5. The Ideal Fermi Gas at Low Temperatures

(M. 11.5, B&S 10.3-4, K&K p183-184)

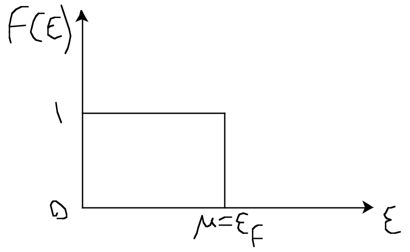
Applications:

- Electrons in metal and semi-conductors
- Liquid helium 3
- Gas of Potassium 40 atoms at $T = 0.3\mu k$
- Electrons in a White Dwarf star
- Neutrons in a Neutron star (pulsar)

5.1 Ideal Fermi Gas at $T = 0$

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

As $T \to 0$, $f(\varepsilon) = 1$ if $\varepsilon < \mu$, or $f(\varepsilon) = 0$ if $\varepsilon > \mu$.



ε_f equals the chemical potential μ at $T = 0$.

All the lowest energy states are occupied. Highest occupied state has energy μ.

For particles in a box:

$$\varepsilon = \frac{\hbar^2 k^2}{2M}, \quad k^2 = k_x^{\ 2} + k_y^{\ 2} + k_z^{\ 2}$$

$$k_x = \frac{l\pi}{L}, k_y = \frac{m\pi}{L}, k_y = \frac{n\pi}{L}$$

l	m	n	Number of particles	Running total
1	1	1	2 (up and down)	2
2	1	1	2	4
1	2	1	2	6
2	1	1	2	8
2	2	1	2	10
…	…	…	…	…
				$\sim 10^{22}$

States form a cubic lattice in k-space of side $\frac{\pi}{L}$.

States inside an octant of k-space of radius k_f are filled.

$$2\frac{1}{8}\frac{\left(\frac{4}{3}\pi k_f^{\ 3}\right)}{\left(\pi/L\right)^3} = N$$

where the 2 is due to spin degeneracy, $= 2s+1$ for double occupancy of k space (up and down).

The $\frac{1}{8}$ is the positive octant $k_x, k_y, k_z > 0$.

$\left(\frac{4}{3}\pi k_f^{\ 3}\right)$ is the volume of the sphere.

$\left(\pi/L\right)^3$ is the volume of k-space per k state.

$k_f = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$ = Fermi wave number

$V = L^3$ = the volume of the box

$k_f = \left(3\pi^2 n\right)^{1/3}$ where $n = \frac{N}{V} =$ particle density.

The surface of a sphere is the boundary between occupied and unoccupied states = Fermi surface.

Energy of particles on the Fermi surfare $= \varepsilon_f =$ Fermi energy $= \frac{\hbar^2 k_f^{\ 2}}{2M}$.

$\varepsilon_f = \frac{\hbar^2}{2M}\left(3\pi^2 k\right)^{2/3} = \mu$ at $T = 0$.

$\varepsilon_f = \frac{1}{2}MV_f^{\ 2}$ defines the Fermi velocity = the speed of the particles on the Fermi surface.

$p_f = \hbar k_f$: defines the Fermi momentum.

5.2 Alternative derivation of k_f, ε_f

$\rho(k)dk = \frac{Vk^2}{2\pi^2}dk =$ the number of k states $k \to k + dk$.

$$N = 2\int_0^{k_f} \rho(k)dk = 2\int_0^{k_f} \frac{Vk^2}{2\pi^2}dk = \frac{V}{\pi^2}\left[\frac{k^3}{3}\right]_0^{k_f} = \frac{V}{\pi^2}\frac{k_f^{\ 3}}{3}$$

where the 2 is once more due to spin degeneracy.

So:

$k_f = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$ as before.

Or work with energy:

$$N = 2\int_0^{\varepsilon_f} g(\varepsilon)d\varepsilon$$

$$g(\varepsilon) = \rho(k)dk$$

$$g(\varepsilon) = \rho(k)\frac{dk}{d\varepsilon}$$

$$\varepsilon = \frac{\hbar^2 k^2}{2M}$$

$$\frac{d\varepsilon}{dk} = \frac{\hbar^2 k}{M}$$

$$\frac{Vk^2}{2\pi^2}dk = \frac{V}{\pi^2}\frac{2M\varepsilon}{\hbar^2}\left(\frac{2M}{\hbar^2}\right)^{1/2}\frac{1}{2}\varepsilon^{-\frac{1}{2}}d\varepsilon$$

$$= \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon$$

$$g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\varepsilon^{1/2} = \text{number of k states per unit energy range}$$

$$N = 2\int_0^{\varepsilon_f} g(k)\,d\varepsilon$$

$$= 2\int_0^{\varepsilon_f}\frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon$$

$$= \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\left[\frac{2}{3}\varepsilon^{3/2}\right]_0^{\varepsilon_f}$$

$$\varepsilon_f = \frac{\hbar^2}{2M}\left(3\pi^2\frac{N}{V}\right)^{2/3} \text{ as before.}$$

$\rho(\varepsilon) = 2g(\varepsilon) =$ number of states allowing for spin degeneracy.

