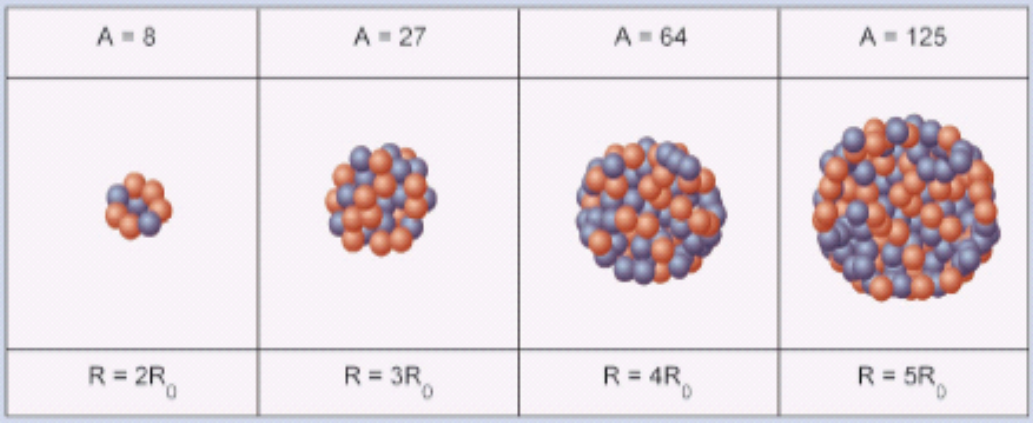
1. Nuclear Size



How does the nuclear radius depends on the number of nucleons? Most nuclei have the same density, so their volume is proportional to the number of nucleons. As a result, the cube of the radius is also proportional to the number of nucleons, as we see here.

We have known since Rutherford's $\alpha-$scattering work at Manchester in 1907, that almost all the mass of the atom is contained in a very small volume with high electric charge. Nucleus with Z protons and N protons. $A = Z + N$ nucleons.

A typical atom radius is a few $\times 10^{-10}\,m$ (Angstroms). The nuclear radius is a few $\times 10^{-15}\,m$ (Fermi).

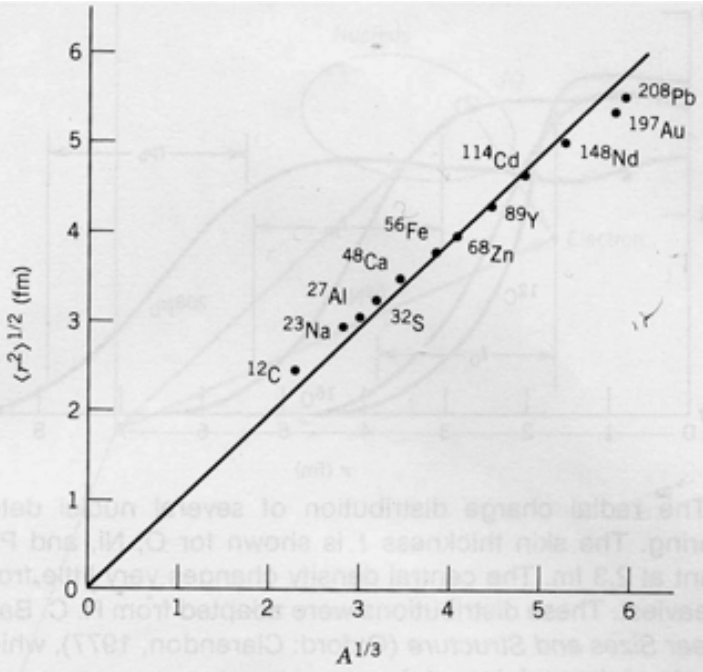


Figure 3.5 The rms nuclear radius determined from electron scattering experiments. The slope of the straight line gives $R_0 = 1.23$ fm. (The line is not a true fit to the data points, but is forced to go through the origin to satisfy the equation $R = R_0 A^{1/3}$.) The error bars are typically smaller than the size of the points (± 0.01 fm). More complete listings of data and references can be found in the review of C. W. de Jager et al., *Atomic Data and Nuclear Data Tables* **14**, 479 (1974).

Nuclear radius: We will see that almost all nuclei have a similar nuclear density.

Therefore the nuclear volume $\left(\frac{4}{3}\pi R^3\right) \propto A$.

$$R = 1.2A^{1/3} fm$$

Experimental methods to measure two types of nuclear distributions:

1. The Proton distribution (also the charge distribution) "nuclear charge".
   Use the Coulomb interaction between the nucleus and charged leptons (e^-, μ^-)
2. Nuclear matter (protons + neutrons)
   Use probes which interact via the Strong interaction, and at high energies so that the weaker coulomb interaction can be ignored. Use p, α, π, …

1.2 Nuclear Matter Radius from total reaction cross-section

A crude method used now for very exotic, short-lived nuclei. Consider a classical collision of two spheres (ignore the Coulomb force).

Take a projectile with radius R_1, striking a stationary target with radius R_2. The distance between the centers of the nuclei is b, the impact parameter. If $b > R_1 + R_2$, no interaction occurs, and the projectile simply continues without any deviation. If $b < R_1 + R_2$, something happens. e.g. elastic scattering, fusion of the two nuclei, exchange of nucleons, …

Whatever happens, it interrupts the path of the projectile and the position of the target. We define the geometric limit for the effective area that will remove particles from the beam

$$\sigma_T = \pi\left(R_1 + R_2\right)^2$$

Example: measuring size of ^{11}Li $\left(\tau_{1/2} \sim 87 ms\right)$.

Take a thin slab of, say, carbon with thickness x, area A (the area drops out later), and n nuclei per unit volume. Each nucleus within this slab has an effective black area with cross-section σ_T. Let there be a fast beam of ^{11}Li incident on this slab, with a flux of N_0 nuclei per second. On the other side of this slab, we have a ^{11}Li flux of $N_0 - \Delta N$, where ΔN is the reaction rate within the slab.

The target is thin so that there are no shadowing effects, i.e. nuclei hiding behind other nuclei. If this weren't the case, there would be an exponential factor to account for this.

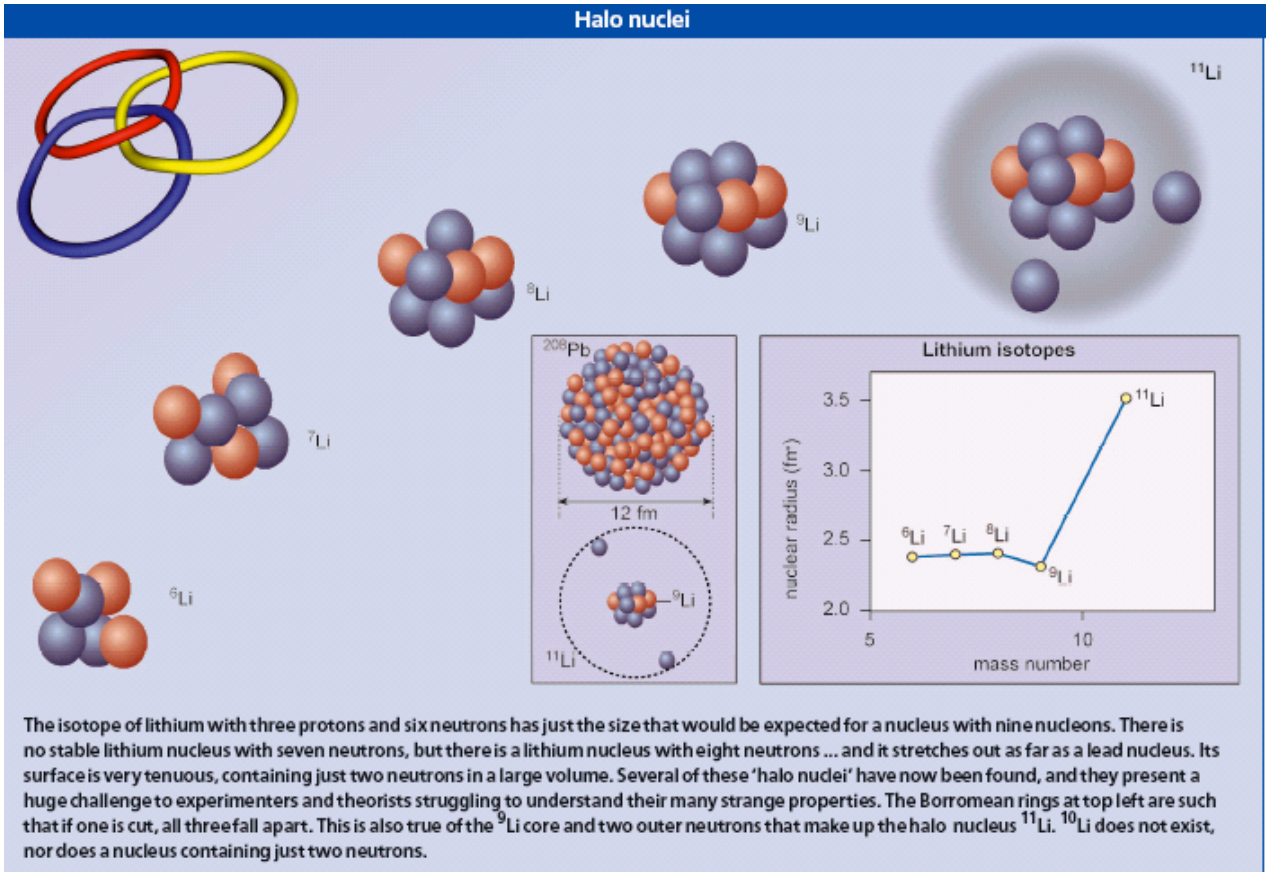
The effective black area across A $= \sigma_T \times$ number of nuclei $= \sigma_T n \underbrace{Ax}_{V}$.

