2. Nuclear Models

A complete theoretical description of the nucleus has not yet been accomplished (beyond $A \sim 10$). The Schrödinger equation must be solved for a many nucleon system with a complex set of interactions between constituents. Nevertheless, there are many properties of nuclei that can be understood using simple models, with some predictive power.

Macroscopic features:

- Nucleus behaves like a liquid drop of incompressible nuclear matter.
- Explains:
  - Nuclear stability
  - α, β fission decays
  - Gross trends in binding energy
  - Collective features of excited nuclei, e.g. rotations and vibrations.

Microscopic features:

- Nucleus is a collection of individual nucleons.
- The Shell model explains:
  - Corrections to gross trends in binding energy
  - Nuclear spins and magnetic moments
  - Non-collective excited states
  - β, γ decay rates

2.1 Liquid Drop Models

2.1.1 The Semi-Empirical Mass Formula

(See handout…)

$$Atom + E_B = Z_{protons} + N_{neutrons} + Z_{electrons}$$

$$\approx Z_{H\ atoms} + N_{neutrons}$$

where we neglect the binding energy, $13.6eV$, of the H atom.

2.1.2

(i) For odd A isotope chains, there is one stable odd-odd isobar.

(ii) For even A isotope chains, there are several stable even-even isobars (One odd Z, odd N. One even Z, even N. Distance between them 2δ).

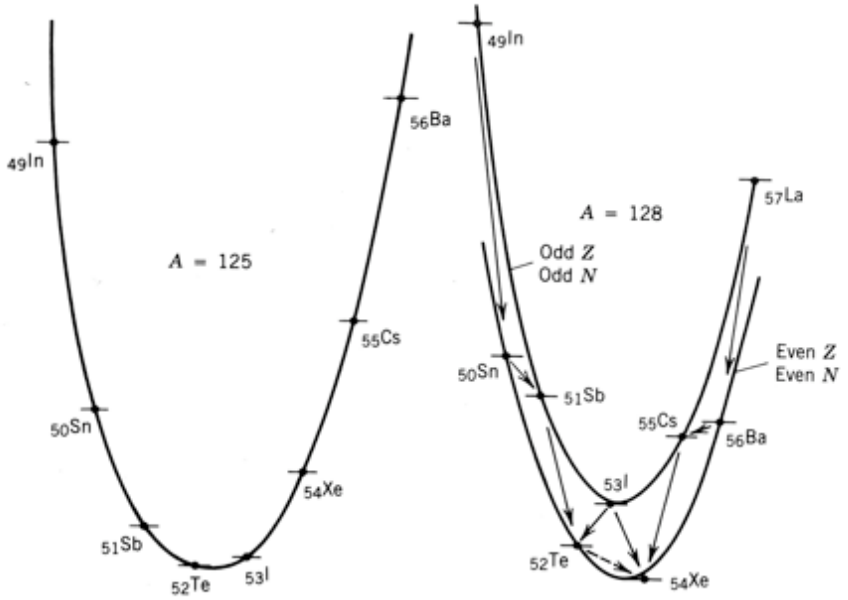


Figure 3.18 Mass chains for $A = 125$ and $A = 128$. For $A = 125$, note how the energy differences between neighboring isotopes increase as we go further from the stable member at the energy minimum. For $A = 128$, note the effect of the pairing term; in particular, ^{128}I can decay in either direction, and it is energetically possible for ^{128}Te to decay directly to ^{128}Xe by the process known as double β decay.

(iii) Energy from fusion:

Example: $^2H + {}^2H \rightarrow {}^4He$

$^2H = n + p + e^-$

$E_B\left({}^2H\right) = 2 \times 1.1 MeV = 2.2 MeV$

$E_B\left({}^4He\right) = 4 \times 7 MeV = 28 MeV$

$^2H + {}^2H + 4.4 MeV = 2m_p + 2m_n + 2m_e = {}^4He + 28 MeV$

Therefore: $^2H + {}^2H \rightarrow {}^4He + 23.6 MeV$ (1)

(iv) Energy from fission:

From the E_B / A vs. A curve, we see that $A \approx 60$ nuclei are most tightly bound. Thus when ^{238}U divides into a lighter Z, more strongly bound nuclei energy is released.

e.g. $\left({}^{100}A_1 + {}^{138}A_2\right) \times 8.2 MeV > \left({}^{238}U\right) \times 7.5 MeV$

Extra energy, here $166MeV$, is released per fission.

