1. Quantum Mechanics of Ideal Gases

Further reading: M 9.1 - 9.2, B&S 6.1 - 6.4, K&K p152 - 153.

Consider the collision of two identical particles. Two particles (1 and 2) enter the interaction region, and two particles leave. The uncertainty principle prevents is from putting labels 1 and 2 on the outgoing particles. We cannot know the position and momentum of the particles accurately enough within the interaction region to distinguish the two possibilities.

This has a profound implication for the wave function of the two particles. We shall reach the conclusion that the particles of nature come in two types – *bosons* and *fermions*.

In quantum mechanics we have studied the wave function of a single particle $\Psi(x)$. $\left|\Psi(x)\right|^2 dx$ is the probability of finding the particle between x and $x+dx$. For two particles, the wave function is a function of two variables x_1 and x_2. Then $\left|\Psi(x_1,x_2)\right|^2 dx_1 dx_2$ is the probability of finding particle 1 between x_1 and x_1+dx_1, and particle 2 between x_2 and x_2+dx_2.

$\Psi(x_1,x_2)$ is found from solving the Schrödinger equation

$$\hat{H}\Psi(x_1,x_2) = E\Psi(x_1,x_2)$$

where $\hat{H}$ is the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^{\,2}} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^{\,2}} + V(x_1) + V(x_2) + V(x_1,x_2)$$

where the first and second parts are the E_k of the two particles, the third and fourth the E_P of particles in an external field, and the last the E_P of field interaction e.g. $\dfrac{e^2}{4\pi\varepsilon_o\left|x_1 - x_2\right|}$ for the Coulomb force.

Indistinguishability restricts the form of $\Psi(x_1,x_2)$.

1.1 Symmetry of $\Psi(x_1,x_2)$

Ψ cannot be measured – only $\left|\Psi\right|^2$. If the particles 1 and 2 are indistinguishable,

$$\left|\Psi(x_1,x_2)\right|^2 = \left|\Psi(x_2,x_1)\right|^2.$$

If this were not the case, then the particles could be identified. E.g. one could identify a particular region of space and say that particle 1 is the particle that has a 10% probability of being in the region, particle 2 has a 15% probability.

The above conclusion $\left|\Psi\left(x_1,x_2\right)\right|^2 = \left|\Psi\left(x_2,x_1\right)\right|^2$ states that these probabilities are *equal* for any region of space.

Deduce
$\Psi\left(x_2,x_1\right) = e^{i\alpha}\Psi\left(x_1,x_2\right)$
But repeating the interchange must give back the original Ψ.
$\Psi\left(x_1,x_2\right) = e^{2i\alpha}\Psi\left(x_1,x_2\right)$
Therefore $e^{2i\alpha} = 1$, and $e^{i\alpha} = \pm 1$
So $\alpha = 0$ or $\alpha = \pi$.
$e^{i0} = +1$; $e^{i\pi} = -1$
For $\alpha = 0$: $\Psi\left(x_1,x_2\right) = \psi\left(x_2,x_1\right)$ - Bosons.
Ψ is symmetric under interchange of particles. Particles with this behavior are Bosons or Bose-Einstein Particles.

For $\alpha = \pi$: $\Psi\left(x_1,x_2\right) = -\Psi\left(x_2,x_1\right)$ - Fermions.
Ψ is anti-symmetric under interchange of particles.
Particles with this behavior are called Fermions or Fermi-Dirac particles.

1.2 – Bosons and Fermions and Spin
Particles such as electrons, photons and neutrons have intrinsic angular momentum or spin. Whether a particle is a boson or fermion is determined by the magnitude of its' spin angular momentum.

$\underline{\hat{S}}$ is the operator for the spin angular momentum of the particle.

$\hat{S}^2$ has eigenvalues $S\left(S+1\right)\hbar^2$ where if $S = 0,1,2,3,...$ we have the case of integral spin, which are always bosons, or if $S = \frac{1}{2},\frac{3}{2},\frac{5}{2},...$ we have the case of half-integral spin, which are always fermions.

