\lambda_{emitted}}$.
z is due to the expansion of space itself; it is not a Doppler shift (something moving through space).
Photons don't have binding energy, therefore don't resist the expansion of the universe (problems 1, question 1.5).

$$\rightarrow 1 + z = \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a(t_{obs})}{a(t_{emit})}$$

This can be proved rigorously in GR.
We often set $a(t_0) = 1$, i.e. now.

$$1 + z = \frac{1}{a(t_{emit})}$$

e.g. if $z = 4$, then $a(t_{emit}) = \frac{1}{5}$, so the universe was $\frac{1}{5}$ its' current size when the light was emitted. This is regardless of the details of the cosmological model.
Cosmology tells us how long ago this was.

Solve Friedmann Equation

We need to understand the properties of constituents. In particular, the "equation of state"; how P and ρ are related (as in thermodynamics).

(i) *Matter*

We can always consider this as a gas (even whole galaxies), since the universe is so big.
(Can be "baryonic", or Dark Matter).
We know that $E_{total} = \gamma m_0 c^2$, or $E_{total} = \underbrace{m_0 c^2}_{rest\ mass} + \underbrace{(\gamma - 1) m_0 c^2}_{E_k}$.

At all times of relevance, matter is non-relativistic → $E_k \ll m_0 c^2$ (moves so slowly), so pressure is (roughly) 0 (pressure is E_k / Volume, i.e. energy density term). Therefore it can be ignored.

(ii) *Radiation*

Photons (+ neutrinos)
Radiation has no rest mass, but lots of energy due to motion. Therefore does have pressure. A standard result for relativistic matter is:

$$P = \frac{\rho_{rad} c^2}{3}$$

where the 3 comes from the 3 dimensions. This is very similar to the result in standard kinetic theory, $P_{gas} = \frac{1}{3}\rho_{gas}\left\langle \bar{U}^2 \right\rangle$, where $\left\langle \bar{U}^2 \right\rangle$ is the mean square speed.

In general, we have both $\rho = \rho_{matter} + \rho_{radiation} = \rho_m + \rho_r$. Consider the effects one at a time.

Fluid equation with $\rho_{matter} \to 0$. (matter only universe)

$$\dot{\rho}_m + 3\left(\frac{\dot{a}}{a}\right)\rho_m = 0$$

$$\frac{\dot{\rho}_m}{\rho_m} = -3\frac{\dot{a}}{a}$$

$$\frac{\rho_m}{\rho_{m,0}} = \left(\frac{a_0}{a}\right)^3$$

using some integration, $\rho_{m,0}$ is the current density of matter.
NB: if a is not written as $a(t)$, it should be. $a_0 \equiv a(t_0)$, where the 0 denotes that it is now.
Since $a_0 = a(t_0) \equiv 1$;

$$\rho_m = \frac{\rho_{m,0}}{a^3}$$

"obvious". Boxes shrink as they go back in time. Volume goes down by a^3.
Now substitute into the Friedmann equation, with $k = 0$ for simplicity (and since observation strongly favours $k = 0$).

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_{m,0}}{a^3}\right)$$

$$\boxed{\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho_{m,0}}{a}}$$

$$\rightarrow \dot{a} \propto \frac{1}{a^{1/2}}$$

The rate of change of the scale factor indicates that the universe is decelerating as gravity is pulling it back to the origin.

Time variation? Integrate the equation… Through separation of variables, we get:

$$a(t) = \left(\frac{3}{2}\right)^{2/3}\left(\frac{8\pi G\rho_{m,0}}{3}\right)^{1/3} t^{2/3}$$

i.e.

$$\boxed{\frac{a(t)}{a(t_0)} = \left(\frac{t}{t_0}\right)^{2/3}}$$

Therefore since $\rho_m = \frac{\rho_{m,0}}{a^3}$, we get

$$\boxed{\rho_m = \rho_{m,0}\left(\frac{t_0}{t}\right)^2}$$

Differentiate the previous equation wrt t:

$$\dot{a}(t) = \frac{2}{3}a(t_0)\frac{1}{t^{2/3}t^{1/3}}$$

$$\boxed{\frac{\dot{a}(t)}{a(t)} = \frac{2}{3}\frac{1}{t} = H(t)}$$

A classic cosmological solution.

<u>Radiation-only universe</u>

$$P_{rad} = \frac{\rho_{rad}c^2}{3}$$

Proceed via the Fluid equation as before (Liddle).
A 'physical' approach is more instructive.
Run the universe backwards, i.e. contract it. The volumes contract by $a(t)^{-3}$, and also the photons change λ by $a(t)^{-1}$. So change energy by $a(t)^{-1}$ (higher in the past; shorter λ). So the energy density $\propto a(t)^{-4}$.

$$\rho_{rad}(t) = \frac{\rho_{r,0}}{a(t)^4}$$

where $a(t_0) = 1$.