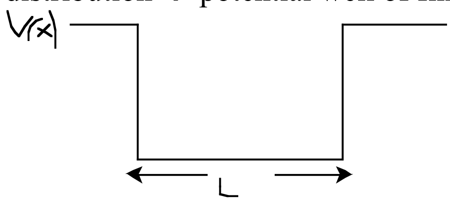
$$\rho(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$$

$V = 1$ for "unit volume".

http://www.BritneySpears.ac

5.3 Free Electron Theory of Metals
Conduction electrons $\approx$ ideal Fermi gas.
Approximations:

1) Positive ions are roughly equal to a uniformly distributed positive charge distribution → potential well of finite depth.



2) Ignore the Coulomb repulsion between electrons
   – justification of approx. difficult
   – departure are visible from ideal gas behavior
   - Free electron theory is a good starting point for a more relativistic theory.

   Typical metal $= n = \frac{N}{V} \approx 10^{28}\,m^{-3}$

   e.g. for Potassium 1.40×10^{28} atoms per metre cubed. $= 1.402\times10^{28}$ electrons per metre cubed.

   $k_f = \left(2\pi^2 n\right)^{1/3} \approx 7.5\times10^{9}\,m^{-1}$ for k, which is roughly 1 / Angstrom.

   Wavelength $\lambda_f = \frac{2\pi}{k_f} \sim 10^{-9}\,m \sim$ atomic spacings. Therefore expect stray quantum effects.

$\varepsilon_f = \frac{\hbar^2 k^2}{2m_e} = 2.1eV$.

Define T_f = degeneracy or Fermi temperature $= \frac{\varepsilon_f}{k_B}$.

T_f is the temperature at which classical gas of particles would have energies of ε_f.

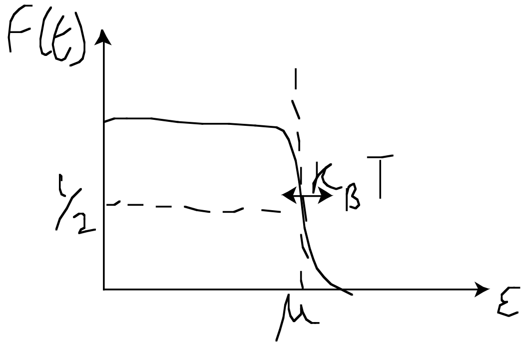
$T_f \approx 2.5 \times 10^4 k$ for K.

At $T = 0$ the electrons near the Fermi surface have kinetic energies corresponding to classical temperatures of $10^4 k$. The Fermi velocity

$v_f = \left(\frac{2\varepsilon_f}{m_e}\right)^{1/2} = \frac{\hbar k_f}{m_e} \approx 8.6 \times 10^5 ms^{-1}$ for k.

Large but still much less than the speed of light.
Relativistic corrections are only needed for heavy metals.

5.4 Ideal Fermi Gas at $T > 0$



If $T << T_F$ (e.g. metal at room temperature) a few electrons in the neighbourhood of the Fermi surface are parallel to the state above ε_f.

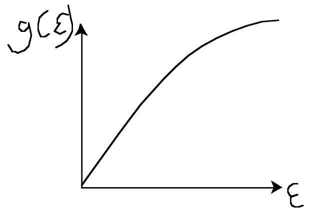
What is μ?

$$\sum_\varepsilon f(\varepsilon) = N = \sum_\varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

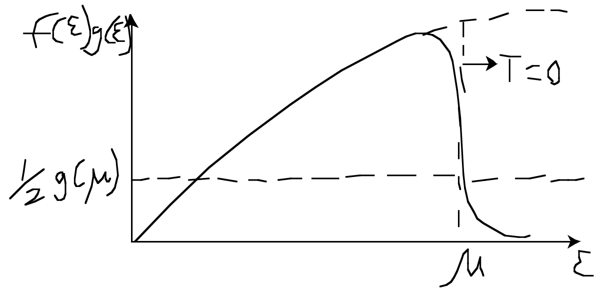
- fixed μ.