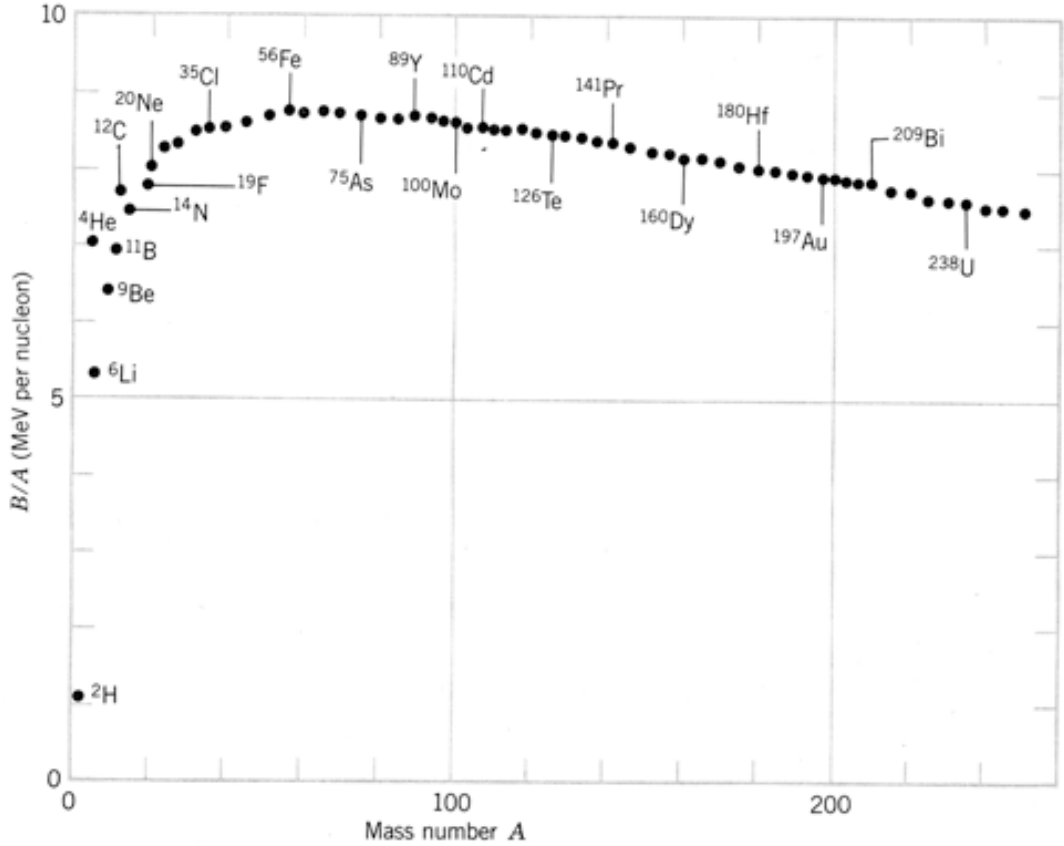


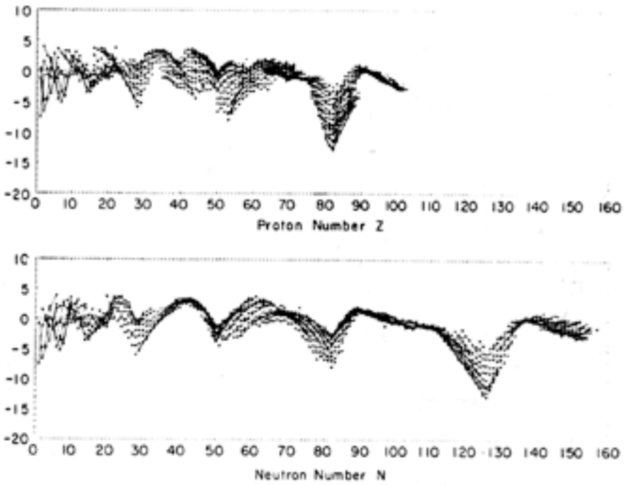
Figure 3.16 The binding energy per nucleon.

(v) Alpha Decay Process:
In heavy nuclei the disruptive Coulomb energy increases at a faster rate $\left(Z^2\right)$ than does the nuclear binding energy $(\propto A)$. The emission of a ${}^4He$ particle is a favoured agent for producing charge.
Energy available for decay = $E_B\left({}^4He\right)+E_B(Z-2,A-4)-E_B(Z,A)$ (refer to equation 1).
Alpha emission is favoured because it has a relatively high binding energy (Krane table 8.1) for ${}^{232}U$ out of all possible light ejectiles, ${}^4He$ emission is the only energetically possible process.

2.1.3 Microscopic effects not explained by the liquid drop model

(Evidence for nuclear shell structure).

(i) Systematic deviations from the smooth curve of binding energy from the semi-empirical mass formula.