The chance of a beam particle not getting through the (thin) target is the black area / the total area $= \sigma_T nx$

Therefore the reaction rate is:

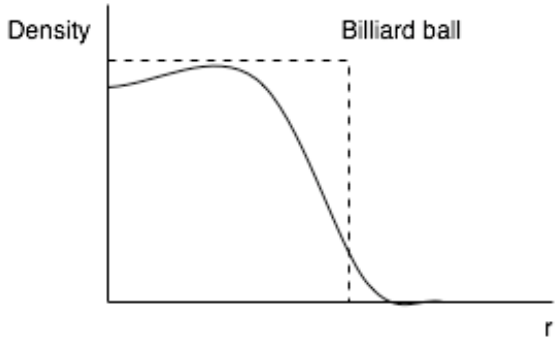
$$\Delta N = N_0 \sigma_T n x .$$

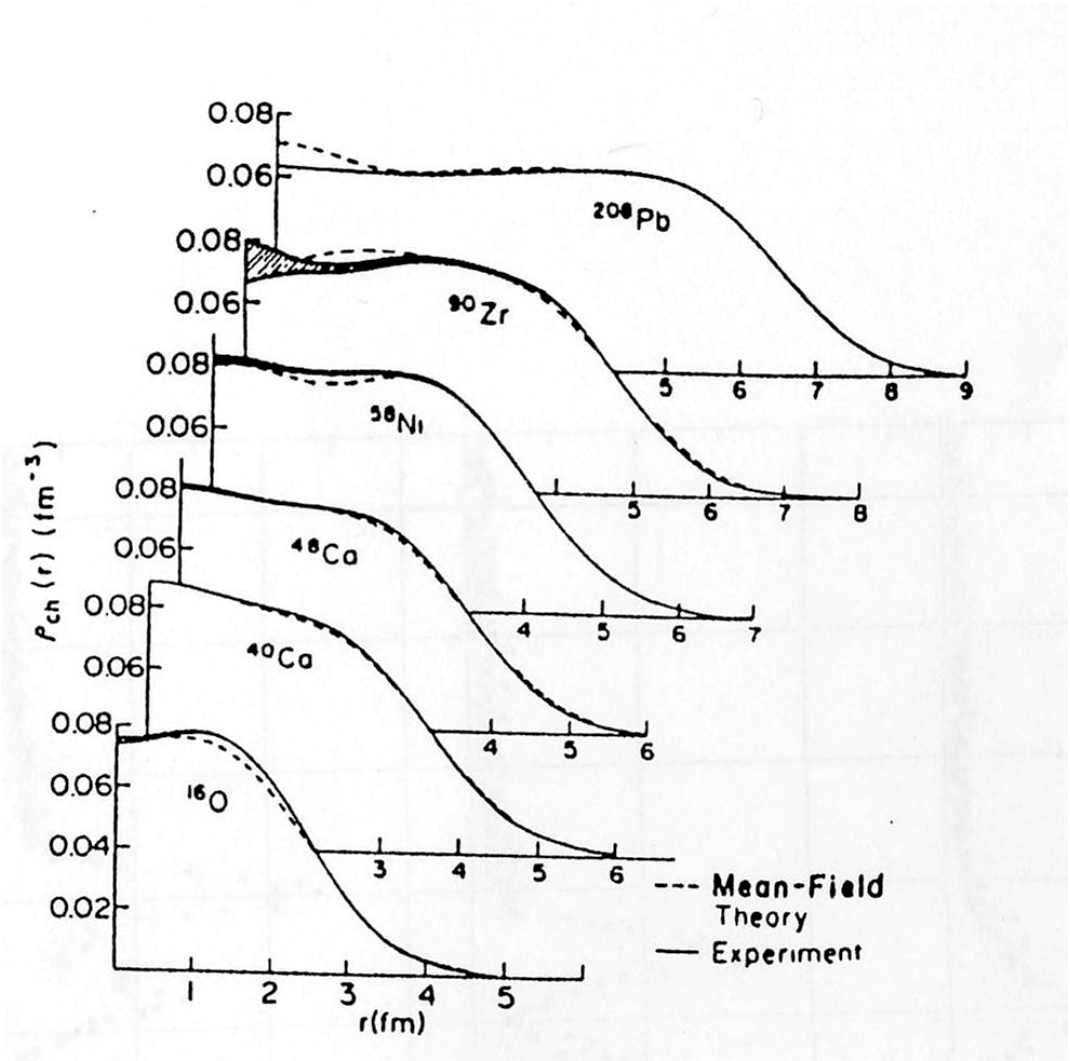
We can measure ΔN and N_0, and we know n and x. Hence we can calculate σ_T.
σ_T has dimensions of area, and order of magnitude $R_1 + R_2 \approx 6 \times 10^{-15} m$, i.e. $6 fm$.
Remember that $R = 1.2 A^{1/3} fm$. So $\pi\left(R_1 + R_2\right)^2 \approx 100 \times 10^{-30} m^2 = \underbrace{10^{-28} m^2}_{1 barn\,(1b)}$.



The isotope of lithium with three protons and six neutrons has just the size that would be expected for a nucleus with nine nucleons. There is no stable lithium nucleus with seven neutrons, but there is a lithium nucleus with eight neutrons ... and it stretches out as far as a lead nucleus. Its surface is very tenuous, containing just two neutrons in a large volume. Several of these 'halo nuclei' have now been found, and they present a huge challenge to experimenters and theorists struggling to understand their many strange properties. The Borromean rings at top left are such that if one is cut, all three fall apart. This is also true of the ^{9}Li core and two outer neutrons that make up the halo nucleus ^{11}Li. ^{10}Li does not exist, nor does a nucleus containing just two neutrons.

A word of caution: the nucleus does not have a well-defined surface.