Change sum to an integral

$$\int_0^\infty f(\varepsilon) g(\varepsilon) d\varepsilon = N$$



$g(\varepsilon) \sim \sqrt{\varepsilon}$



Integral = everything below the line.
At $T = 0$, $\mu = \varepsilon_f$

$T > 0$, $\mu < \varepsilon_f$

$$\mu = \varepsilon\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_f}\right)^2 + ...\right)$$

Mandl problem 11.2
If $T << T_f$, $\mu \approx \varepsilon_f$

For metals at $300k$, can take $\mu = \varepsilon_f$.

Because the electrons are crammed into their lowest possible energy states and this dominates their behaviour, an ideal Fermi gas at low temperatures is said to be highly degenerate.

5.4.1 Heat Capacity of Fermi gas $T << T_f$

Exact calculation:
Number of electrons $\varepsilon \to \varepsilon + d\varepsilon = f(\varepsilon)g(\varepsilon)d\varepsilon$

Energy of electrons $\varepsilon \to \varepsilon + d\varepsilon = \varepsilon f(\varepsilon)g(\varepsilon)d\varepsilon$

Total energy E of electrons

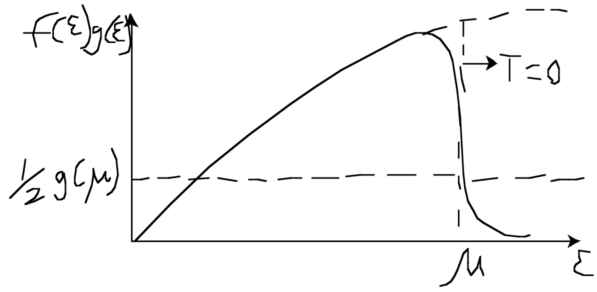
$$E = \int_0^\infty \varepsilon f(\varepsilon)g(\varepsilon)d\varepsilon$$

$$C_v = \left(\frac{dE}{dT}\right)_v$$

For $T << T_f$;

$$C_v = \frac{\pi^2}{2} Nk_B \frac{T}{T_f}$$

This is the exact result.
Approximate calculation – ignore changes in μ.



Now looking at the regions between the solid line, and the vertical line at μ.

Electrons from the left side of the line $\left(\varepsilon < \varepsilon_f\right)$ are thermally excited to the region to the right side of the line, $\varepsilon > \varepsilon_f$. The two regions are the same area.

Let them be triangles. $\frac{1}{2} base \times height$.

$$= \frac{1}{2}\left(k_B T\right)\frac{1}{2}g\left(\varepsilon_f\right) = \frac{1}{4}\left(k_B T\right)g\left(\varepsilon_f\right)$$

Every electron that has moved has changed its' energy by $\sim k_B T$.

Therefore the increase in the energy $\Delta E \approx \frac{1}{4}\left(k_B T\right)^2 g\left(\varepsilon_f\right)$

$$C_v = \frac{d\Delta E}{dt} \approx \frac{1}{2}k_B{}^2 T g\left(\varepsilon_f\right)$$

What is $g\left(\varepsilon_f\right)$?

$$g\left(\varepsilon_f\right) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_f{}^{1/2}$$

But $\varepsilon_f = \frac{\hbar^2}{2m}k_f{}^2 = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$

Eliminate V:

$$g\left(\varepsilon_f\right) = \frac{3N}{2\varepsilon_f}$$

Hence

$$C_v \approx \frac{3}{4}N\frac{k_B{}^2 T}{\varepsilon_f} \approx \frac{3}{4}Nk_B\frac{T}{T_f}$$

This result $\left(C_v \sim Nk_B\frac{T}{T_f}\right)$ solves a mystery.

If electrons behave classically,

$$E = \frac{3}{2}Nk_B T$$

$$C_v = \frac{3}{2}k_B$$

Early success of quantum theory of Fermi gas was to explain why such a large contribution to the heat capacity was not seen.

Quantum theory reduces classical value by a factor $\frac{T}{T_f}$. Only a fraction of the electrons $N\frac{T}{T_f}$ are involved in the heat capacity.

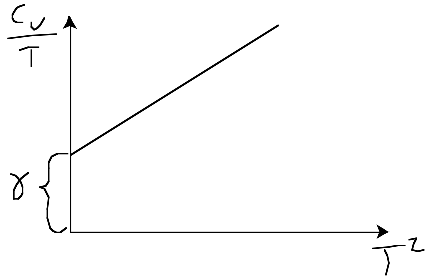
Observation of Electronic Contribution to C_v

It is difficult at high temperature because contributions are $\sim 3Nk_b$ from lattice vibrations. At low temperatures, lattice vibrations contribute $\sim T^3$.

$$C_v = \gamma T + \beta T^3$$

$\gamma =$ electronic specific heat coefficient $= \frac{\pi^2}{2} N \frac{k_B}{T_f}$ for ideal gas

$$\frac{C_v}{T} = \gamma + \beta T^2$$



Slope is β.