Deviations of nuclear masses from their mean values

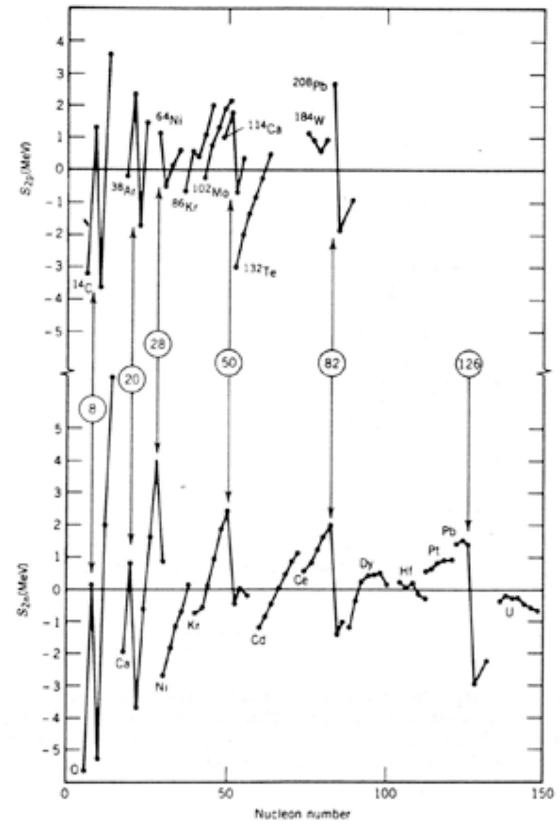


Figure 5.2 (Top) Two-proton separation energies of sequences of isotones (constant N). The lowest Z member of each sequence is noted. (Bottom) Two-neutron separation energies of sequences of isotopes. The sudden changes at the indicated "magic numbers" are apparent. The data plotted are differences between the measured values and the predictions of the semiempirical mass formula. Measured values are from the 1977 atomic mass tables (A. H. Wapstra and K. Bos, *Atomic Data and Nuclear Data Tables* **19**, 215 (1977)).

Extra binding is seen when Z or N are near 28, 50, 82, 126.

(ii) Proton and neutron separation energies: s_p, s_n, s_{2p}, s_{2n}.

(Equivalent to "ionization energy" of atomic electrons.
E.g. s_{2n} = energy to pull a neutron pair off a nucleus (a measure of the binding energy of the last pair of neutrons).
s_{2n}, s_{2p} are greatest at $N, Z = 8, 20, 28, 50, 82, 126$.

(iii) Nuclear charge radii do not increase smoothly but kink upwards as N passes 28, 50, 82 and 126.

(iv) Energy required to raise nucleus to first excited state is a maximum when $N = 8$, 20, 28, 50, 82, 126.

2.2 Shell Model

The available evidence suggests protons and neutrons fill orbits and shells similar to atomic electrons. It has been possible to map out the density distribution, $|\psi(r)|^2$, of the last (82nd) proton on ^{206}Pb by comparing the charge distribution for $^{205}_{81}Tl$ and $^{206}_{82}Pb$ by electron scattering (fig. 5.13 in Krane)

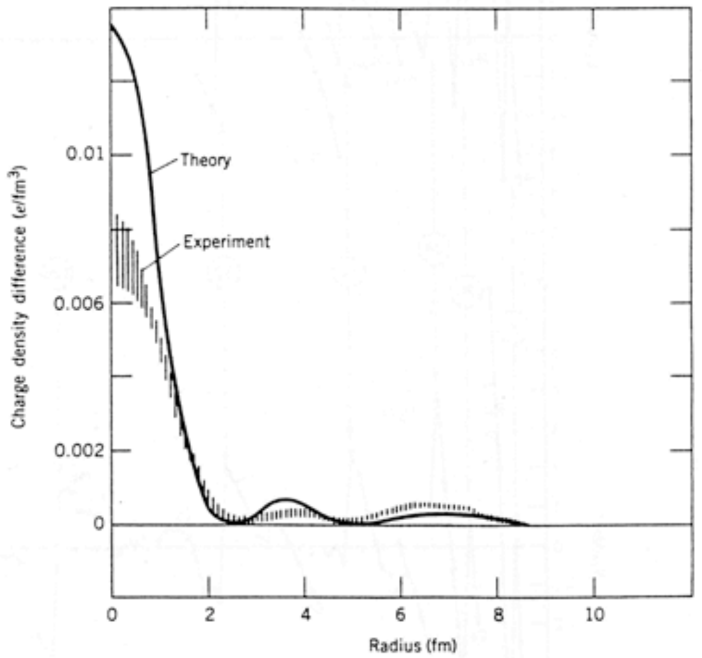
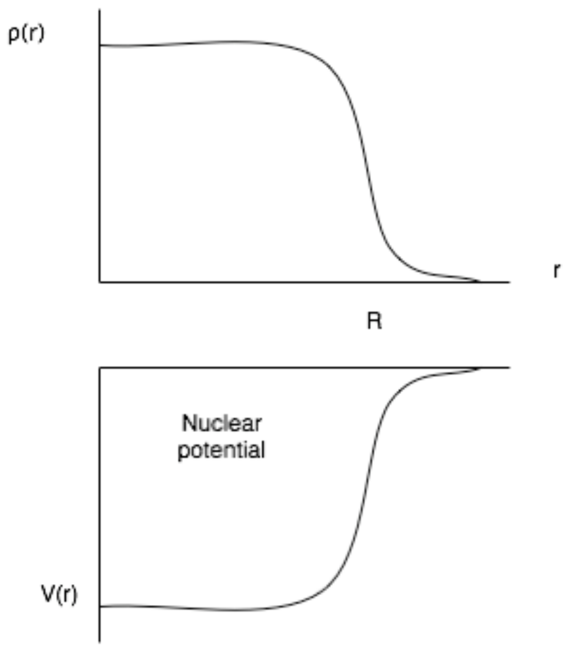