Spin degeneracy $= 2S+1$. For an electron, spin degeneracy = $2s+1 = 2$. $\hat{S}_z$ eigenvalues $\pm\frac{\hbar}{2}$ - electron spin can be up or down.

Pauli confirmed this through relativistic formulation of quantum field theory, which provides a rigorous proof of the connection between the spin and the statistics.

<table>
<tr><td>Fermions</td><td>Bosons</td></tr>
<tr><td>Electron, Quark and Neutrino all have spin $\frac{1}{2}$</td><td>Photon $S = 1$
Graviton $S = 2$</td></tr>
<tr><td colspan="2">The above are all “really fundamental” particles.
Composite particles consist of even numbers of fermions have integral spin and behave as bosons.
Composite particles consist of odd numbers of fermions have half-integral spin and</td></tr>
</table>

behave as fermions.	
Neutron, proton - $s=\frac{1}{2}$ each mode of 3 quarks. ${}^{40}_{19}k$ $\left(10p+21n+19e\right)=59$ $S=\frac{1}{2}$	π meson $S=0$ (2 quarks) H atom (2 fermions, $e+p$) $S=0$ α particle $\left(2n+2p\right)$ $S=0$ He^4 atom $\left(2n+2p+2e\right)$ $S=0$ ${}^{87}_{37}Rb$ $\left(37p+50n+37e\right)=124$ $S=0$ H_2 $2e+2p$ $S=0$

Why should the H_2 molecule $\left(2e+2p\right)$ behave as a boson?

$$\begin{aligned}\Psi\left(x_1,x_2\right) &= \Psi\left(\left[x_{e1},x_{p1}\right],\left[x_{e2},x_{p2}\right]\right)\\ &= -\Psi\left(\left[x_{e1},x_{p2}\right],\left[x_{e2},x_{p1}\right]\right)\\ &= \Psi\left(\left[x_{e2},x_{p2}\right],\left[x_{e1},x_{p1}\right]\right)=\Psi\left(x_2,x_1\right)\end{aligned}$$

Therefore symmetric under exchange of the atoms.

1.3 Fermions obey the Pauli Exclusion Principle

No two fermions in the same single particle states. Single particle states, $\phi_a\left(x\right)$, $\phi_b\left(x\right)$,… with energies E_a, E_b, are solutions of TISE for one particle in the system.

If more than one particle in a system and particles do not interact too strongly – then states still useful.

For two non-interacting particles, $\Psi\left(x_1,x_2\right)=\phi_a\left(x_1\right)\phi_b\left(x_2\right)$ satisfies the TISE with $E=E_a+E_b$. However it is not correct for identical particles – it is not anti-symmetric or symmetric, under $x_1 \Leftrightarrow x_2$. Could distinguish between the particles – 1 is in state a, 2 is in state b.

Hence:

$\Psi\left(x_1,x_2\right)=\frac{1}{\sqrt{2}}\left(\phi_a\left(x_1\right)\phi_b\left(x_2\right)+\phi_a\left(x_2\right)\phi_b\left(x_1\right)\right)$ Bosons → symmetric under $x_1 \Leftrightarrow x_2$.

$\Psi\left(x_1,x_2\right)=\frac{1}{\sqrt{2}}\left(\phi_a\left(x_1\right)\phi_b\left(x_2\right)-\phi_a\left(x_2\right)\phi_b\left(x_1\right)\right)$ Fermions → anti-symmetric under $x_1 \Leftrightarrow x_2$.

Now we cannot say which particle is in which state.

If both particles are in the same state, $a=b$, then;

$\Psi\left(x_1,x_2\right)=\phi_a\left(x_1\right)\phi_a\left(x_2\right)$ for bosons.

$\Psi\left(x_1,x_2\right)=0$ for fermions. i.e. this is not possible → Pauli principles.

Many bosons can live in the same state – they prefer it as it lowers their energies.