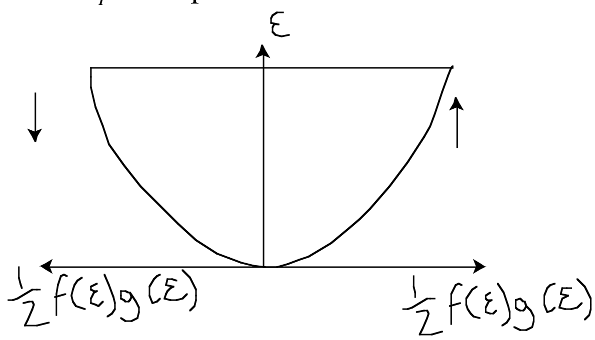
For potassium, $\gamma_{theory} = 1.67 mJ\, mol^{-1} k^{-2}$, $\gamma_{experiment} = 2.08 mJ\, mol^{-1} k^{-2}$.

Discrepancy due to the failure of the assumption of the free electron theory. We ignored:

a) Periodic potential of the positive ion cores – these produce band structure effects.
b) Coulomb repulsion between electrons

5.4.3 Paramagnetic Susceptibility of Conduction Electrons

$T << T_P$ - temperatures have a small effect. Assume $T = 0$.



$B = 0$

Previous diagram, split into up and down, and rotated through 90 degrees.

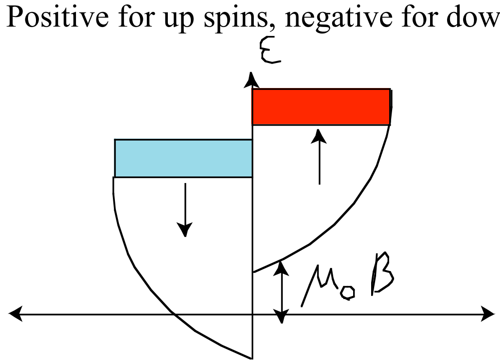
Spin ups have $\mu_z = \mu_B$.

Spin downs have $\mu_z = -\mu_B$

Apply a field B along z: $dE = -\underline{\mu} \cdot \underline{B}$ where $\underline{\mu}$ is the magnetic moment of the electron.

$dE = -\mu_z B = \pm \mu_B B$

Positive for up spins, negative for down spins.



Spin up electrons in the red area move into the blue area, where the energy is reduced. If $\mu_0 B << \varepsilon_f$, one can approximate the shaded area by a rectangle

For a metal, $B << 3\times 10^4 T$ - an enormous field.
→ energies of the highest occupied state are the same as before $= \varepsilon_f$.

→ the number of electrons moving is base times height $= \frac{1}{2} g(\varepsilon)(\mu_0 B)$

Magnetization per unit volume $= \frac{(N_\uparrow - N_\downarrow)\mu_B}{V} = \frac{2\times\frac{1}{2} f(\varepsilon)\times\mu_0 B\mu_B}{V}$.

Use $g(\varepsilon_f) = \frac{3N}{2\varepsilon_f}$ → use if $\varepsilon = \frac{p^2}{2m}$

$$M = \frac{3\mu_B{}^2}{2\varepsilon_f}\frac{N}{V}B$$

χ_p is the Pauli paramagnetic susceptibility $= \frac{\mu_0 M}{B}$.

$$\chi_p = \frac{3\mu_0\mu_B{}^2}{2\varepsilon_f}\frac{N}{V}$$

Note that this is independent of temperature.
Cf. the Curie law result for classical spins:

$$\chi_{curie} = \frac{N}{V}\frac{\mu_0\mu_B{}^2}{k_B T}$$

$$\frac{\chi_p}{\chi_{curie}} = \frac{3k_B T}{2\varepsilon_f} = \frac{3}{2}\frac{T}{T_f}$$

Like the heat capacity, the magnetic susceptibility is reduced by a factor of order T/T_f from the classical result.