Figure 5.13 The difference in charge density between ^{205}Tl and ^{206}Pb, as determined by electron scattering. The curve marked "theory" is just the square of a harmonic oscillator 3s wave function. The theory reproduces the variations in the charge density extremely well. Experimental data are from J. M. Cavedon et al., *Phys. Rev. Lett.* **49**, 978 (1982).

It turns out that $|\psi(r)|^2$ is very close to that of the expected 3s orbit.

2.2.1 Solving Schrödinger equation for a simple central potential

We assume that each nucleon moves independently in a potential $V(r)$ that represents the average interaction with all the other nucleons.



The Pauli exclusion principle suggests that nucleons (in closed shells) should have long mean free paths in the nucleus because there are no vacant states that they can scatter into. (the only place it could go to is a higher energy state, but it hasn't got enough energy to get up to that…)

The potential depicted is called the Woods-Saxon Shape, and is:

$$V(r) = \frac{-V_0}{1 + e^{\frac{r-R}{a}}}$$

where a is the length to go from 90% to 10% of the potential, and R is the point at which the potential is at 50% of its maximum.

The single particle levels can be calculated by solving the 3D Schrödinger Equation.

$$H\psi(r) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(r) = E\psi(r)$$

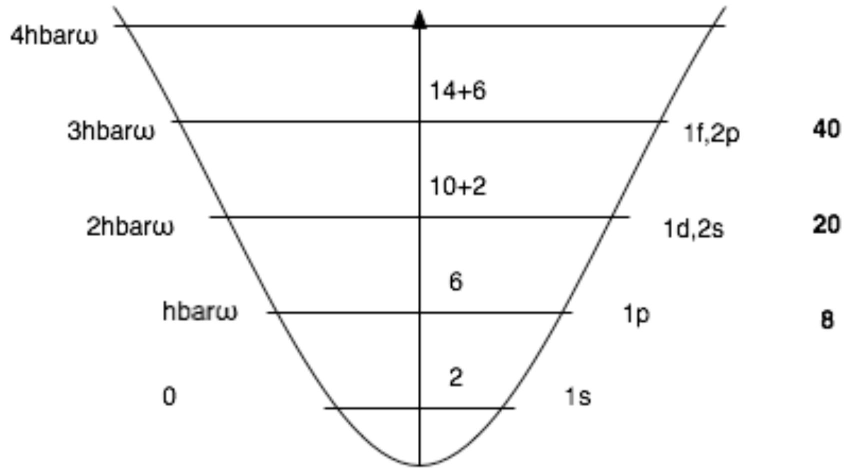
We can separate variables if $V(r)$ has no θ or ϕ dependence (Exactly like the H atom approach).

Ignoring nucleon spin,

$$\psi_{n,\ell,m}(r,\theta,\phi) = R_{n,\ell}(r)Y_{\ell,m}(\theta,\phi)$$

where $m \equiv m_\ell$ (the z-component of ℓ with $2\ell+1$ values $m = +\ell, \ell-1, \ldots, -\ell$), $R_{n,\ell}$ is the radial function, and $Y_{\ell,m}$ is the angular functions in the form of spherical harmonics (angular momentum eigenstates).

The solutions for the nuclear (Woods Saxon) potential are similar to the 3D harmonic oscillator.



We do get some magic numbers right, but it goes wrong higher up.

NB:

i. Self-consistency of $V(r)$: as the orbits are filled, we want the density distribution $\left(\rho = \sum_{i=1}^{A} |\psi(n)|^2\right)$ to match the chosen potential $V(r)$. We can normally choose the parameters V_0, R to achieve this.

ii. Technically, we should include the Coulomb potential for protons – but this has only a small effect in light nuclei – ignore for the present.

2.2.2 Spin-Orbit Interaction

Early on, it was apparent that a spin-orbit interaction was present (e.g. nucleon scattering experiments.