Going back to the Friedmann equation, with $k=0$ again:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{r,0}}{a(t)^4}$$

Without worrying about constant factors;

$$\frac{1}{a} \propto \frac{da}{dt}$$

$$a^2 \propto t$$

$$\boxed{a \propto t^{\frac{1}{2}}}$$

(which is different to the matter case) or;

$$\boxed{\frac{a(t)}{a(t_0)} = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}}$$

and again, $\rho_r(t) = \rho_{r,0}\left(\frac{t_0}{t}\right)^2$ (which is the same as the matter case).

This is another classic cosmological solution. Note that the case when radiation dominates implies that expansion rate is slower $\left(t^{\frac{1}{2}}\right)$ than when the matter dominates. $\left(t^{\frac{2}{3}}\right)$.

→ relativistic effect: energy has mass, therefore has gravity. Pressure (E_k terms) does not "blow up the universe"; it is exactly the opposite.

Mixtures:

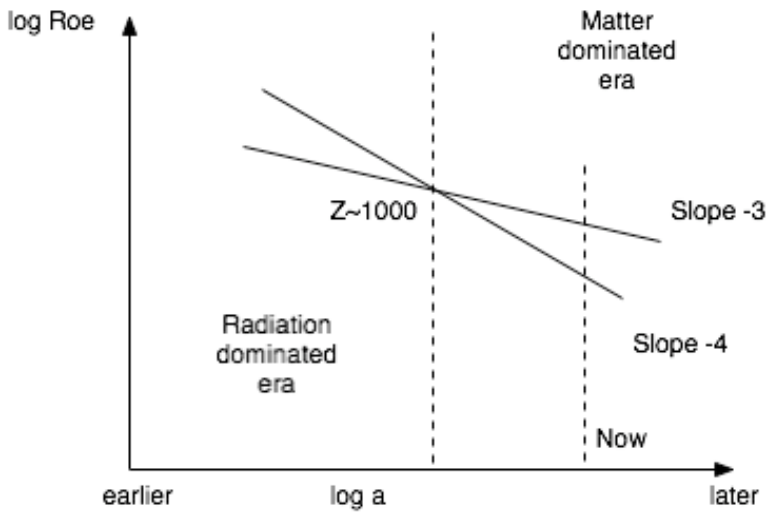
$\rho = \rho_{matter} + \rho_{rad}$.

A general solution of the Friedmann equation is complicated. However, one term dominates in "regions of most interest", i.e. now and the future (matter term), or at early times (radiation term).

1. Consider when radiation dominates, therefore sets the scale factor change with time.
   $a(t) \propto t^{\frac{1}{2}}$
   $\rho_{rad}(t) \propto t^{-2}$
   (as above)
   So the smaller constituent $\rho_{matter} \propto \frac{1}{a(t)^3} \rightarrow t^{-\frac{3}{2}}$.
   Therefore ρ_{matter} falls off more slowly than $\rho_{radiation}$ as universe expands. So eventually, matter will come to dominate.
2. Consider when matter dominates.
   $a(t) \propto t^{\frac{2}{3}}$
   $\rho_{matter} \propto t^{-2}$

→ the smaller constituent, $\rho_{radiation} \propto \frac{1}{a(t)^4} \propto t^{-8/3}$.

Therefore ρ_{matter} falls off more slowly than ρ_{rad} (again). Once matter dominates, it stays dominant.



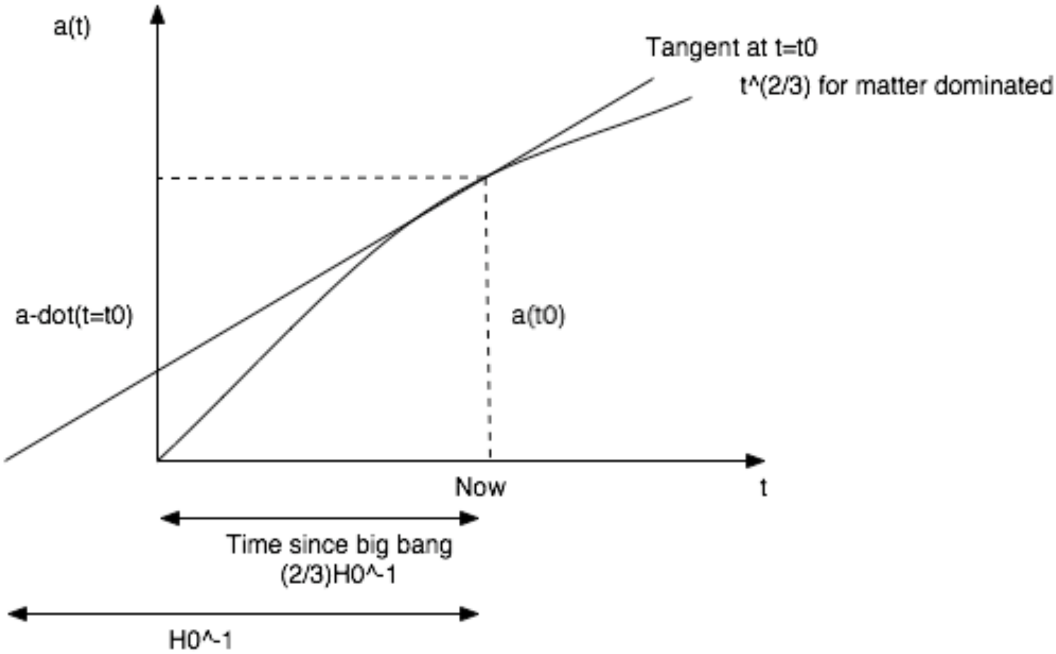
When the two lines cross, $Z \sim 1000$; each bit of the universe is 1000 times smaller than it is now in scale. Radiation dominates earlier, due to photons having higher energy at early times.

