Lectures 5-6

From the last lecture:

$$U = \frac{1}{2}m\dot{r}^2 - \frac{4}{3}\pi G\rho r^2 m \quad (2),$$

where U is the total energy, $\frac{1}{2}m\dot{r}^2$ the kinetic energy, and $\frac{4}{3}\pi G\rho r^2 m$ the potential energy.

If $U > 0$, then KE dominates and the universe expands forever.
If $U < 0$, then PE dominates → expansion stops and then contracts.

If the Cosmological Principle is correct, then the stretch in the universe is the same everywhere. Hence you get self-similar expansion. The distribution of the galaxies will remain the same; just their separation will change. *(Picture 22)*

Coordinates will be expanding, while galaxies remain in a "fixed position" with respect to an expanding coordinate system. Coordinates which are carried along with the expansion are called *co-moving coordinates*, here denoted by $\underline{x}$ (which is constant in time). Their physical distance (or "proper distance" if measured at the same "cosmic" time) is

$$\underline{r}(t) = a(t)\underline{x} \quad (3),$$

where $a(t)$ is the scale factor. This is the same everywhere. While the physical and co-moving distances cannot be found, it is possible to measure the scale factor through redshift.

Now substitute equation 3 into equation 2, and expressing it in terms of co-moving coordinates:

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4}{3}\pi G\rho a^2x^2 m$$

$$\frac{2U}{a^2} = m\left(\frac{\dot{a}}{a}\right)^2 x^2 - \frac{8\pi G}{3}x^2 m\rho$$

$$m\left(\frac{\dot{a}}{a}\right)^2 x^2 = \frac{8\pi G}{3}x^2 m\rho + \frac{2U}{a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \left(\frac{2U}{mx^2}\right)\frac{1}{a^2}$$

Note: $\frac{\partial r}{\partial t} = a\frac{\partial x}{\partial t} + \frac{\partial a}{\partial t}x = \frac{\partial a}{\partial t}x$. This is the Newtonian derivation. To get the same result as the full GR derivation, $\frac{2U}{mx^2}$ should be $-kc^2$.

The standard form of the Freedman equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}.$$

(NB; things like ρ change with time, and should be denoted $\rho(t)$ …)

Comments:

1. This is an energy relationship.
2. U in the Newtonian approach is the total energy of a particle / galaxy. In the GR approach, the k factor tells us about the curvature of space. It is equivalent but more sophisticated idea.
   U and k are constant in time. They were set at the beginning of the universe.
3. In the standard form, we only see the stretch / scale factor. The geometry of curved space is seen in k.
   (It seems that $k = 0$ from modern observations.)

This is the first equation expressing relative change in a under the effect of gravity. How is the stretch changing with time? We now need to know $\rho(t)$ to get a gravitating mass. → fluid equation.

Note that the universe contains atoms, radiation (including neutrinos), "dark matter", [and "dark energy"] (ignore the last one for the time being). All these are forms of energy, and they are distributed, so they have an energy density. So we need to consider pressure.

Define that Pressure = energy / volume = energy density. This definition is much more useful than force per unit area, as the universe has nothing for this force to apply against. Note that this is a jargon term; it is not the pressure that is expanding the universe. We really always translate this to "energy density".

Energy considerations → first law of thermodynamics.

$$dE = dQ - PdV$$

where dE is the change in internal energy, PdV is work done by the system, and dQ is the energy added to the system ($dQ = TdS$). In this case, $dQ = 0$ (energy does not enter the universe post-big-bang…).

$$\frac{dE}{dt} + p\frac{dV}{dt} = 0 \quad (5)$$

Let us consider a sphere of unit co-moving radius $x = 1$ (no loss of generality. All x's would cancel out.)

$$V = \frac{4}{3}\pi a^3$$

remembering that $r = ax$. We know that

$$E = mc^2$$

So

$$E = \frac{4}{3}\pi c^2 a^3 \rho$$

for a sphere in co-moving coordinates.

$$\frac{dE}{dt} = 4\pi a^2 \rho c^2 \dot{a} + \frac{4\pi}{3} a^3 \dot{\rho} c^2$$

through the chain rule.

$$\frac{dv}{dt} = 4\pi a^2 \dot{a}$$

Substitute into 5…

$$4\pi a^2 \rho c^2 \dot{a} + \frac{4}{3}\pi a^3 \dot{\rho} + 4\pi p a^2 \dot{a} = 0$$

