7. Ideal Gas of Photons – Black Body Radiation
(M ch. 10; B&S 8.1-6, 8.10; K&K p87-91)

1. Cannot use $\varepsilon = \frac{\hbar^2 k^2}{2m}$ → photons are massless. Use $\varepsilon = pc = \hbar kc$ instead.
2. Photon number is not conserved. In thermal equilibrium, walls absorb and emit photons.

7.1 The Simple Harmonic Oscillator

Quantum mechanical energy levels $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where ω is the classical vibration frequency, and $n = 0,1,2,\ldots$.
Consider a SHO in equilibrium at temp T.

$$p(E_n) = \frac{e^{-\frac{E_n}{k_B T}}}{Z}$$

where Z is the partition function $= \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\left(n+\frac{1}{2}\right)\hbar\omega\beta}$, i.e. a geometric series.

First term $a = e^{-\frac{1}{2}\hbar\omega\beta}$.
Ratio $r = e^{-\beta\hbar\omega}$

Sum to infinity $= \frac{a}{1-r} = \frac{e^{-\frac{1}{2}\hbar\omega\beta}}{1-e^{-\beta\hbar\omega}} = Z$

Average energy of the SHO $\bar{E} = \sum_n E_n p(E_n) = -\frac{\partial \ln Z}{\partial \beta}$

$$\ln Z = -\frac{1}{2}\beta\hbar\omega - \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$\bar{E} = \frac{1}{2}\hbar\omega + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \left(\bar{n} + \frac{1}{2}\right)\hbar\omega$$

where $\bar{n} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = e^{\frac{1}{\beta\hbar\omega}} - 1$

$$\boxed{\bar{n} = \frac{1}{e^{\frac{\varepsilon}{k_B T}} - 1}}$$

$\bar{n}$ is the average number of quanta of energy $\varepsilon = \hbar\omega$ that the SHO possesses at temperature T.
But this is just the Bose-Einstein distribution function with $\mu = 0$.
This makes some sense – chemical potential relates to where there are a fixed number of particles, which is not the case here.
Conclusion: quanta of energy $\hbar\omega$ of a SHO behave like bosons with $\mu = 0$.
At low T $\left(k_B T << \varepsilon\right)$:

$$\bar{n} \sim e^{-\varepsilon/k_B T}$$

$$\bar{E} \approx \frac{1}{2}\hbar\omega + \underbrace{\hbar\omega e^{-\varepsilon/k_B T}}_{small}$$

High T $\left(k_B T >> \varepsilon\right)$:

$$\bar{n} \approx \frac{1}{\left(1+\frac{\varepsilon}{k_B T}+...\right)-1}$$

$\left[e^x = 1 + x + ...\right]$

$$\bar{n} = \frac{k_B T}{\varepsilon}$$

Note that this is linear at higher T.

$$\bar{E} = \underbrace{\frac{1}{2}\hbar\omega}_{small} + \hbar\omega\frac{k_B T}{\hbar\omega}$$

→ $\bar{E} \approx k_B T$

This is the classical equipartition theorem – 2 degrees of freedom.
(If you don't get this back, then something's gone wrong.)

$\bar{\mathrm{E}}$ vs. T will always be above the line $\bar{E} = k_B T$ by $\frac{1}{2}\hbar\omega$.

Free Energy of SHO

$$F = -k_B T \ln Z = \frac{1}{2}\hbar\omega + k_B T \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

(Use later to compute pressure and entropy.)

7.2: EM waves in a box (Classical)

Walls of the box are at some temperature T. Sides of the box are of length L.
Assume perfectly conducting walls. → $E_{||} = 0$ at walls.

Solve Maxwell's equations. Will get standing wave solutions for EM field (normal modes of the wave equation). These can be represented by the wave vector $\underline{k}$, which should be quantized.

Remember that $\sin\left(\frac{n\pi x}{L}\right) = \sin\left(kx\right)$, where $k = \frac{n\pi}{L}$, is the solution in the 1D case.

States represented by $\underline{k}$ vectors in k-space → simple cubic lattice in k-space of side π/L in the positive octant of k-space.

→ $\frac{Vk^2 dk}{2\pi^2}$ states in the range $k \to k + dk$. ($V = L^3$ the volume of the box.)

Note that each k value represents two normal modes, which arise from the two different polarizations of EM waves.