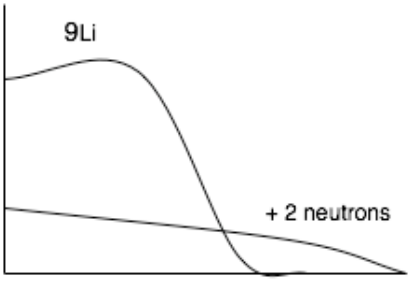


The actual density is parameterized by quantities such as the "half-value radius"; the "skin thickness" (10% → 90% range) and the "mean square radius"

$$\langle r^2 \rangle = \frac{\int r^2 \rho(r) dV}{\int \rho(r) dV}.$$

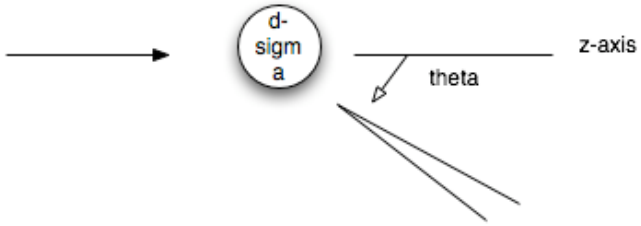
This is equal to $\frac{3}{5}R^2$ for a billiard ball.

This is even worse for ^{11}Li …



The differential cross-section

This is used to describe the scattering of particles.



$d\sigma$ is the effective area of the target nucleus that scatters a particle into a cone of solid angle $d\Omega$ in direction θ.

Total scattering cross-section $\sigma = \int_0^{2\pi}\int_0^{\pi}\frac{d\sigma}{d\Omega}\underbrace{\sin\theta d\theta d\phi}_{d\Omega}$

1.3 Nuclear Charge Distribution $\rho_{ch}(r)$

For stable nuclei, the best way to find this is by electron scattering. To measure the size of any object, we observe radiation scattered by it (e.g. visible light, X-rays [crystal lattice], electron microscope).

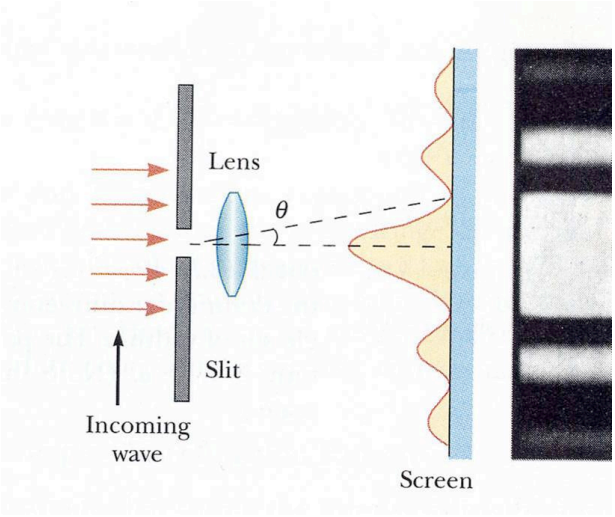
Details smaller than the wavelength / the de Broglie wavelength cannot be resolved, so to see details of nuclear charge distributions we need electrons where $\lambda << 10\,fm$. This means that the electron must be in the relativistic regime. For high energy electrons, the total energy $E \approx pc$. The rest mass energy of the electron is $0.511MeV$,which will turn out to be a lot smaller than that due to momentum.

$$\lambda = \frac{h}{p} = 2\pi\frac{\hbar c}{pc} = 2\pi\frac{\hbar c}{E} \approx 6\frac{200\,MeVfm}{E}$$

NB: $\hbar c = 197.3MeV\ fm$.

So $E >> 100\,MeV$ for the required wavelength, to see the details of the nucleus.

An analogue to electron scattering from a nucleus is light scattering around a 2D black disk.



$$\sin\theta = 1.22\frac{\lambda}{D}$$

gives the point where the first dark point is on the curve.

Electron scattering as a function of angle does show a diffraction-like pattern, but the first minimum does not fall to 0 as the nucleus is not an opaque disk with a sharp edge.

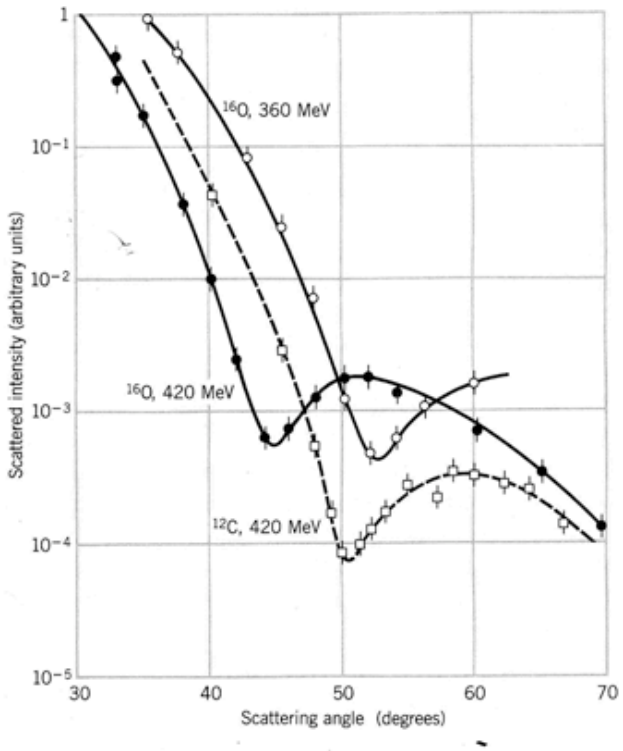


Figure 3.1 Electron scattering from ^{16}O and ^{12}C. The shape of the cross section is somewhat similar to that of diffraction patterns obtained with light waves. The data come from early experiments at the Stanford Linear Accelerator Center (H. F. Ehrenberg et al., *Phys. Rev.* **113**, 666 (1959)).

Outline of e^- scattering theory (ignore electron spin)

According to quantum theory, the electron scattering probability is controlled by an "overlap integral" containing the initial and final states of the electron and the scattering potential $V(r)$.

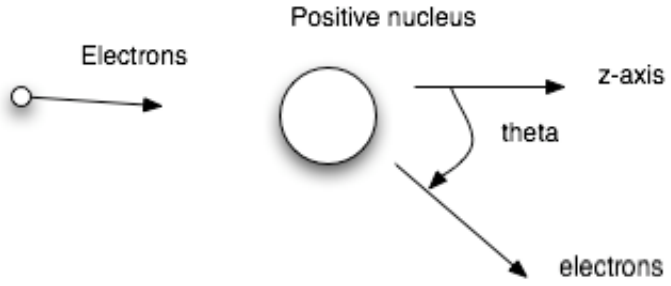
Matrix element M_{if}, where i stands for initial, and f for final.

$$M_{if} = \int \psi_f^* V(r) \psi_i dV$$

In words, "$V(r)$ acting on ψ_i converts it into a superposition of outgoing scattered waves. M_{if} picks out the amplitude of component ψ_f."

The scattered intensity $\frac{d\sigma}{d\Omega}(\theta)$ is thus proportional to the matrix element squared,

$$\frac{d\sigma}{d\Omega}(\theta) \propto \left|M_{if}\right|^2 .$$

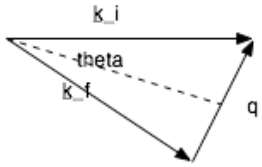


Incoming electron has $\underline{p_i} = \hbar \underline{k_i}$. $\psi_i = e^{+i\underline{k_i}\cdot\underline{r}}$

Outgoing electron has $\underline{p_f} = \hbar \underline{k_f}$, $\psi_f{}^* = e^{-i\underline{k_f}\cdot\underline{r}}$

We introduce the momentum transfer $\underline{q} = \underline{k_i} - \underline{k_f}$.

For elastic scattering, $\left|\underline{k_i}\right| = \left|\underline{k_f}\right| = k$ (to a good approximation).



$$q = 2k\sin\frac{\theta}{2} .$$

$$M_{if} = \int \psi_f{}^* v(r) \psi_i dv = \int V(r) e^{i\underline{q}\cdot\underline{r}} dV$$

This is a 3D Fourier transform of the potential $V(r)$.

(1) Special case: point nucleus.