χ_p is comparable to the Landau diamagnetic susceptibility.
For free electrons,

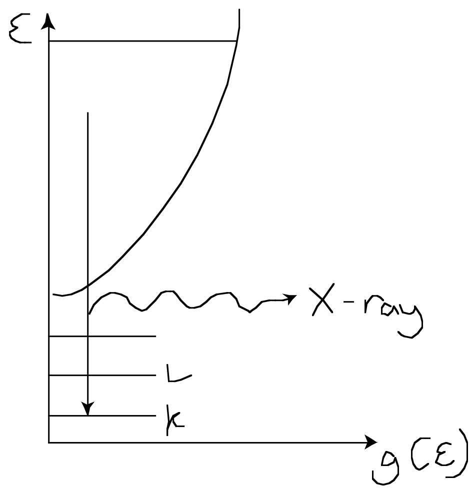
$$\chi_L = -\frac{1}{3}\chi_p$$

→ makes χ_p hard to measure.

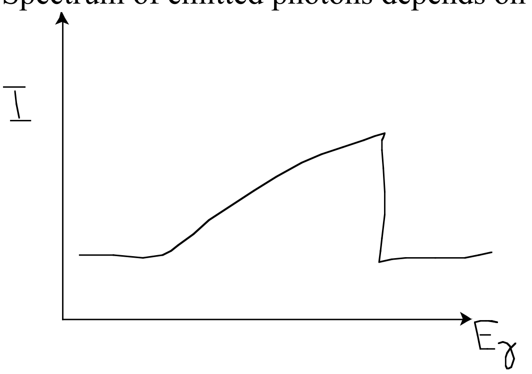
Metal	$10^5\times\chi_p$ (experiment)	$10^5\times\chi_p$ (theory)
Li	2.5	1.01
Na	1.4	0.83
K	1.1	0.67
Rb	1.0	0.63
Cs	1.0	0.58

Discrepancies are due to the neglection of band structure effects, as well as Coulomb repulsion of the electron.

5.4.4 X-Ray Emission Spectrum



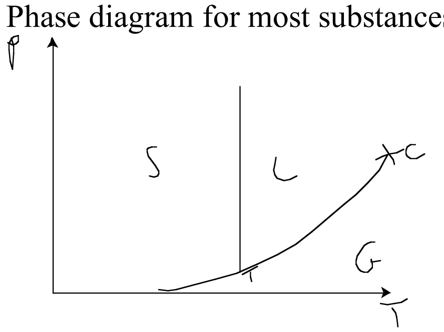
The lines are electron energy states in iron cores. The curve/line is the conduction electrons.
Knock out electrons from the K-shell, then (say) conduction electrons fall down into vacant states emitting photons.
Spectrum of emitted photons depends on the number of electrons of each energy.



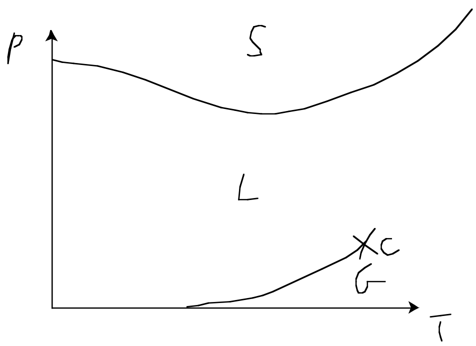
This plot is the intensity of the emitted X-rays (proportional to $g(\varepsilon)$), vs. the photon energy. There is a sharp cut-off at the Fermi energy.

5.5 Liquid ^3He

This is the Fermi form of a He ion. It is a byproduct of the nuclear weapons program.
Phase diagram for most substances:



S = Solid
L = Liquid
G = Gas
C = Critical Point
T = Triple Point
The substance is only crystal at $T = 0$.
For ^3He:



At lower T, the liquid acts as a degenerate Fermi liquid. It turns into a solid at around 34 Atmosphere. The minimum point of the solid curve is at around 1k.
S = Solid BCC.