Frequency of mode is $\omega = ck$.

7.3 EM Waves in a Box (Quantum Treatment)

Each mode behaves like a SHO of frequency ω. The allowed energy for such operator is

$$E = \left(n + \frac{1}{2}\right)\hbar\omega$$

for mode of frequency ω. So the mean number of excited quanta is

$$\bar{n} = \frac{1}{e^{\beta\hbar\omega} - 1}$$
at temperature T. The quanta are Photons.

The number of modes between $k \to k + dk$ is:
$$2\frac{Vk^2 dk}{2\pi^2}$$
where the 2 arises from the 2 polarizations. We can change variables such that
$$w = ck$$
$$d\omega = cdk\ ,$$
in which case the number of modes between $\omega \to \omega + d\omega$ is
$$\frac{V\omega^2 d\omega}{\pi^2 c^3}\ .$$

The mean energy of Black-Body Radiation for photons of frequency in the range $\omega \to \omega + d\omega$ is
$$\underbrace{\left(\bar{n} + \frac{1}{2}\right)\hbar\omega}_{energy\ of\ each\ \mathrm{mod}\, e} \ \underbrace{\frac{V\omega^2 d\omega}{\pi^2 c^3}}_{number\ of\ \mathrm{mod}\, es}\ .$$
Ignore the Zero-Point Energy, i.e. the ½ in the first part of the equation.
$$= \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\frac{V\omega^2 d\omega}{\pi^2 c^3} \equiv V\rho_\omega d\omega\ ,$$
where ρ_ω is the energy per unit volume per unit frequency range.

NB: an equivalent view-point is that photons $(s = 1)$ are Bosons with spin degeneracy of 2 rather than 3 $(2s + 1)$ because photons are massless, and $\mu = 0$ because the photon number is not conserved. Here, the number of photons in mode of frequency ω is $\frac{1}{e^{\beta\hbar\omega} - 1}$, and their energy is $\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$. So the energy of black-body radiation in the range $\omega \to d\omega$ is:
$$\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\frac{V\omega^2 d\omega}{\pi^2 c^3}\ .$$
At low ω, the curve is $\propto \omega^2$. It reaches $\hbar\omega_{\max} \sim k_B T$ before falling $\propto \omega^3 e^{-\beta\hbar\omega}$. Note that in the classical limit (the earlier Rayleigh-Jeans formula), it doesn't reach a maximum but constantly rises as ω^2, the ultra-violet catastrophe.

To get $\omega_{\max}$, simply differentiate the above function wrt ω. You get:
$\hbar\omega_{\max} = 2.82 k_B T$
(actually $= x k_B T$, where $e^x = \frac{3}{3 - x}$.)

(what happened to 7.4…?)

7.5 Total Thermal Energy of BBR

$E = V\int_0^\infty e_\omega d\omega$

(neglecting Zero Point Energy $\int_V^\infty \frac{1}{2}\hbar\omega\rho(\omega)d\omega = \infty$)

$$E = V\int_0^\infty \frac{\hbar\omega}{e^{\beta\hbar\omega}}\frac{\omega^2}{\pi^2c^2}d\omega$$

$$x = \beta\hbar\omega$$

$$\rightarrow\ E = \frac{V}{\pi^2}\frac{\hbar}{c^3}\left(\frac{k_BT}{\hbar}\right)^4\int_0^\infty \frac{x^3}{e^x-1}dx$$

$\int_0^\infty \frac{x^3}{e^x-1}dx$ is just a number $= \frac{\pi^4}{15}$.

Energy density, $\frac{E}{V} = \frac{\pi^2}{15\hbar^3c^3}(k_BT)^4 = bT^4$

Where $b = \frac{\pi^2}{15(\hbar c)^3}{k_B}^4$.

How to sample BBR – Stefan's Law

Take an insulated box with internal temperature T. Let there be a small holl, out of which radiation can be emitted without disturbing the thermal equilibrium inside.

Stefan's law: radiation emerges at a rate, σAT^4, where

$\sigma = \frac{\pi^2}{60\hbar^3c^2}{k_B}^4 = 5.67\times10^{-8}Wm^{-2}k^{-4}$, Stefan's constant.

Proof: from first year's Gases, Liquids and Solids, gas particles hit container walls at rate $\frac{1}{4}n\bar{v}$ per unit area per second. Photons strike hole at rate of $\frac{1}{4}ncA$ where n is the photon density, and A the hole area.