$V(r) =$ Coulomb potential energy $= \frac{Ze^2}{4\pi\varepsilon_0 r}$

It turns out that $\frac{d\sigma}{d\Omega}(\theta) \propto \left|M_{if}\right|^2 = \left(\frac{Ze^2}{\varepsilon_0}\right)^2 \frac{1}{q^4} \propto \frac{1}{\sin^4\left(\theta/2\right)}$

This is the Rutherford scattering formula.

(2) Real (extended) nucleus

Take a non-spherical nucleus, with charge density $\rho_{ch}(\underline{r})$. Let there be a small box of charge $\underline{r}'$ from the centre of the nucleus, with volume dV'. Look from a distance $\underline{r}$ from the centre of the nucleus. The box will be at distance $\underline{r} - \underline{r}'$.

$$V(r) = \frac{Ze^2}{4\pi\varepsilon_0}\int \frac{\rho_{ch}(\underline{r}')dV'}{|\underline{r} - \underline{r}'|}$$

(Normalization: $\int \rho_{ch}(r')dV' = 1$)

After some manipulation, we get:

$$M_{if} = \frac{Ze^2}{\varepsilon_0}\frac{1}{q^2}\int e^{i\underline{q}\cdot\underline{r}'}\rho_{ch}(\underline{r}')dV',$$

where the dV' is over the nuclear volume. The first two terms are for the point nucleus result. The integral is a Fourier transform of the charge density distribution. This is called the "electric form factor of the nucleus", and can be written $F(q)$ (actually a function of q^2).

In summary: probability of scattering at angle $\theta = \frac{d\sigma}{d\Omega}(\theta) = \left(\frac{d\sigma}{d\Omega}\right)_{point\ nucleus} F(q)^2$

Note:

$F(q)$ depends on the electron beam momentum and scattering angle <u>only</u> through the momentum transfer q. So we can combine data from different electron energies. (i.e. we can combine data from different laboratories, even though it's been done at a different energy). This is done by plotting $\frac{d\sigma}{d\Omega}$ against q.

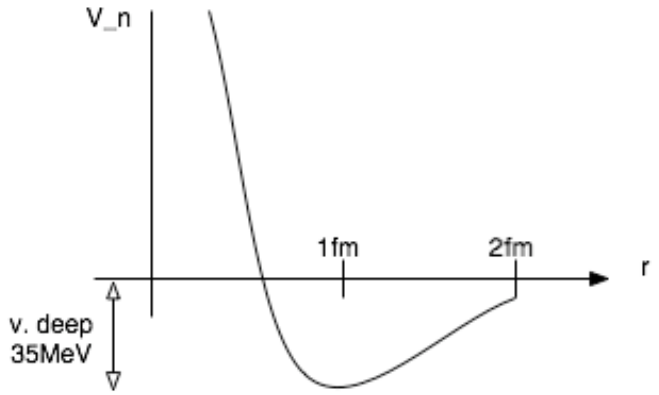
Since $F(q)$ is the Fourier transform of the charge density distribution $\rho_{ch}(\underline{r}')$, then we get $\rho_{ch}(\underline{r}')$ by an inverse transform of $F(q)$.

In principle, we need to know $F(q)$ up to $q = \infty$. This is not possible – this leads to some error on the charge density at small values of distance from the nuclear origin $\underline{r}'$.

<u>Conclusions</u>

From electron scattering data on stable nuclei:

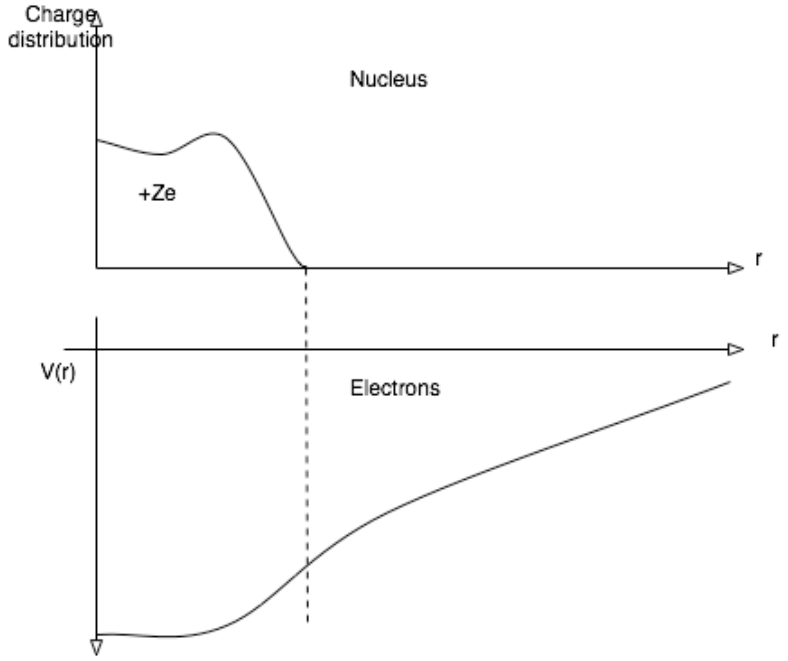
1. The charge density $\rho_{ch}(\underline{r})$ has similar central density in all nuclei. So nucleons like to keep a particular distance from each other – like atoms in a solid. The nucleon-nucleon potential must look like this:



2. Surface thickness is similar in all nuclei, $\sim 2.3\,fm$ for $10\% \rightarrow 90\%$ of the central density.

1.3 (ii) $\rho_{ch}(r)$ from perturbations of atomic energy levels

The finite spread of the nuclear charge modifies the Coulomb potential in which the atomic electrons move – but only within the small volume that the nucleus occupies.



$$V_0(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

for point nucleus. $V(r)$ for finite nucleus – shallower than V_0. This reduces the binding energy of an atomic electron by ΔE.

1st order perturbation theory:

$$\Delta E = \int \psi_e^*(r)\underbrace{\left(V(r) - V_0(r)\right)}_{\Delta V}\psi_e(r)\,d\tau$$

where τ is the volume.

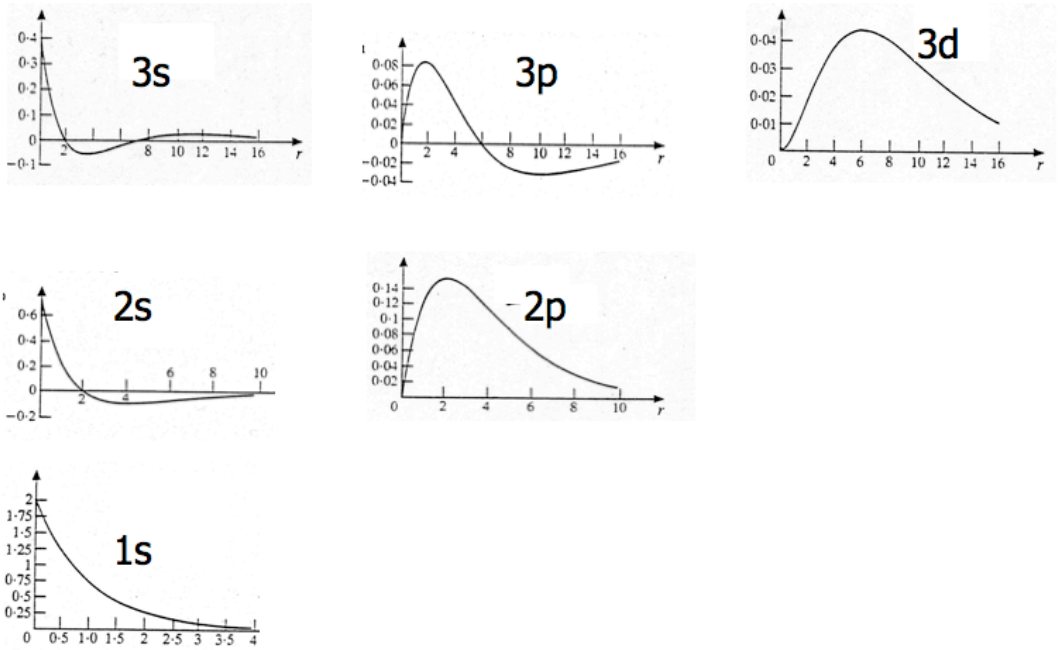
Since the electron wavefunctions vary only slowly over $10^{-15}\,m$ (Bohr radius is more like $10^{-10}\,m$ for electron wavefunctions), then we can approximate the wavefunction at r as $\psi_e(r) = \psi_e(r=0)$.