	Radiation dominated	Matter dominates
$a(t)$	$\propto t^{1/2}$	$\propto t^{2/3}$
$\rho_m \propto a^{-3}$	$\propto t^{-3/2}$	$\propto t^{-2}$
$\rho_{rad} \propto a^{-4}$	$\propto t^{-2}$	$\propto t^{-8/3}$

As we go from radiation dominated to matter dominated, the expansion rate speeds up $\left(\propto t^{1/2} \to \propto t^{-2/3}\right)$.

<u>Evolution including the k-terms</u>

1. $k = 0$



2. $k < 0$ (i.e. $U > 0$, E_k dominates)

   $FE = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$

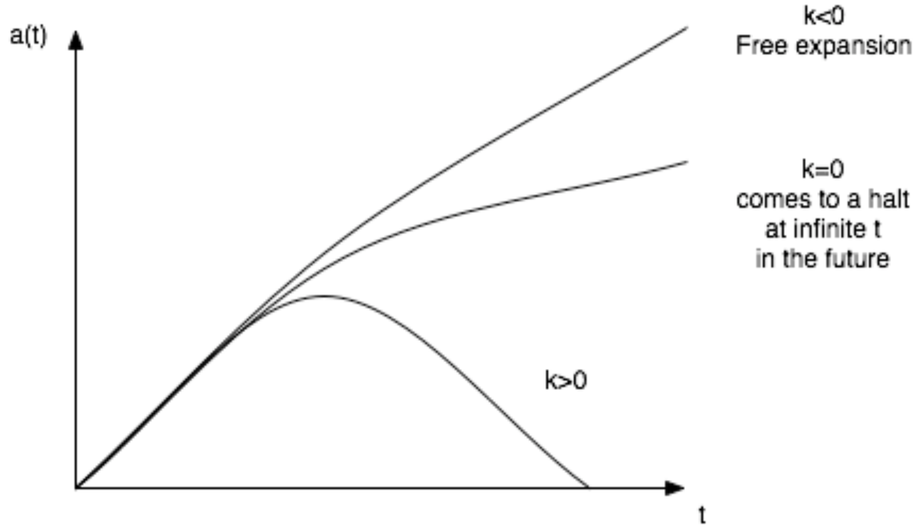
   this is always positive, and hence expands forever. At late times, $\frac{kc^2}{a^2}$ term dominates since $\rho \propto \frac{1}{a^3}$ at best.

   → $\left(\frac{\dot{a}}{a}\right)^2 = -\frac{kc^2}{a^2}$ at late times; $a \propto t$ (E_k dominating).

   → velocity tends to become constant. "Free expansion".

3. $k > 0$, i.e. U negative, so E_p "wins".

   Terms on the RHS of the Friedmann equation cancel. Eventually, the RHS becomes negative, i.e. positive curvature, i.e. gravity dominates.
   → universe "grinds to a halt", and then re-collapses. "Big crunch".



(*pictures 28, 29, 30)*

Observable Parameters

Hubble's constant (now): $H_0 \approx 70 \pm 10 kms^{-1}Mpc^{-1}$

Uncertainty arises from the difficulty in measuring the distances to better than 10%.

Use various forms of standard lengths, and have to calibrate.

Often parameterize the uncertainty.

$H_0 = 100h\, km\, s^{-1}Mpc^{-1}$

H_0 larger → universe expanding faster.

The Density Parameter

Friedmann equation without k (universe is flat):

$$\left(\frac{\dot{a}}{a}\right)^2 = H(t)^2 = \frac{8\pi G}{3}\rho$$

Critical density $\rho_{crit}(t) = \frac{3H(t)^2}{8\pi G}$

Put H into SI units:

→ $\rho_{crit}(t_0) = 1.88h^2 \times 10^{-26} kg\, m^{-3}$

This is very small, i.e. $11h^2$ hydrogen atoms per meter cubed.