1) System remains liquid down to $T = 0$ at low pressures (at least for pressures less than 34 atmospheres.)
   Quantum effects cause this.
   Solid: atoms localized → positions are known.
   → large uncertainty in momentum → large momentum.
   → large kinetic energy $\frac{p^2}{2m}$. This will encourage a liquid to form. This can overcome the energy level gained by going into a crystalline structure, as the liquid state now has lower energy.
2) Liquid solid phase boundary at low T has $\frac{dp}{dT}$ negative.
   Clausius-Clapepron → $\frac{dp}{dT} = \frac{S_L - S_S}{V_L - V_S}$.
   S_L and S_S are the liquid and solid entropies per kg.
   V_L and V_S are the liquid and solid volume per kg.
   For most materials, $\frac{dp}{dT} > 0$, $S_L > S_S$ and $V_L > V_S$.
   For water, $\frac{dP}{dT} < 0$, $S_L > S_S$ → liquid more disordered than solid. $V_L < V_S$ → liquid more dense than solid (ice floats on water).
   For ${}^3He$, $V_L > V_S$ i.e. contracts on freezing like most materials. But $S_L < S_S$ → the liquid is more ordered than the solid.
   ${}^3He$ is a Fermi liquid at low temperatures, and is very ordered at low temperatures. Particles in the Fermi sphere are filled by ${}^3He$ particles with up and down spins. (i.e. each k state is filled. The particles are not so ordered.). Each k state less than k_f contains two atoms, one with spin up, and one with spin down.
   Ordering is in k-space, not real space.
   For solid, the spins (which lie on the nuclei of the ${}^3He$) are disordered – each spin can be up or down because the atoms are located on crystal sites. This gives the contribution to the entropy of $Nk_B \ln 2$ $\left(\Omega = 2^N\right)$ $\left(S = k \ln \Omega\right)$.
   At $1mK$, the solid undergoes a magnetic ordering transition to the anti-ferromagnetic phase, $S_S \to 0$.

$$\frac{dP}{dT} \sim \frac{0}{V_L - V_S} \to 0$$

As $T \to 0$, $\frac{dP}{dT} \to 0$.

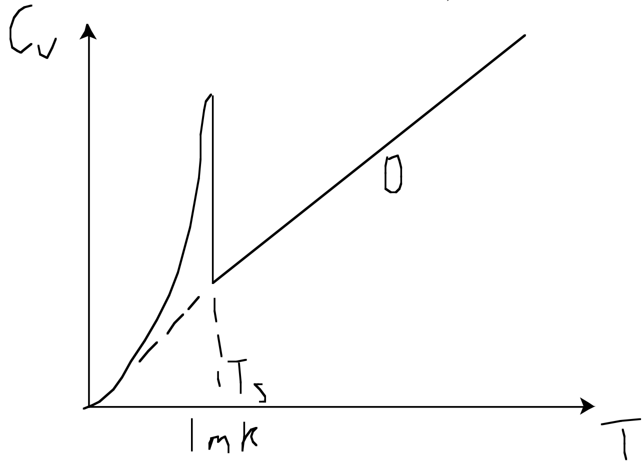
What is the Fermi temperature T_F of ^3He?

At $P = 0$, $n = 1.6 \times 10^{28} m^{-3}$.

$T_f = \frac{\varepsilon_f}{k_B} = \frac{1}{k_B}\frac{\hbar^2}{2M}\left(3\pi^2 n\right)^{2/3} = 5k$.

NB: $m = 3 \times 1.66 \times 10^{-27} kg$

For $T << T_f$, ^3He should behave like a Fermi system. $C_V = \gamma T$. (no lattice, therefore no T^3 term as in metals.)



D = Degenerate Fermi fluid.

$T < T_S$, liquid makes a transition to the A-phase of superfluid ^3He.

5.6 Electrons in Stars

(A.C. Phillips, "The Physics of Stars")

What is the sun made of?

Mostly Hydrogen.

Atomic hydrogen?

No – temperatures are too high. The proton and the electron have been dis-associated, i.e. it is a plasma.

How hot is the sun?

Central temperature $10^7 k$.

Do the electrons form a degenerate Fermi gas, i.e. is $T << T_f$?

Mass of the Sun is roughly $2 \times 10^{30} kg$.

The radius of the sun is $7 \times 10^8 m$.

Assume that the sun has a uniform density (not true.)

The electron density in the Sun is equal to the number of Hydrogen atoms, which is the same as the number of Protons. So:

$$n = \frac{M_\odot}{M_H V_\odot} = \frac{2\times10^{30}}{1.66\times10^{-27}\times\frac{4}{3}\pi\times\left(7\times10^{8}\right)^{3}} = 8\times10^{29}m^{-3}$$

This is a little bit larger than in metals.

$$T_f = \frac{\varepsilon_f}{k_B} = \frac{1}{k_B}\frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} = 4\times10^{5}k$$

Therefore the electrons in the sun are classical.

Why does the sun not collapse under its own gravity?
Partly gas pressure, but also the outward pressure of the radiation.

Where do we find degenerate Fermi gas in stars?
We need to increase the density. So look at dense stars – neutron stars, or white dwarfs.

5.6.1 White Dwarf Stars

Entering the white dwarf phase in the ultimate fate of light stars such as the sun.
H burns at $10^7 k$ (we need the kinetic energy to overcome the coulomb repulsion).
After all the H has burnt, the star shrinks and heats until He burns to Carbon at $10^8 k$. If the star is then heavy enough, it can progress to burn C at $5\times10^8 k$, etc.