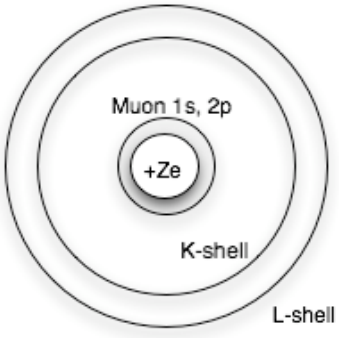
$$\boxed{\Delta E = \left|\psi_e(0)\right|^2 \int \Delta V d\tau = \left|\psi_e(0)\right|^2 \frac{Ze^2}{6\varepsilon_0}\left\langle r^2 \right\rangle}$$

where $\left|\psi_e(0)\right|^2$ is the electron density at the nucleus. See Examples Sheet 1 for the math behind this. $\left\langle r^2 \right\rangle$ is the mean square charge radius of the nucleus.

NB; it is only the s electron shells which will experience this shift, as they are the only ones in the nuclei range.

Wavefunctions Ψ(r) of atomic electrons





If an electron drops down from the L-shell to the K-shell, then an X-ray will be emitted.

a) Measure ΔE of k-shell electrons (measure the x-ray energies).

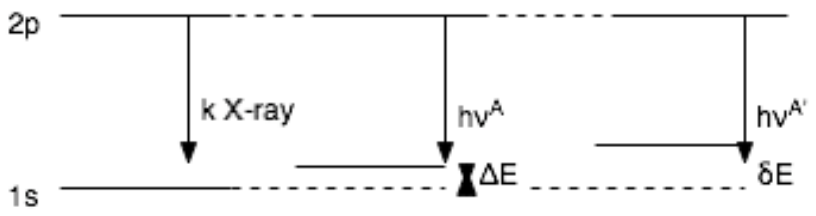
Look at the 6s and 6p shells. Fire a laser at them…

b) Measure ΔE e.g. of the 6s valence electron by laser spectroscopy.

Look at Muon energy levels close to the nucleus. Change between the 2p and 1s shells, with the emission of an X-ray.

c) ΔE of "muonic" atoms (measure muonic X-rays).

1.3 (ii) a) X-ray isotope shifts



Atomic energy levels. The first is for a point nucleus of charge $+Ze$, the second for a real nucleus with area A and charge $+Ze$, the third for a nucleus with A' and $+Ze$ (an isotope of the same element).

ΔE or p-electrons ≈ 0 because $\left|\psi_p(0)\right|^2 \approx 0$.

$$\delta E = \Delta E^{A'} - \Delta E^{A}.$$

This has 3 problems:

1. Shift is small $\left(10^{-5}\right)$ because of small electron density at nucleus.
2. We don't have a point nucleus as a reference.
3. Calculations for the electron wave functions $\psi_e(r)$ in heavy elements are not sufficiently accurate to estimate the K X-ray energy of the theoretical point nucleus (one part in a million, 10^{-6} precision).

But what can be done is a comparison of the $\langle r^2 \rangle$ between the different isotopes.

$$\delta E = h\underbrace{\left(\nu^{A} - \nu^{A'}\right)}_{\delta\nu^{A'A}} = \frac{Ze^2}{6\varepsilon_0}\left|\psi_{1s}(0)\right|^2 \underbrace{\left(\langle r^2 \rangle^{A'} - \langle r^2 \rangle^{A}\right)}_{\delta\langle r^2 \rangle^{A'A}}.$$

$\delta\nu^{A'A}$ is called the isotope shift of the X-ray lines. $\delta\langle r^2 \rangle^{A'A}$ is the change in mean square charge radius between isotopes.

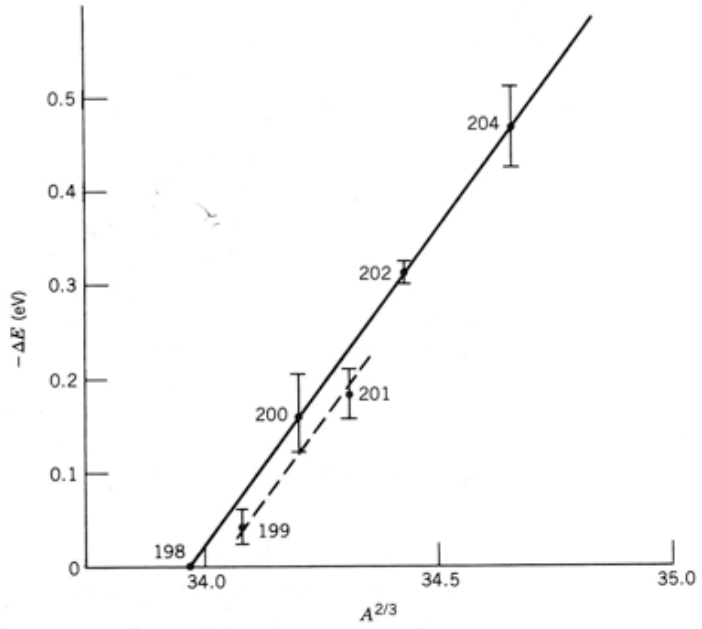
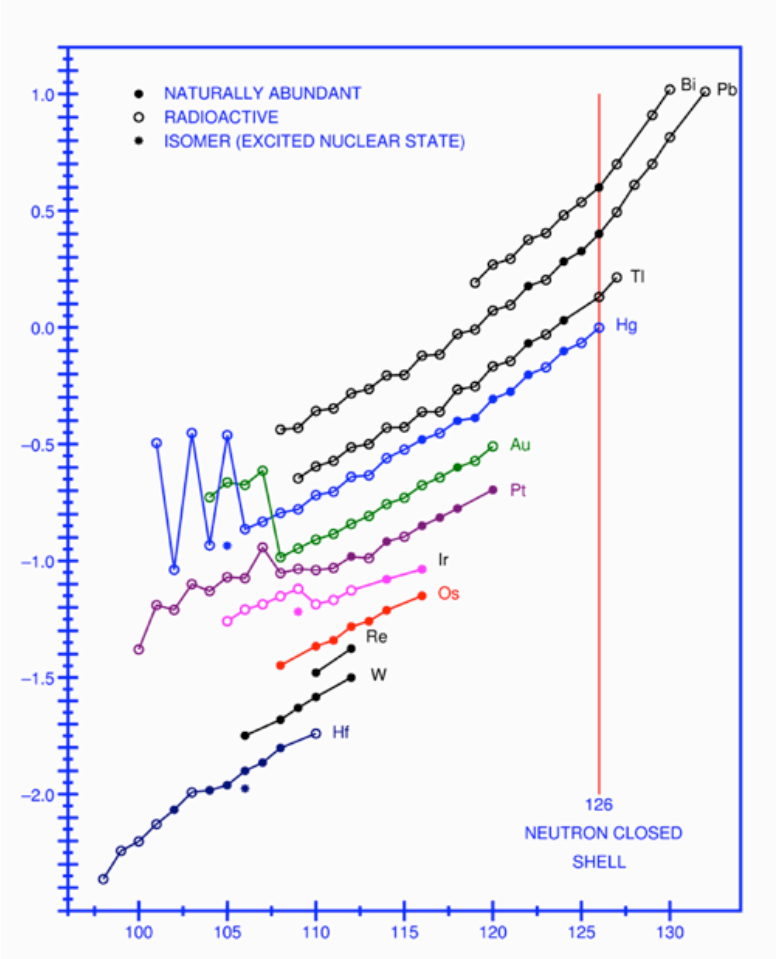


Figure 3.6 K X-ray isotope shifts in Hg. The energy of the K X ray in Hg is about 100 keV, so the relative isotope shift is of the order of 10^{-6}. The data show the predicted dependence on $A^{2/3}$. There is an "odd-even" shift in radius of odd-mass nuclei relative to their even-A neighbors, brought about by the orbit of the odd particle. For this reason, odd-A isotopes must be plotted separately from even-A isotopes. Both groups, however, show the $A^{2/3}$ dependence. The data are taken from P. L. Lee et al., *Phys. Rev. C* **17**, 1859 (1978).