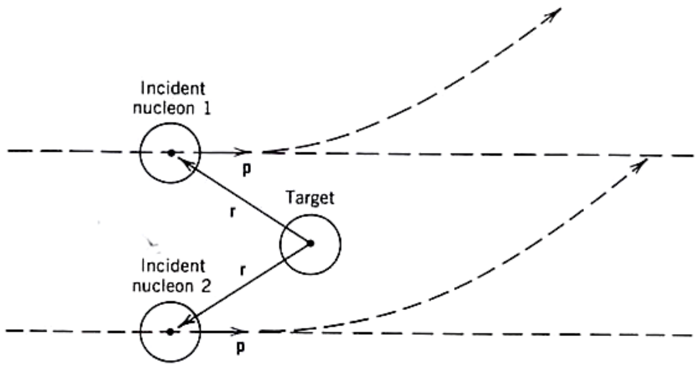
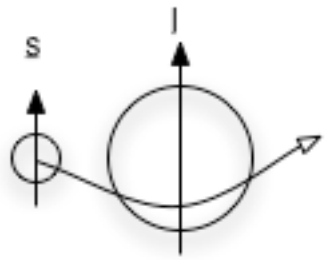


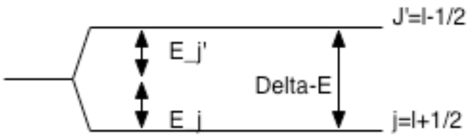
Figure 4.14 Top view of nucleon–nucleon scattering experiment. All spins point up (out of the paper). Incident nucleon 1 has $\boldsymbol{r} \times \boldsymbol{p}$ into the paper, and thus $\boldsymbol{\ell} \cdot \boldsymbol{S}$ is negative, giving a repulsive force and scattering to the left. Incident nucleon 2 has $\boldsymbol{r} \times \boldsymbol{p}$ out of the paper, resulting in an attractive force and again scattering to the left.

The force is more attractive when nucleon's spin $\underline{s}$ is parallel to its orbital angular momentum $\underline{\ell}$.



$\underline{s} \parallel \underline{\ell}$ is favoured.
We must add a term $-V_{so}\underline{\ell}\cdot\underline{s}$ to the $V(r)$ term in the Hamiltonian.
The result in each nucleon orbit ℓ (except s-states $(\ell\cdot s)$ is split into two components, labeled by the total spin j.
$\underline{j} = \underline{\ell} + \underline{s}$

m has $(2\ell+1)$ substates for one orbit. $m = +\ell, (\ell-1), ..., -\ell$. As $s = \frac{1}{2}$, there is also $m_s = +\frac{1}{2}, -\frac{1}{2}$. So in total, we have $(4\ell+2)$ separate combinations of (m_s, m) quantum numbers.
Adding the $-V_{so}\underline{\ell}\cdot\underline{s}$ state:



In the lower energy state: $m_j = j, j-1, ..., -j$, so $(2\ell+2)$ states.
In the higher energy state, $m_{j'} = j', j'-1, ..., -j'$ so 2ℓ distinct states.
Total number of states $= 2\ell + 2 + 2\ell = 4\ell + 2$, which is the same as before.

J can be viewed as a new angular momentum formed from $\underline{j} = \underline{\ell} + \underline{s}$.

Calculation of ΔE
We want the energy shifts $\varepsilon_j = -V_{so}\langle\underline{\ell}\cdot\underline{s}\rangle$ for both j, j' states.
Standard procedure:
$\underline{j} = \underline{\ell} + \underline{s}$
Square both sides.
$j^2 = (\underline{\ell}\cdot\underline{s})(\underline{\ell}\cdot\underline{s}) = \ell^2 + s^2 + 2\underline{\ell}\cdot\underline{s}$
Therefore $\underline{\ell}\cdot\underline{s} = \frac{1}{2}(j^2 - \ell^2 - s^2)$.

$$-V_{so}\langle\underline{\ell}\cdot\underline{s}\rangle = -\frac{V_{so}}{2}\left(j(j+1) - \ell(\ell+1) - s(s+1)\right)\hbar^2$$

$j' = \ell - \frac{1}{2}$, so $\varepsilon_{j'} = -\frac{V_{so}}{2}\left(\left(\ell-\frac{1}{2}\right)\left(\ell+\frac{1}{2}\right) - \ell(\ell+1) - \frac{3}{4}\right)\hbar^2$

$j = \ell + \frac{1}{2}$ so $\varepsilon_j = -\frac{V_{so}}{2}\left(\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right) - \ell(\ell+1) - \frac{3}{4}\right)\hbar^2$

$$\Delta E=\varepsilon_{j'}-\varepsilon_j=\frac{V_{so}}{2}\left(\left(\ell+\tfrac{1}{2}\right)\left(\ell+\tfrac{3}{4}-\ell+\tfrac{1}{2}\right)\right)\hbar^2=\frac{V_{so}}{2}(2\ell+1)\hbar^2$$

Therefore ΔE increases with ℓ.

The introduction of the spin-orbit interaction is able to count for the experimental shell closures which occur at $2,8,20,28,50,82,126$.
NB: all magic numbers (shell gaps) above $N,Z=20$ are produced by large spin-orbit splittings.

