6. The Ideal Bose Gas at Low Temperatures
M 11.6; B&S 10.5, 13.5; K&K 119-217

$$N = \int_0^\infty d\varepsilon f(\varepsilon) g(\varepsilon)$$

is used to fix μ.

$$f(\varepsilon) = \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}} - 1}$$

This is the average number of bosons in the state of energy ε.
We must have $\varepsilon > \mu$, but as ε is positive, μ must be negative.

As $T \to 0$, $\mu \to 0$ in order to get area $\sim (k_B T)^{3/2}$: eventually $\mu \to 0$ at $T = T_c$.
(strictly, $\mu =$ energy of lowest state).
For $T < T_c$, cannot make the area = N.
Problem is replacement of sum over states by the integral

$$\sum_{\varepsilon} \to \int_0^\infty d\varepsilon g(\varepsilon)$$

Integral gives 0 weight, i.e. $\varepsilon = 0$, $g(t=0) = 0$
What are the bosons going to do at low temperatures?
- They'll start to crowd into state of lowest energy → $\ell = 1$, $m = 1$, $n = 1$.

$$\psi = \left(\sqrt{\frac{2}{L}}\right)^3 \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

- T_c is the temperature at which "macroscopic" occupancy of the state becomes important.
→ Bose-Einstein condensation (BEC)

6.2 Bose-Einstein Condensation (BEC)
What is T_c? Temperature at which

$$\int_0^\infty d\varepsilon f(\varepsilon) g(\varepsilon) = N \text{ for } \mu = 0.$$

$$N = \int_0^\infty dk \frac{Vk^2}{2\pi^2} \frac{1}{e^{\frac{\hbar^2 k^2}{2mk_B T}} - 1}$$

$$= \int_0^\infty d\varepsilon \frac{1}{e^{\frac{\varepsilon}{k_B T}} - 1} \underbrace{\frac{V}{4\pi^2}\left[\frac{2M}{\hbar^2}\right]^{3/2} \varepsilon^{1/2}}_{g(\varepsilon)}$$

In the last part, $g(\varepsilon)$ is the $g(\varepsilon)$ for spin 0 bosons. ½ electron value because no spin degeneracy.

Change variable to $x = \frac{\varepsilon}{k_B T_c}$.

$$dx = \frac{1}{k_B T_c} d\varepsilon$$

$$\frac{N}{V} = \frac{1}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\left(k_B T_c\right)^{3/2} \underbrace{\int_0^\infty \frac{x^{1/2}dx}{e^x - 1}}_{=2.31...}$$

The last part of this is just a number, which is 2.31 to 3 decimal places.

$$k_B T_c = \frac{\hbar^2}{2M}\left(\frac{4\pi^2}{2.31}\frac{N}{V}\right)^{2/3}$$

Another way of writing this:

$d = \left(\frac{V}{N}\right)^{1/3}$, the typical distance between the bosons.

$$d = \left(\frac{\sqrt{\pi}}{2t2.31}\right)^{1/3}\lambda_{T_c}$$

Where λ_{T_c} is the de Broglie wavelength $= \frac{h}{\left(2\pi M k_B T_c\right)^{3/2}}$.

i.e. T_c is the temperature at which d and λ_T are comparable. NB: at high temperatures $d >> \lambda_T$. When $d \sim \lambda_T$, quantum effects become important.

What happens below T_c?

$$N = N_0(T) + \int_0^\infty \frac{1}{e^{\frac{\varepsilon}{k_B T}} - 1} g(\varepsilon) d\varepsilon$$

where N_0 is the number of particles in lowest energy state, which is the number of Bose-Einstein condensed particles, and the integral part is where μ stays 0 for $T \le T_c$. The integral gives the number of particles not in the lowest energy state at temperature T.

The integral part $= \frac{V}{4\pi^2}\left[\frac{2M}{\hbar^2}\right]^{3/2}\left(k_B T\right)^{3/2}\int_0^\infty \frac{x^{1/2}}{e^x - 1}dx$, where $k = \frac{\varepsilon}{k_B T}$.

Use $\frac{N}{V} = \frac{1}{4\pi^2}\left[\frac{2M}{\hbar^2}\right]^{3/2}\left(k_B T_C\right)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x - 1}$

$$N = N_0(T) + N\left(\frac{T}{T_c}\right)^{3/2}$$

$$N_0(T) = N\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)$$

At $T = 0$, all the particles are in the lowest state. The amount of particles in the lowest energy state will hit 0 at T_c. This is completely different from what happens with fermions, where you can only have one particle in each energy state.

6.2.2 How to observe BEC

The problem is to find a boson system which remains gaseous at low T. This can be achieved in alkali metal vapours.

$$K_B T_c = \frac{\hbar^2}{2M}\left(\frac{4\pi^2}{2.31}\right)^{2/3} n^{2/3}$$

e.g. $^{7}_{3}Li$, $^{23}_{11}Na$, $^{87}_{37}Rb$ - since 1995.

For $^{23}_{11}Na$, $n = 10^{20} m^{-3}$

$T_c = 1.5\mu k$

Start with atoms in oven at 600k. Therefore cooling by a factor of 4×10^8 required. This takes around 10 seconds.

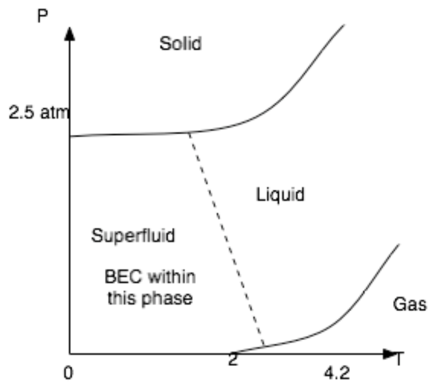
More about this can be found in Physics World, March 1997.