We can see that the increase of $\langle r^2 \rangle$ with N is seen → proton distribution does increase as neutrons are added. This is because of the strong attraction between protons and neutrons.

1. Fractional shift $= \dfrac{0.15eV}{100keV} = 10^{-6}$.
2. Proton distribution does spread out as neutrons are added because of the strong interaction between protons and neutrons.
   But:
3. Despite Krane's comment, the rate of increase is about half that expected from $R = 1.2A^{1/3}$ fm.

(Mass range of A is so limited across Hg isotopes that almost any power law of A can fit)



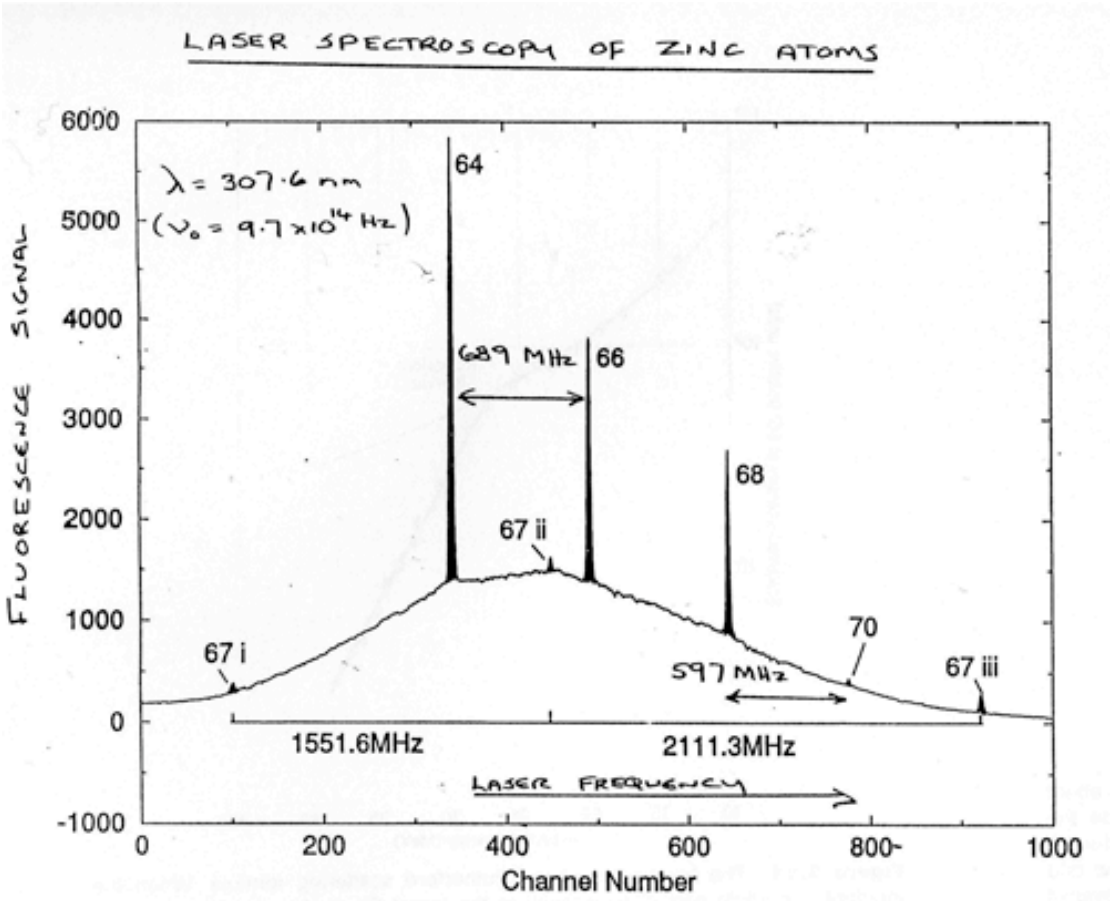
1.3(ii) ΔE for valence electrons

The shifts that we will be looking at are much smaller than those before, as $|\psi_{6s}(0)|^2 << |\psi_{1s}(0)|^2$, and $\delta E \sim 10^{-6} eV$ (compared with $0.15eV$ for X-rays).

However, the transition energies are $\sim 3eV$ (not $100keV$), and these shifts can be measured with great precision by laser spectroscopy.

Tunable laser, excites an electron from the 6s to 6p level. Resonant excitation will lead to the emission of a fluorescent photon, which can be detected with a photomultiplier tube.

$\nu_{laser} = \nu_0$



Isotope shift between peaks.

We must also consider the electron density for the upper state.

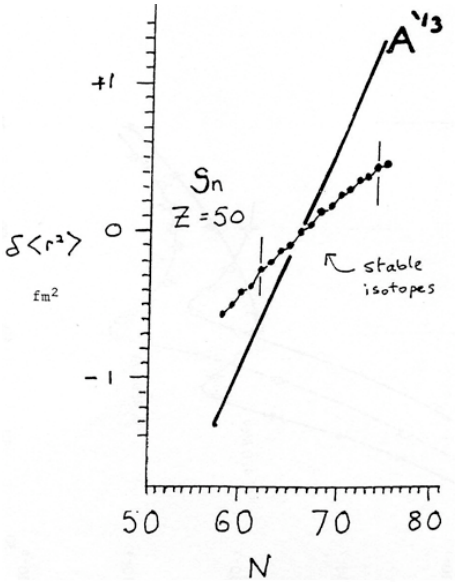
Isotope shift $\delta\nu^{A'A} = \frac{Ze^2}{6\varepsilon_0}\Delta|\psi(a)|^2\,\delta\langle r^2\rangle^{A'A}$ + small mass shift.