Energy leaves the hole at rate $\frac{1}{4}\frac{E}{V}cA$, i.e. just replace the number density by energy density.

Therefore $\sigma AT^4 = \frac{1}{4}bT^4cA$, where $\sigma = \frac{bc}{4} = \frac{\pi^2}{60\hbar^3c^2}{k_B}^4$.

Cannot study BBR by heating up any old object, i.e. not everything's a black body.

BB absorbs all the energy that falls on it – hole has this property.

A general object, such as tungsten wire, only absorbs a fraction of the light falling onto it.

Let a = absorbency of the object, and e = emissitivity. At temperature T, energy emitted per unit area = $e\sigma AT^4$. But $e = a$ → container in equilibrium with BBR has to reach the same temperature as the BBR. Therefore absorption would have to happen at $a\sigma AT^4$.

7.7 Thermodynamics of Black-Body Radiation

$E = bvT^4$ is the energy.

$F = \frac{1}{2}\hbar\omega + k_BT\ln\left(1-e^{-\beta\hbar\omega}\right)$ for SHO.

Each mode = SHO.

Ignore ZPE.

$$F_{tot} = 2k_BT\int_0^\infty \ln\left(1-e^{\beta\hbar\omega}\right)\frac{V\delta^2}{2\pi^2c^3}d\omega$$

where the 2 comes from polarizations (or left / right ness), and $\frac{V\delta^2}{2\pi^2 c^3}$ is the density of states $= \frac{Vk^2}{2\pi} dk$.

We define $x = \beta\hbar\omega$, hence $dx = \beta\hbar d\omega$.

$$F_{tot} = \frac{V\left(k_B T\right)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \ln\left(1 - e^{-x}\right) x^2 dx$$

$\int_0^\infty \ln\left(1 - e^{-x}\right) x^2 dx = -\frac{\pi^4}{45}$, i.e. just a number.

$$F = -\frac{V\pi^2 \left(k_B T\right)^4}{45\hbar^3 c^3} = -\frac{E}{3}$$

Therefore $S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4\pi^2 V}{45\hbar^3 c^3} k_B{}^4 T^3$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{\pi^2 \left(k_B T\right)^4}{45\hbar^3 c^3} = \frac{1}{3}\frac{E}{V}$$

NB: for non-relativistic particles, $p = \frac{2}{3}\frac{E}{V}$, $E = \frac{3}{2} N k_B T$.

Tidy up some of these formulae, so we have:

$E = bVT^4$, $S = \frac{4}{3} bVT^3$, $P = \frac{1}{3} bT^4$

Check: $dE = TdS - pdV$.

$$dV = bdVT^4 + 4bVT^3$$
$$= T\underbrace{\left(\frac{4}{3} bdVT^4 + 4bVT^2 dT\right)}_{=dS} - \underbrace{\frac{1}{3} bT^4}_{=p} dV$$

- photon gas behaves like a conventional gas.

Reversible Adiabatic Expansion

Therefore $S = const. = \frac{4}{3} bVT^3$ is fixed.

$$V_i T_i^3 = V_f T_f{}^3$$

$T_f = T_i \left(\frac{V_i}{V_f}\right)^{\frac{1}{3}}$ → in an adiabatic expansion.

7.8 The Ultimate Adiabatic Expansion of a Photon Gas – Cosmic Microwave Background Radiation

300,000 years after the Big Bang, the temperature of the universe has fallen t, about 4000k. At this temperature, stable atoms could form.

Binding Energy of at atom $\sim 1eV$: $T = \frac{BE}{k_B} \sim 10,000k$.

Before this time, the matter of the universe consisted of a plasma of electrons, protons and alpha particles interacting strongly, with photons in thermal equilibrium.

But neutral atoms do not interact strongly with photons.
Ever since the photon gas has been expanding and cooling, essentially decoupled from the matter. It is a relic of the universe as it was 300,000 years after the Big Bang.
Adiabatic expansion – if the photons are essentially in the state they were in then, no change in order takes place, i.e. no entropy change; no change in occupation number.
Therefore $T^3V = const$.
T is now $= 2.726k$ cosmic background radiation.
Can see tiny variations in T of order of 1 part in 10^5 - indication of the structure of the early universe → galaxy formation.