2.2.3 Filling orbits

Notation: $n\ell_j$, where n is the number of times the ℓ-value has occurred in the level sequence $(n=1,2,3,...)$.
ℓ:

Letter	ℓ	Parity
s	0	+
p	1	-
d	2	+
f	3	-
g	4	+
h	5	-

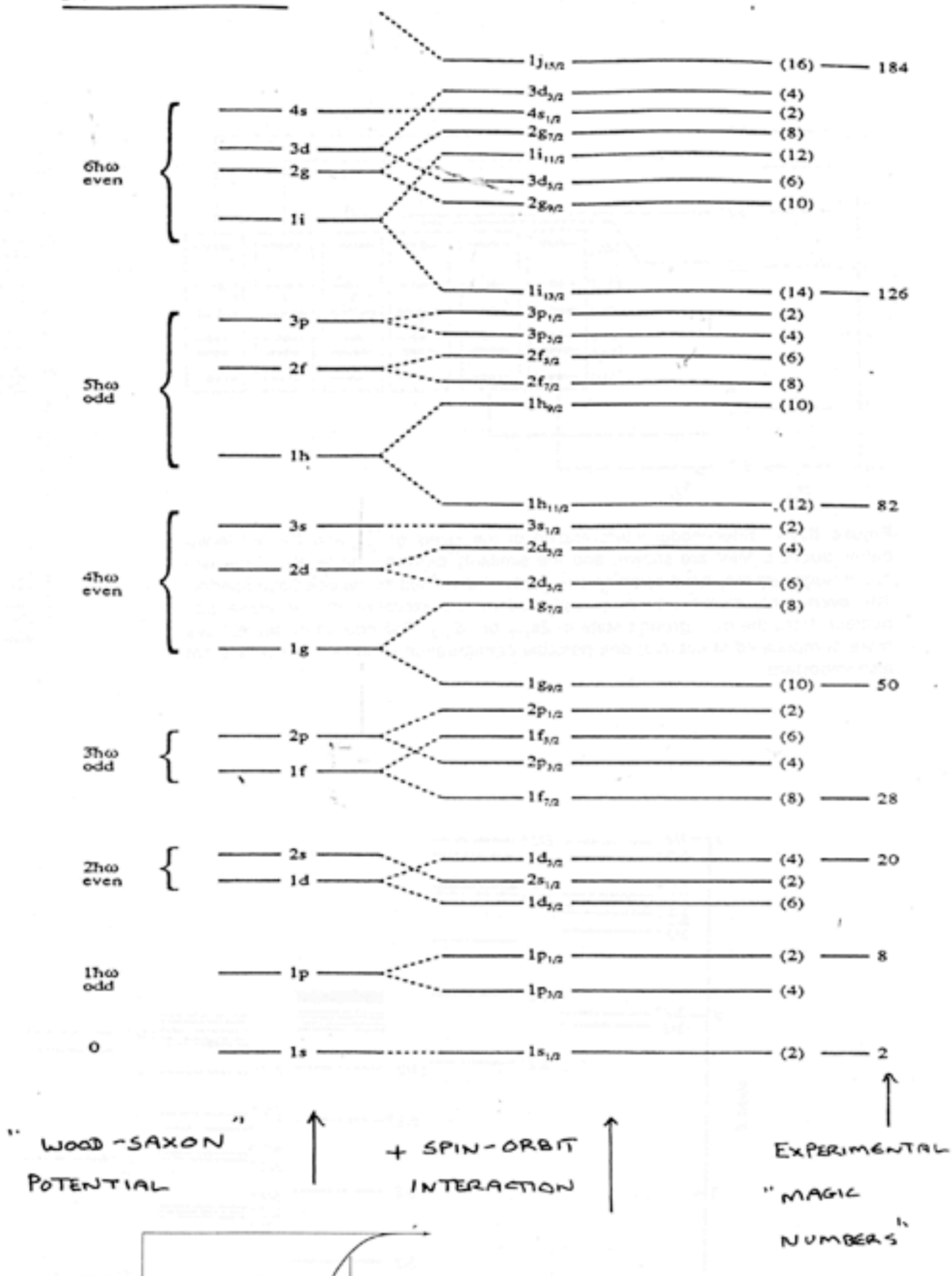
Parity π: $\psi(-r)=$ either $+\psi(r)$ or $-\psi(r)$. It is the spatial symmetry of $\psi(r)$.
"reflection through the origin".

$$\boxed{\pi=(-1)^\ell}$$

Since the Hamiltonian is invariant under the parity operation ($(-p)^2=p^2$ so the kinetic energy doesn't change. $V(-\underline{r})=V(\underline{r})$ as we have a simple potential. So then ψ must have the property of + or - $\psi(\underline{r})$.

We fill the levels according to the Pauli exclusion principle (like fermions must have distinct quantum numbers n,ℓ,j,m_j).