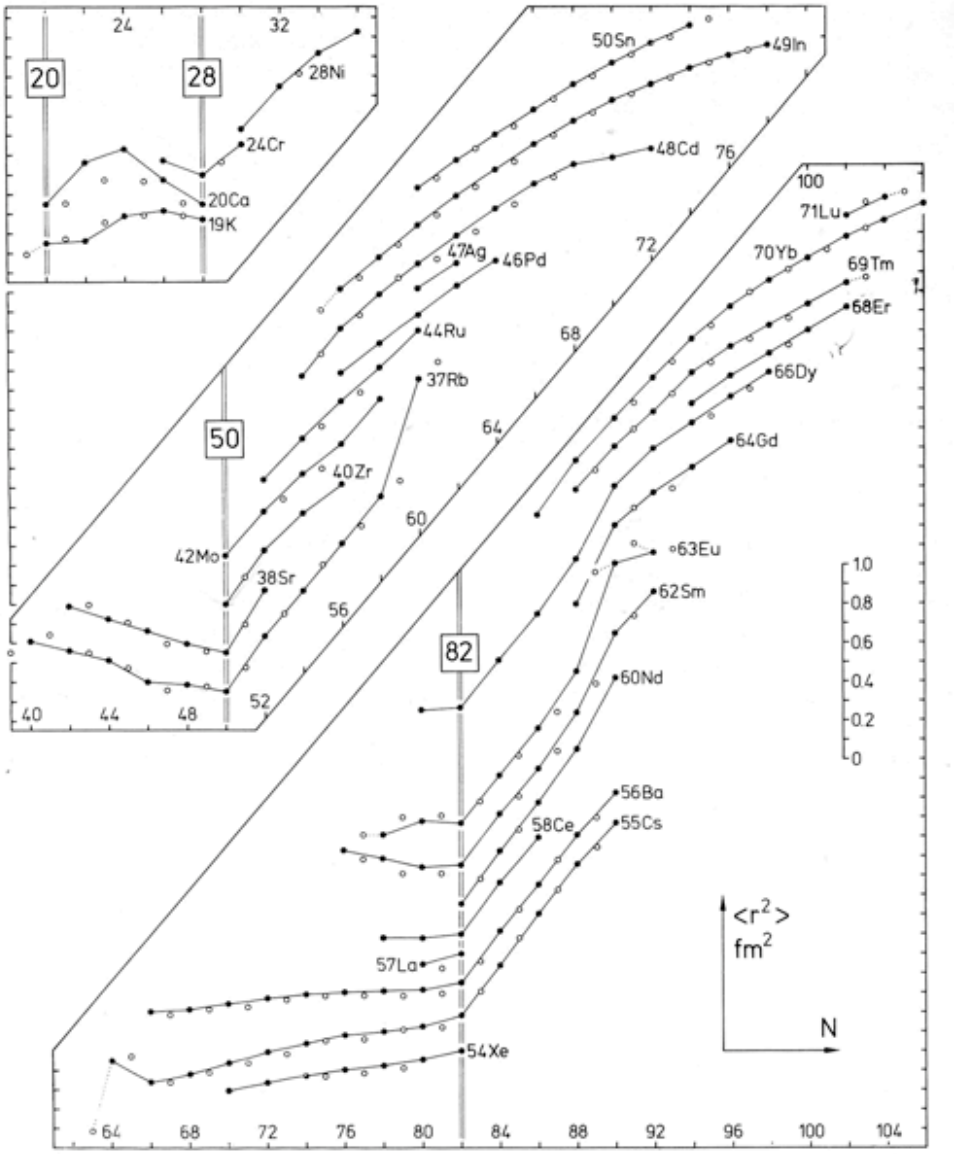
Where $\Delta|\psi(a)|^2$ is the chance in electron density between the two atomic states. The small mass shift is connected to the change in electron reduced mass.

If $\mu =$ reduced mass, then $\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{M_A}$.

Example: Zinc isotopes (see above diagram).

- The frequency shift $\frac{\delta E}{E} \approx \frac{700\,MHz}{10^{15}\,Hz} \sim 7\times10^{-7}$.
- Odd-A nuclei have several components (hyperfine structure)
- If $\langle r^2\rangle \propto A^{2/3}$, then we should have approximately evenly-spaced peaks – but they aren't in Zn.


A^1/3
+1
0
−1
δ⟨r²⟩
fm²
Sn
Z = 50
stable
isotopes
50
60
70
80
N


20
28
50
82
24
32
28Ni
24Cr
20Ca
19K
50Sn
49In
48Cd
47Ag
46Pd
44Ru
37Rb
40Zr
42Mo
38Sr
40
44
48
52
56
60
64
68
72
76
100
71Lu
70Yb
69Tm
68Er
66Dy
64Gd
63Eu
62Sm
60Nd
56Ba
58Ce
55Cs
57La
54Xe
1.0
0.8
0.6
0.4
0.2
0
⟨r²⟩
fm²
N
64
68
72
76
80
84
88
92
96
100
104

Notes:

1. As neutrons are added along isotope chains (constant Z), the proton distribution expands but only at about half the rate expected from $R = 1.2A^{1/3} fm$. → nuclei with a neutron excess must develop a neutron-rich skin which is not seen in stable nuclei.
2. Not all nuclei are spherical. Deformed nuclei will have noticeably larger mean squared charge radii $\langle r^2 \rangle$ - e.g. light Hg isotopes. (2 or 3%)

1.3 (ii) (c) Muonic Atoms

The muon is a lepton with 207 times the mass of the electron. If muons from a decaying pion beam are stopped in a target of your chosen isotope, the negatively-charged muons spiral down through Bohr-like orbits emitting photons (X-rays) finally from the $2p \rightarrow 1s$ transition.

These orbits are much smaller (by a factor of 1/207) than electron orbits. Bohr radius

$$a_0 = \frac{1}{\alpha} \frac{\hbar}{m_e c} .$$

$$r_\mu = \frac{r_e}{207}$$

In ${}_{82}Pb$ the muon 1s orbit is inside the nucleus.

Thus $|\psi_{1s,\mu}(0)|^2 >> |\psi_{1s,e}(0)|^2$ and muonic X-ray shifts due to nuclear size effects are substantial.

$$\frac{\delta E}{E} \approx \frac{1}{30} .$$

Wavefunction calculations are much better (we can ignore the electrons, to a good approximation). They are accurate enough to derive absolute $\langle r^2 \rangle$ for the nucleus. Up to now, we have only been able to compare isotope sizes.

$$\Delta E = \frac{Ze^2}{6\varepsilon_0} |\psi_{1s}(0)|^2 \langle r^2 \rangle$$

The approximation that the muon density is constant across the nuclear volume no longer holds and proper wavefunctions are used.

Muonic X-ray energies:
$\sim 1 MeV$ in Fe
~ 55MeV in Pb (55, or 5.5?)
Comparable to γ -rays, so detectors are accurately calibrated with γ -ray sources. NB: stable isotopes only.

Example: Fe isotopes

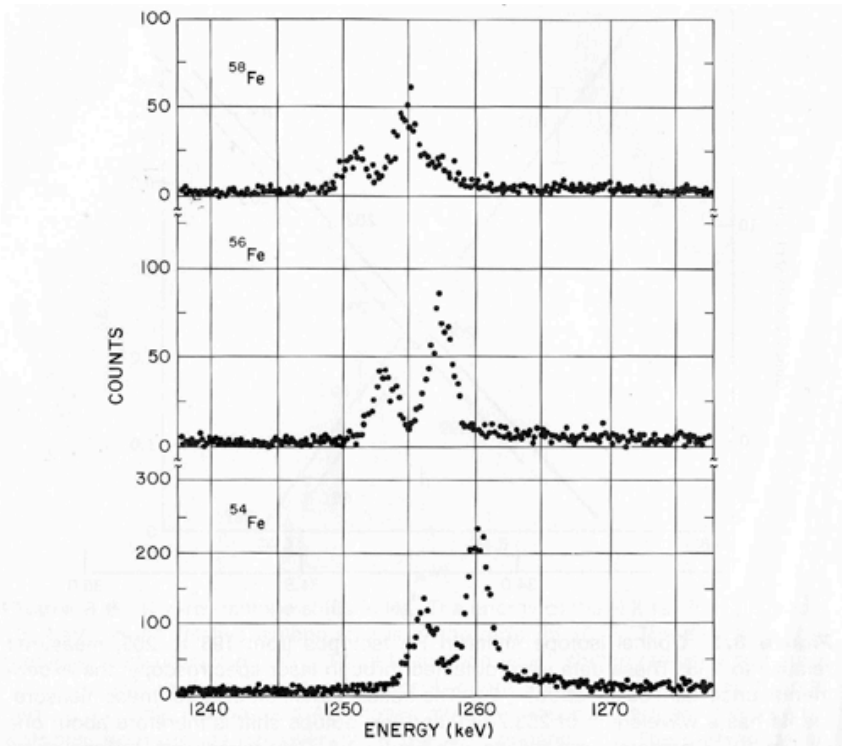


Figure 3.8 The muonic K X rays in some Fe isotopes. The two peaks show the $2p_{3/2}$ to $1s_{1/2}$ and $2p_{1/2}$ to $1s_{1/2}$ transitions, which have relative intensities in the ratio 2 : 1 determined by the statistical weight $(2j + 1)$ of the initial state. The isotope shift can clearly be seen as the change in energy of the transitions. The effect is about 0.4%, which should be compared with the 10^{-6} effect obtained with electronic K X rays (Figure 3.6). From E. B. Shera et al., *Phys. Rev. C* **14**, 731 (1976).