1) Atoms effuse from oven into a high vacuum container. A laser beam is then fixed at them, and slow them through laser traps to $T \sim 1k$
   → 10^{10} atoms in magneto-optical trap $0.5m$ from oven.
2) MOT uses laser beams and a magnetic field to trap atoms in a small region of space.
   → $100\mu k$
   The moving atom absorbs a Doppler-shifted photon, hence promoting an electron to a higher state than the energy of the photon would normally allow (through resonance). When it collapses back down from the excited state, it emits the proper energy – hence it looses some energy overall. As the emitted photon has energy $h\nu' > h\nu$, the atom is cooled.
   Six laser beams are employed, so atoms moving in all directions are slowed. ("optical molasses")
3) Turn off laser beam.
   The magnetic field traps atoms of a given spin orientation (they have a magnetic moment $E = -\underline{\mu} \cdot \underline{B}$). You can then apply an RF field to flip spins of the higher energy particles, which will allow them to escape. The remaining atoms are cooled by evaporation. Now, $T < 1\mu k$.

Questions:

1) Are the BEC atoms superfluid? Yes.
2) Interactions between the atoms have been studied.

Liquid ^4He



Similarities to ^3He:

- Remains liquid down to $T = 0$ because ZPE of solid

Differences:

- No minimum in the melting curve $\frac{dp}{dt} > 0$ (because no large (spin) outcrops in solid).

Estimate of BEC transition temperature (ignores interaction!):

$$\rho = 145 kgm^{-3}$$

$$M = 4 \times 1.66 \times 10^{-27} kg$$

$$\frac{N}{V} = \frac{\rho}{M} = 2.2 \times 10^{28} m^{-3}$$

$$k_B T = \frac{\hbar^2}{2m}\left(\frac{4\pi^2}{2.31}\frac{N}{V}\right)^{2/3}$$

$$T_c = 3.12k$$

So we are in the right region for liquid → superfluid transition.

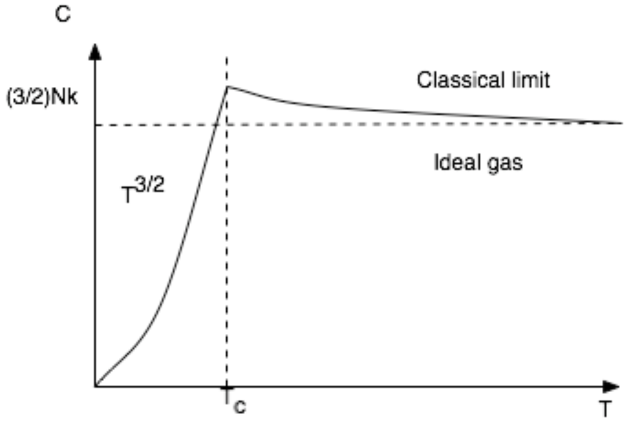
Heat capacity of ideal Bose gas

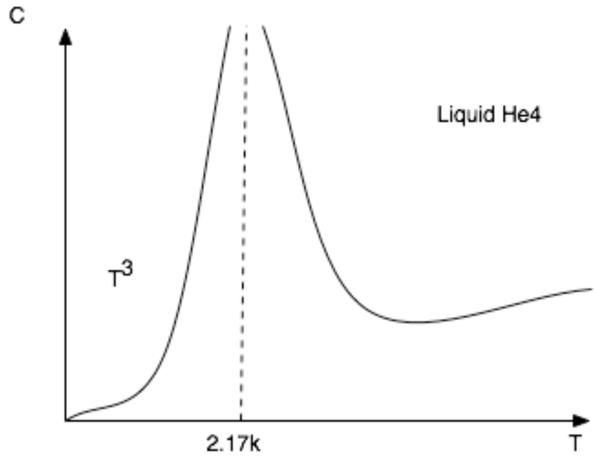
$$E = \int_0^\infty \varepsilon f(\varepsilon) g(\varepsilon) d\varepsilon$$

$$C = \left(\frac{dE}{dT}\right)_V$$

μ chosen so $N = \int_0^\infty d\varepsilon f(\varepsilon) g(\varepsilon)$, $T > T_c$

$\mu_0 = 0$, $T < T_c$





Similarity to the Greek letter λ, therefore called λ-transition.

Below $T_\lambda = 2.17k$, ^4He is a superfluid.

About 10% of the atoms are in the lowest energy state at $T = 0$ - not 100% as for ideal gas.
NB: if all the atoms were in the state:

$$\psi = \left(\sqrt{\frac{2}{L}}\right)^3 \sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L}$$

Density $|\psi|^2$
Atoms would be predominantly in the center of the box, with none near to the walls.
You end up with a state of uniform density across the system. In fact is a liquid due to the interactions.
Experiment to demonstrate BEG in Helium II:
(Adrian Wyatt, Nature 391, 57 (1998)
Employs quantum evaporation – analogue of photoelectric effect.
Sound waves become phonons (quantized). They come in at an angle θ_{ph} through the liquid. Atoms are then evaporated out of the liquid with momentum p and angle θ_a.

Energy of evaporated atom $\frac{p^2}{2m} = h\nu - L$

Energy of the evaporated atom = phonon energy – latent heat per atom.
Therefore for monochromatic phonons, p is known.
Component of momentum parallel to surface conserved.

$$\hbar q_{ll} + P_{i,atom,ll} = p_{f,atomll}$$

$$\frac{h\nu}{v}\sin\theta_{ph} + p_{i,atomll} = p\sin\theta_a$$

Therefore by measuring angular distribution of evaporated atoms determine distribution of $p_{i,atomll}$ for liquid atoms.