So the temperature needs to rise in order to set kinetic energies to overcome the larger coulomb barrier of the heavier nuclei.

But suppose that the star never gets hot enough for Helium to burn – or for Carbon to burn? What steps come from shrinking the star?
Inward gravitational pressure is balanced by pressure from the degenerate electron gas. → white dwarf → electron gas.
White dwarf slowly cools and dies.
If mass $= M_s$, what is its' radius R?
Gravitational pressure $= f\left(G, M_s, R\right)$
Pressure = force per unit area

$$P = \frac{GM_s^{\,2}}{R^3}\frac{1}{4\pi R^2} \sim \frac{GM_s^{\,2}}{R^4}$$

Assume the star is of constant density.
Gravitational potential energy:

$$E_p = -\frac{3}{5}\frac{GM_s^{\,2}}{R}$$

$$pA\delta R = \Delta V = \frac{3}{5}\frac{GM_s^{\,2}}{R^2}\Delta R$$

$$p = \frac{3}{5}\frac{GM_s^{\,2}}{R^2}\frac{1}{4\pi R^2}$$

This inward gravitational pressure is counteracted by pressure being exerted outwards (excited by electrons).
What is the pressure of the electrons?
Assume $T \ll T_f$, and take $T = 0$ for simplicity.

$$dE = TdS - pdV = -pdV \text{ (at } T = 0\text{)}$$
$$p = -\left(\frac{\partial E}{\partial V}\right)_N \text{ (at } T = 0\text{)}$$
$$E = \int_0^{t_f} \varepsilon g(\varepsilon) d\varepsilon$$
$$= V\int_0^{t_f} 2\frac{k^2}{2\pi^2} dk \frac{\hbar^2 k^2}{2m}$$
$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\varepsilon_f} \varepsilon^{3/2} d\varepsilon$$
$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\varepsilon_f^{5/2}}{5/2} = \frac{3}{5}N\varepsilon_f$$
$$\varepsilon_f = \frac{\hbar^2}{2m}k_f^{\,2} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$
has been used to eliminate the volume term V.
$$\varepsilon = \frac{3}{5}N\varepsilon_f$$
$$p = -\left(\frac{\partial \varepsilon}{\partial V}\right)_N = -\frac{3}{5}N\left(\frac{\partial \varepsilon_f}{\partial V}\right)_N$$
$$\left(\frac{\partial \varepsilon_f}{\partial V}\right)_N = -\frac{\hbar^2}{2m}\left(3\pi^2 N\right)^{2/3} \frac{1}{V^{5/3}}\frac{2}{3} = -\frac{2}{3}\frac{\varepsilon_f}{V}$$
$$P = \frac{2}{5}\varepsilon_f \frac{N}{V} = \frac{2}{3}k_0 T_f \frac{N}{V}$$
$$= \frac{\left(3\pi^2\right)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}$$
N is the number of electrons in the star $= \dfrac{M_s}{2M_H}$, where M_H is the mass of a H atom.

For either carbon or helium white dwarfs, these will be approximately one electron for every 2 nucleons.
$$V = \frac{4}{3}\pi R^3$$
$$\rho = A\frac{\hbar^2}{m}\left(\frac{M_s}{M_H}\right)^{5/3}\frac{1}{R^5}$$
$$A = \frac{\left(3\pi^2\right)^{2/3}}{5}\left(\frac{3}{8\pi}\right)^{5/3} \approx 0.06$$
NB: this is the pressure pushing outwards.

Hence $\dfrac{GM_s^{\,2}}{R^4} \approx \dfrac{\hbar^2}{m}\left(\dfrac{M_s}{M_H}\right)^{5/3}\dfrac{1}{R^5}$