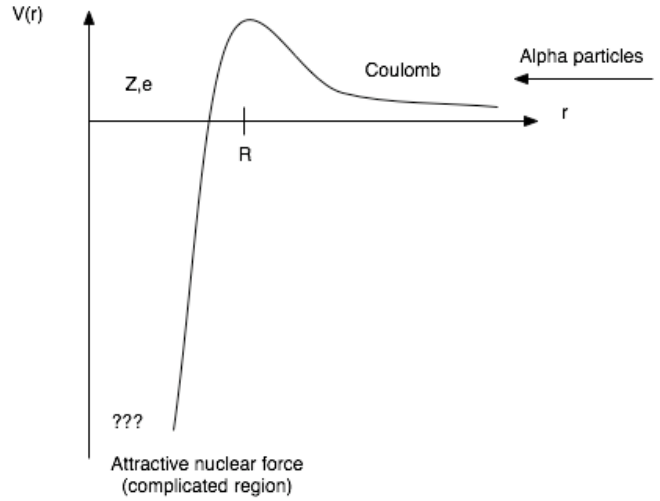
The 2p level has a fine structure, $P_{3/2}$ and $P_{1/2}$, which decay down to the $1s$ $s_{1/2}$ state. Ratio of $(2j+1) = 2:1$, which explains the relative heights of the peaks.

1.4 Nuclear Matter Distributions

1.4 (a) Elastic Scattering of Protons, α 's

Probes must interact via strong force to see both protons and neutrons: $p-n$ and $p-p$ forces are very similar (the neutral and charged pions have slightly differing masses).

The scattering potential looks like this:



The Coulomb potential is $V_C = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r} = 1.44 \frac{Z_1 Z_2}{r} MeV$ (when it is in fermi).

If $r > R$, Gauss's law tells us the scattering will be the same as a "point nucleus" – Rutherford scattering.

If the projectile is high enough to overcome the Coulomb repulsion, strong attractive nuclear forces come into play. Clearly at this point, we should expect deviations from the Rutherford cross-section σ_R because other outcomes (e.g. absorption) are possible.

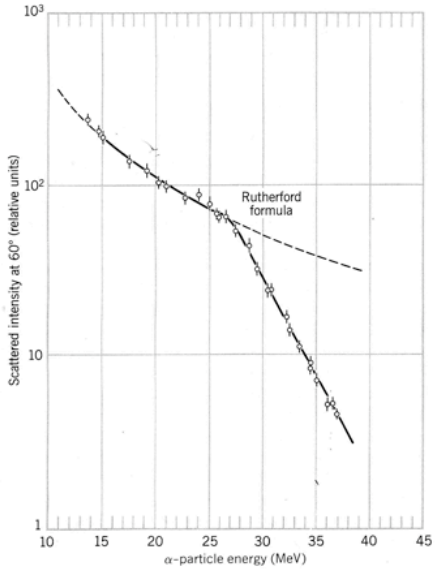
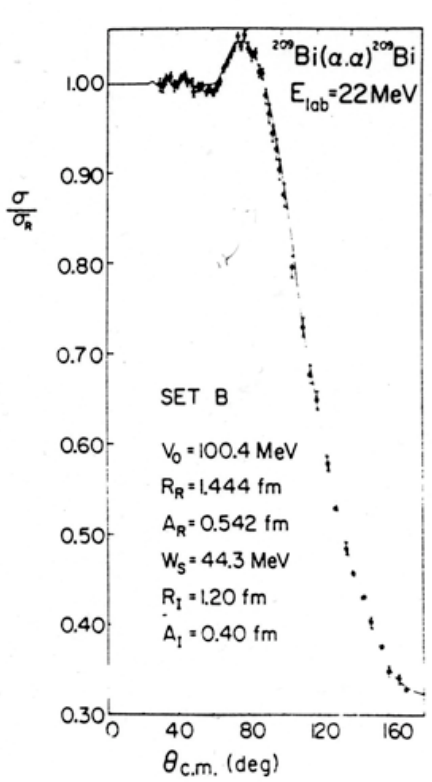
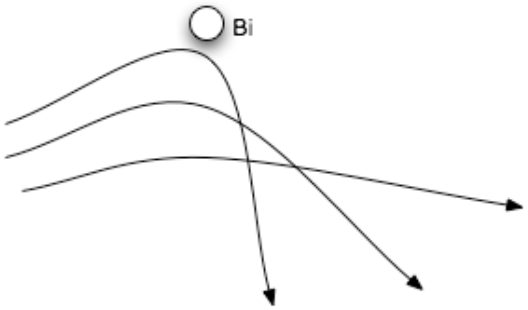


Figure 3.11 The breakdown of the Rutherford scattering formula. When the incident α particle gets close enough to the target Pb nucleus so that they can interact through the nuclear force (in addition to the Coulomb force that acts when they are far apart) the Rutherford formula no longer holds. The point at which this breakdown occurs gives a measure of the size of the nucleus. Adapted from a review of α particle scattering by R. M. Eisberg and C. E. Porter, *Rev. Mod. Phys.* **33**, 190 (1961).

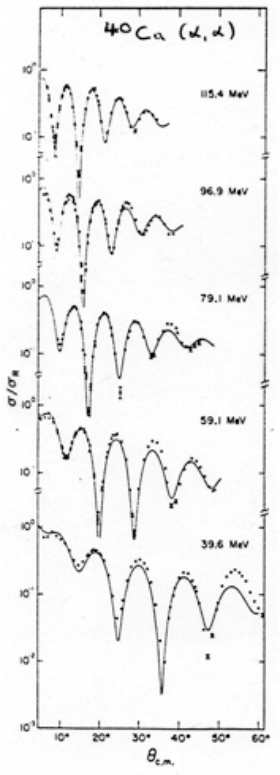
The analysis of these data is far more difficult than electron scattering because of the more complex nature of the strong interaction.

At low energy, where absorption of the particle is strong, we can only measure the matter radius (nucleus like a black disc)
Example: $^{209}B(\alpha,\alpha)$ (alpha scattering on Bismuth)





At higher energies, where nuclear matter appears more transparent, the scattering pattern has a fraunhoffer-like appearance which provides information on the internal structure of the nucleus – the internal density distribution. Example: $^{40}Ca(\alpha,\alpha)$ (see image on left).



1.5 Summary of results:

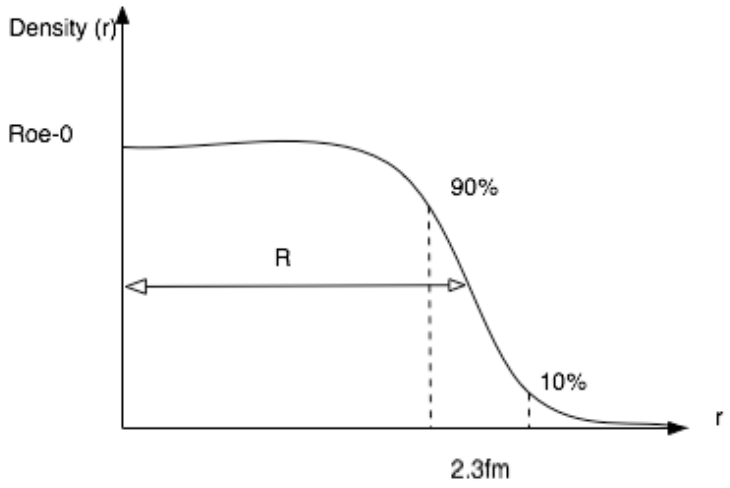
(i). Stable nuclei (along the line of stability): charge and matter radii are very similar. This is because the neutron-proton attraction keeps them in the same place.

(ii) Nuclei with large neutron excess appear to have a neutron-rich skin (even a halo in ^{11}Li). (similar things appear to happen with proton excess, but that data's only just coming in.)

(ii) Central density is similar in all stable nuclei.

(iv) All have a similar surface diffuseness.

We have a common description of the charge distribution:



$$\rho(r) = \frac{\rho_0}{1 + e^{\left(\frac{r-R}{a}\right)}}$$

$$R = 1.2A^{1/3} fm$$

$a = 0.6 fm$ - controls surface diffuseness.