SHELL MODEL: ORDERING OF LEVELS



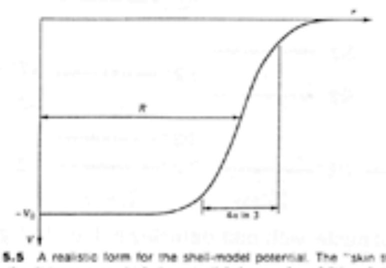
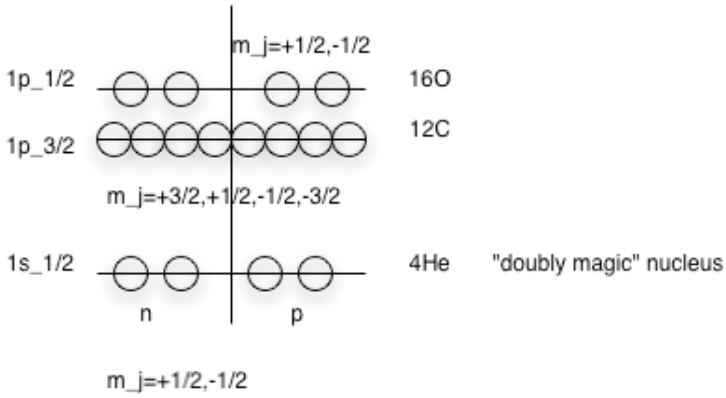


Figure 5.5 A realistic form for the shell-model potential. The "skin thickness" $4a \ln 3$ is the distance over which the potential changes from $0.9V_0$ to $0.1V_0$.



Expected to remember up to 20.

2.2.4 Predictions of ground state spins and parities

We assume the pairing interaction couples pairs of protons and pairs of neutrons into spin and parity $I^\pi = 0^+$ pairs. This is not always a good assumption, but <u>all</u> even nuclei have $I^\pi = 0^+$ ground states.

<u>Off odd-nuclei</u>

Spin and parity of nucleus is determined by the last unpaired nucleon.

Examples:

${}^{9}_{4}Be_5$ $\frac{3}{2}^-$ Last neutron is in the $1P_{3/2}$ level → $I^\pi = \frac{3}{2}^-$ (negative parity as it's a P state with $\ell = 1$)

${}^{13}_{7}N_6$ $\frac{1}{2}^-$ last proton is $1P_{1/2}$ → $I^\pi = \frac{1}{2}^-$

${}^{17}_{9}F_8$ $\frac{5}{2}^+$ last proton is $1D_{5/2}$ → $I^\pi = \frac{5}{2}^+$, positive parity as $\ell = 2$.

${}^{41}_{18}Ar_{23}$ $\frac{7}{2}^-$ last neutron is $1f_{7/2}$ → $I^\pi = \frac{7}{2}^-$

But:

${}^{19}_{9}F_{10}$ $\frac{1}{2}^+$ last proton is $1d_{5/2}$, which means we predict $I^\pi = \frac{5}{2}^+$, which is wrong.

<u>For odd-odd nuclei</u>

The low-lying states (including the ground state) are made by vector coupling the spins of the unpaired proton and neutron.

Possible spins $\underline{I} = \underline{j_p} + \underline{j_n}$,

where j_p is the spin of the last unpaired proton, and j_n the spin of the last unpaired neutron.

$I = |j_p - j_n|, \ldots, (j_p + j_n)$

All these states will exist, but it is difficult to predict which one is the ground state.

Parity (multiplicative) $= \pi_p \times \pi_n$.

Example: ${}^{16}_{9}F_7$ → $1d_{5/2}$ proton. $1P_{1/2}$ neutron. $\frac{5}{2}^{+} + \frac{1}{2}^{-} \rightarrow \begin{cases} 3^- \\ 2^- \end{cases}$ *states*

We also expect at low energy: $2s_{1/2}$ proton, $1p_{1/2}$ neutron. $\frac{1}{2}^{+} + \frac{1}{2}^{-} \rightarrow \begin{cases} 1^- \\ 0^- \end{cases}$.

Excited states in all nuclei can be made:

1. By promoting nucleons up to higher levels (figure 5.11 Krane)

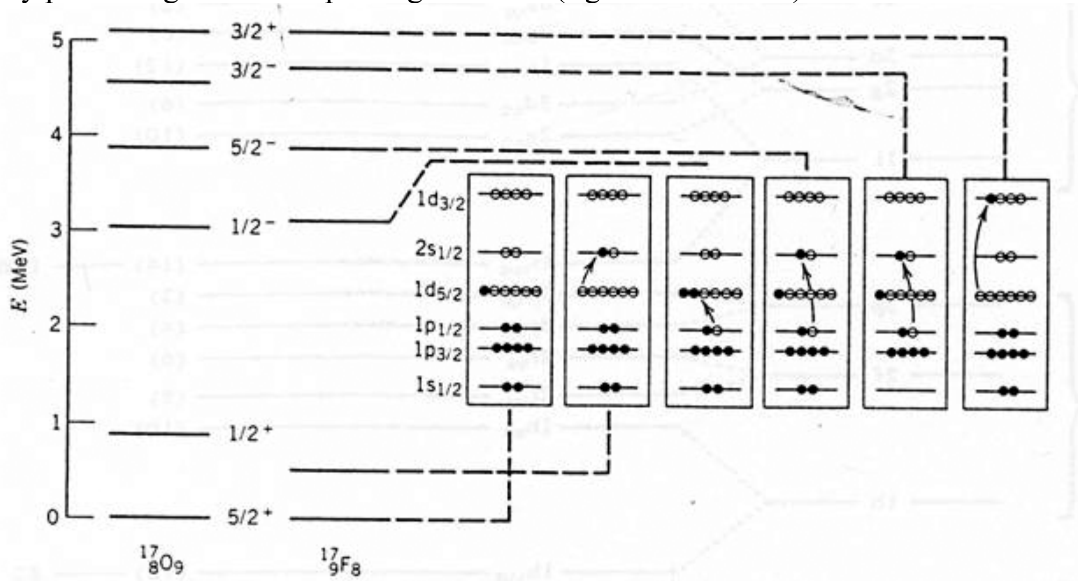


Figure 5.11 Shell-model interpretation of the levels of ^{17}O and ^{17}F. All levels below about 5 MeV are shown, and the similarity between the levels of the two nuclei suggests they have common structures, determined by the valence nucleons. The even-parity states are easily explained by the excitation of the single odd nucleon from the $d_{5/2}$ ground state to $2s_{1/2}$ or $1d_{3/2}$. The odd-parity states have more complicated structures; one possible configuration is shown, but others are also important.

Dark dots: filled states. Light dots: empty states.

and / or:

2. By breaking a 0^+ pair of nucleons and re-coupling their spins:

$$I = \underbrace{j_1 + j_2}_{were\, 0^+} + j_{odd}$$

This happens at low excitation in ${}^{43}_{20}Ca_{23}$, ${}^{43}_{21}Sc_{22}$, but can't occur in ${}^{41}_{20}Ca_{21}$.

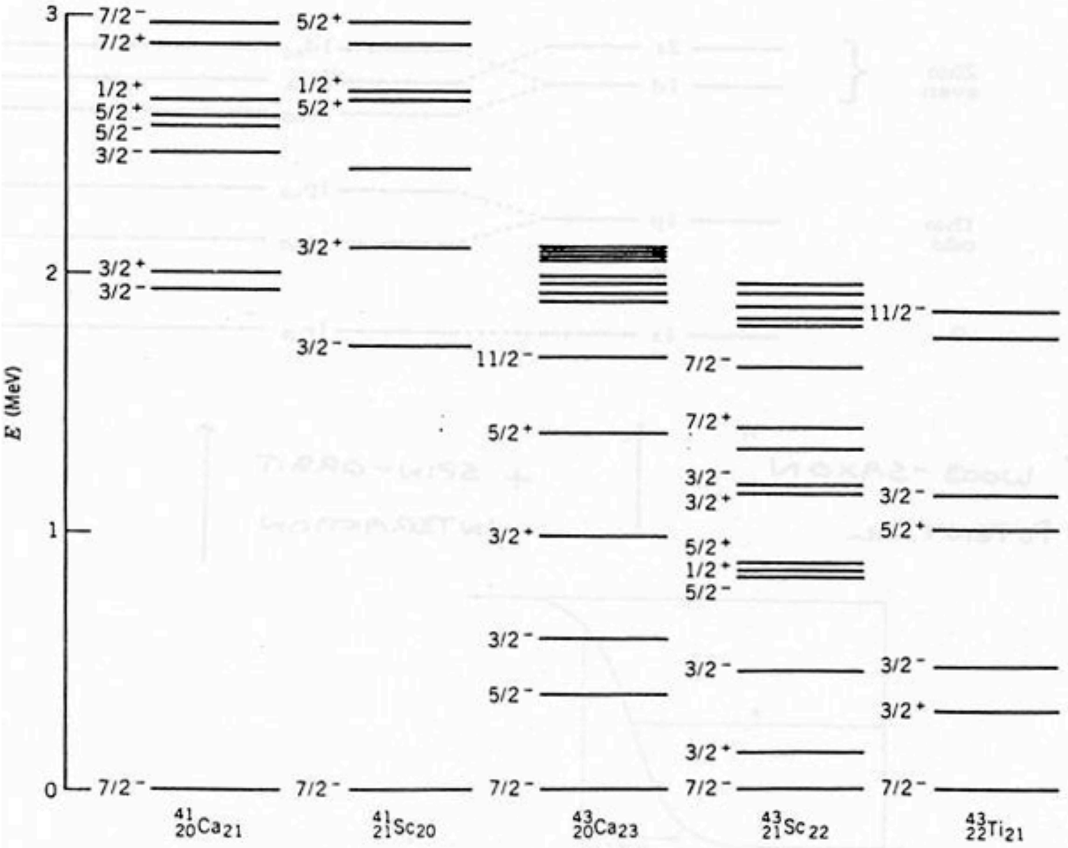


Figure 5.12 Energy levels of nuclei with odd particles in the $1f_{7/2}$ shell.