With manipulation:

$$3\left(\frac{\dot{a}}{a}\right)\rho + \dot{\rho} + 3p\left(\frac{\dot{a}}{a}\right)\frac{1}{c^2} = 0$$

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\left[\rho + \frac{P}{c^2}\right] = 0 \quad (6)$$

This is called the Fluid equation. Notice that P/c^2 has units of mass density, therefore has the same units as ρ. It describes how the density changes in an expanding universe due to an increase in volume, and a decrease in thermal energy ("pressure" falls → grav. PE) (Adiabatic expansion).

NB: "pressure" actually increases gravitational attraction, or gravitational potential energy, since it is equivalent to a mass density $(E = mc^2)$, and mass gives rise to gravity in Einstein's relativity.

Pressure does not cause expansion; pressure forces imply pressure gradients (i.e. a wall or piston), but there are no gradients in a homogeneous universe. P and ρ are the same everywhere.

We now need a relation between P and ρ (the equation of state; they are different for matter and radiation) to work out the universe dynamics. We will come to this later…

There is an acceleration of a scale factor (differential Freedman equation). Equations 4 and 6 give another equation, which contains no new physics but is in a useful form. $\frac{d}{dt}(4)$ (taking care of the chain rule). Then substitute for $\dot{\rho}$ from (6), and rearrange. Substitute for $\left(\frac{\dot{a}}{a}\right)$ from (4) again.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \quad (7).$$

This is the differential Freedman equation. It is a force equation. k or U is not present – total energy does not affect forces (change in energies only).

We now have 2 (+1) basic equations. We will come back to these soon…

Geometry of the Universe

What is the k in the Friedman equation? It is proportional to U, the total energy, in Newtonian derivation. In GR, it tells us about the curvature of space under the influence of (mass-energy). See PC4771 for all the maths…

It turns out that the cosmological principle restricts 3D space to having only one of three basic characteristics.

1. "Flat" geometry, i.e. Euclidean $(k = 0)$. (it looks like this is correct)
   Axioms:
   a. straight line is the shortest distance between points.
   b. Parallel lines never meet
   c. The sum of the angles of a triangle are $180°$
   d. The circumference of a circle is $2\pi r$.
2. Spherical geometry $(k = +1)$
   a. "Parallel" (initially) lines do not stay a fixed distance apart.
   b. Circumference of a circle is $< 2\pi r$
   c. Sum of the angles of a triangle $> 180°$

   A good 2D example is the geometry on the surface of a sphere. 2D beings could recognize that they were on a sphere by making triangle and circle measurements (Don't have to go into 3D to see the sphere).
   3D space can be "curved", but it's impossible to visualize – but again we could tell by triangle and circle measurements. Light beams would diverge then converge.
   Note that if we do live in a positively curved space, then if we travel in a fixed direction we will return to the starting point, just as in 2D on the surface of a sphere.
   This gives rise to a finite but unbounded universe. This seems philosophically attractive, but it doesn't look like it's correct.
   "Space curves in on itself". → closed universe.
3. Hyperbolic geometry $k = -1$.
   This has the opposite properties to spherical.
   a. The sum of angles of a triangle $< 180°$
   b. Circumference of a circle is $> 2\pi r$
   c. Parallel lines diverge

   "Space curves away from itself". → open universe.

How do we determine the geometry of space? It is impossible via triangle and circle measurements. The scales are much greater than 10MPc, (30 million lightyears). So examine the properties of the universe at great distances, e.g.:

- The counts of galaxies as a function of distance: if it is spherical, then the numbers grow faster than for Euclidean, then peak and decrease (examples sheet 1, problem 2.2). If the universe is "spherical" (closed) grow less fast than if it's flat.
- Measuring a "standard rod" at great distance – know its' intrinsic size, and measure its' apparent size → geometry. Use statistics of major "blobs" in the CMBR (see later). Physics of the early universe → intrinsic size. The apparent size → $k = 0$ (very closely). These "blobs" are around $1°$ in size, which is 2x the full moon.

There is always the problem that the universe's content evolve with time, so the statistical counting methods aren't very useable. See later for the best approaches: look at the angular size of objects whose physical size we can calculate from first principles: CMBR.
(Pictures 23, 24, 25)