$$R \approx \frac{\hbar^2}{mGM_s^{1/3}M_H^{5/3}}$$
(the more massive the star, the smaller R gets.)
For example, 40EriB:
$M_s \approx 10^{30}\,kg$
$R_{theory} = 7.9 \times 10^6\,m$
$R_{actual} = 8.7 \times 10^6\,m$
Note that this is around the size of the Earth.
→ good agreement.
$R \downarrow$ as $M_s \uparrow$.
$R \propto \hbar^2$, therefore QM effects work on enormous length scales.
Check assumptions that electron gas is degenerate:
$$T_f = \frac{\varepsilon_f}{k_B} = \frac{1}{k_B}\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$
$N = \frac{M_s}{2M_H}$, $V = \frac{4}{3}\pi R^3$
→ $10^9\,k$
But $T \sim 10^7\,k$.
Therefore $T << T_f$, i.e. cannot be hot enough to burn He or C.
Now, $P_{electrons} \propto \frac{1}{R^5}$, $P_{grav} \sim \frac{1}{R^4}$.
i.e. electron pressure increases faster at small R.
Electron pressure $\left(P_{electrons}\right)$ increases faster at small R. It ought to be able to resist collapse of heavier: wrong.
Work out V_f for electrons at Fermi surface for $T_f = 10^9\,k$
$$\frac{1}{2}mV_f^{\,2} = k_B T_f = 87\,keV$$
$V_f = 1.8 \times 10^8\,ms^{-1}$
So what's the problem? – electrons are relativistic and it is important to treat them as such for heavier stars.
(non-relativistic approximation is OK for white dwarfs)
$\varepsilon^2 = p^2c^2 + m^2c^4$
We will work in the extreme relativistic limit, $p^2c^2 >> m^2c^4$.
$\varepsilon^2 = pc = \hbar ck$
At $T = 0$,
$$E = \int_0^{\varepsilon_f} \varepsilon g(\varepsilon)\,d\varepsilon = \int_0^{k_f} 2\frac{V}{2\pi^2}k^2 dk\,\hbar ck$$
$$N = \int_0^{k_f} 2\frac{V}{2\pi^2}k^2 dk = \frac{V}{3\pi^2}k_f^{\,3}$$
→ $k_f = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$

$\varepsilon_f = \hbar c k_f$

$$E = \frac{V\varepsilon_f^{\ 4}}{4\pi^2(\hbar c)^3} = \hbar c\left(\frac{9\pi}{8}\right)^{2/3}\left(\frac{N}{V}\right)^{4/3} V$$

$$p = -\left(\frac{\partial E}{\partial V}\right)_N = \frac{\hbar c}{3}\left(\frac{9\pi}{8}\right)^{2/3}\left(\frac{N}{V}\right)^{4/3}$$

$\sim \frac{M_s^{\ 4/3}}{R^4} \sim$ electron pressure

Gravitational pressure $M_s^{\ 2}/R^4$ → therefore for heavy stars, gravity will always win if 1.4.

For heavy stars:

$1.4 \times M_\odot$ → star becomes a neutron star

$5.8 \times M_\odot$ → star becomes a black hole.

5.5.2 Neutron Stars

When mass is $> 1.4 M_\odot$, then the star goes through all the stages of nuclear fusion to reach iron core.

Then get catastrophic collapse via a supernova explosion – this releases some of the star's original mass.

As star collapses, electron density n rises.

Electron KE rises $\sim \frac{1}{v^{1/3}}$ (relativistic).

Therefore KE becomes so high that inverse beta decay occurs.

$e^- + p \rightarrow n + \nu$

Also, atomic nuclei break up – left with a giant lump of nuclear matter.

Predominantly made of neutrons – only a few e and p left.

Gravitational pressure can now be resisted by the pressure of gas of degenerate neutrons with

$T \sim 10^7 k$

(Star is initially at $10^4 k \rightarrow 10^8 k$ in 100 years → neutron star or pulsar.)

Solid crystalline crust of nuclei and neutrons, with a neutron fluid inside.

$$p = \frac{2}{5}\frac{N}{V}\varepsilon_f$$

$$\varepsilon_f = \frac{\hbar^2}{2m_n}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$V = \frac{4}{3}\pi R^3$$

$$N = \frac{m_s}{m_n}$$

$$p \sim \frac{\hbar^2}{m_n}\frac{N^{5/3}}{R^5} \approx \frac{GM_s^{\ 2}}{R^4} \text{ (gravitational pressure)}$$

$$R \approx \frac{\hbar^2}{m_n G m_n^{5/3} m_s^{1/3}}$$

For $m_s \sim 3\times10^{30} kg \ \sim 1.5 M_{\odot}$, $R \sim 10 km$

$$\frac{N}{V} = 4.3\times10^{44} m^{-3}$$

$$\varepsilon_f = 1.8\times10^{-11} J = 0.11 GeV$$

$$V_f = \sqrt{\frac{2\varepsilon_f}{m_n}} = 1.5\times10^{8} ms^{-1}$$

i.e. about ½c. So it is non-relativistic – only just.
For heavier stars, relativistic effects become more important – as with white dwarfs and electrons.
Eventually, pressure of neutrons can no longer withstand gravity.
$M_{\max}$ ~ 5.8 solar masses.
Collapse continues → black holes.