8. Addition of Angular Momenta

8.1 General Results

$\underline{\hat{J}_1}, \underline{\hat{J}_2}$ angular momenta with orthogonal eigenfunctions $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ respectively.

$\hat{J}^2 |j_1, m_1\rangle = \hbar^2 (j_1 + 1)|j_1, m_1\rangle$

$\hat{J}_{1Z} |j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle$

$\hat{J}^2 |j_2, m_2\rangle = \hbar^2 (j_2 + 1)|j_2, m_2\rangle$

$\hat{J}_{2Z} |j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle$

We assume that all the components of $\underline{J_1}$ commute with the components of $\underline{\hat{J}_2}$.

$\left[\hat{J}_1^2, \hat{J}_2^2\right] = \left[\hat{J}_1^2, \hat{J}_{2Z}\right] = \left[\hat{J}_2^2, \hat{J}_{1Z}\right] = \left[\hat{J}_{1Z}, \hat{J}_{2Z}\right] = 0$

We also know that $\left[\hat{J}_1^2, \hat{J}_{1Z}\right] = 0$, $\left[\hat{J}_2^2, \hat{J}_{2Z}\right] = 0$.

$\left\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1Z}, \hat{J}_{2Z}\right\}$ are a set of compatible observables with common eigenfunctions $|j_1, j_2, m_1, m_2\rangle$.

$\hat{J}_1^2 |j_1, j_2, m_1, m_2\rangle = \hbar^2 j_1 (j_1 + 1)|j_1, j_2, m_1, m_2\rangle$

$\hat{J}_{1Z} |j_1, j_2, m_1, m_2\rangle = \hbar m_1 |j_1, j_2, m_1, m_2\rangle$

$\hat{J}_2^2 |j_1, j_2, m_1, m_2\rangle = \hbar^2 j_2 (j_2 + 1)|j_1, j_2, m_1, m_2\rangle$

$J_{2Z} |j_1, j_2, m_1, m_2\rangle = \hbar m_2 |j_1, j_2, m_1, m_2\rangle$

We define the total angular momentum $\underline{\hat{J}} = \underline{\hat{J}_1} + \underline{\hat{J}_2}$.

$\hat{J}_Z = \hat{J}_{1Z} + \hat{J}_{2Z}$

$\hat{J}_Y = \hat{J}_{1Y} + \hat{J}_{2Y}$

$\hat{J}_X = \hat{J}_{1X} + \hat{J}_{2X}$

$\hat{J}^2 = \underline{\hat{J}} \cdot \underline{\hat{S}} = \left(\underline{\hat{J}_1} + \underline{\hat{J}_2}\right)^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\underline{\hat{J}_1} \cdot \underline{\hat{J}_2}$

$\left[\hat{J}^2, \hat{J}_1^2\right] = \left[\hat{J}^2, \hat{J}_2^2\right] = 0$

$\left[\hat{J}_z, \hat{J}_1^2\right] = \left[\hat{J}_z, \hat{J}_2^2\right] = 0$

$\left\{\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2\right\}$ are a set of compatible observables. There is a set of common eigenfunctions $|j, m, j_1, j_2\rangle$.

$\hat{J}^2 |j, m, j_1, j_2\rangle = \hbar^2 j(j+1)|j, m, j_1, j_2\rangle$

$\hat{J}_z |j, m, j_1, j_2\rangle = \hbar m |j, m, j_1, j_2\rangle$

$\hat{J}_1^2 |j, m, j_1, j_2\rangle = \hbar^2 j_1 (j_1 + 1)|j, m, j_1, j_2\rangle$

$\hat{J}_2^2 |j, m, j_1, j_2\rangle = \hbar^2 j_2 (j_2 + 1)|j, m, j_1, j_2\rangle$

Given j_1 and j_2 we want to find the possible values of j and m.

Eigenfunctions of $\hat{J}_z$

$\hat{J}_z |j_1, j_2, m_1, m_2\rangle = \left(\hat{J}_{1Z} + \hat{J}_{2Z}\right)|j_1, j_2, m_1, m_2\rangle = \hbar\left(m_1 + m_2\right)|j_1, j_2, m_1, m_2\rangle$

→ The states $|j_1, j_2, m_1, m_2\rangle$ are eigenfunctions $\hat{J}_z$ with eigenvalue $(m_1 + m_2)\hbar$.
→ $m = m_1 + m_2$.
$-j_1 \le m_1 \le j_1$
$-j_2 \le m_2 \le j_2$
So:
$-(j_1 + j_2) \le m \le (j_1 + j_2)$

The state space governed by $|j_1, j_2, m_1, m_2\rangle$ is the same as that governed by $|j, m, j_1, j_2\rangle$.
→ the number of linearly independent eigenfunctions has to be the same.

m	j	$\|j_1, j_2, m_1, m_2\rangle$
$j_1 + j_2$	$j_1 + j_2$	j_1, j_2, j_1, j_2
$j_1 + j_2 - 1$	$j_1 + j_2$	$\|j_1, j_2, j_1 - 1, j_2\rangle$
	$j_1 + j_2 - 1$	$\|j_1, j_2, j_1, j_2 - 1\rangle$
$j_1 + j_2 - 2$	$j_1 + j_2$	$\|j_1, j_2, j_1 - 2, j_2\rangle$
	$j_1 + j_2 - 1$	$\|j_1, j_2, j_1 - 1, j_2 - 1\rangle$
	$j_1 + j_2 - 2$	$\|j_1, j_2, j_1, j_2 - 2\rangle$
$j_1 + j_2 - n$	$j_1 + j_2$	$\|j_1, j_2, j_1 - n, j_2\rangle$
	$j_1 + j_2 - 1$	$\|j_1, j_2, j_1 - n + 1, j_2 - 1\rangle$
	…	…
	$j_1 + j_2 - n$	$\|j_1, j_2, j_1, j_2 - n\rangle$

We will have the same number of $|j, m, j_1, j_2\rangle$ states.
$-(j_1 + j_2) \le m \le j_1 + j_2$
$-j \le m \le j$
$m = m_1 + m_2$
For $j_1 \ge j_2$:

m	j	$\|j_1, j_2, m_1, m_2\rangle$
$j_1 + j_2 - 2j_2$	$j_1 + j_2$	$\|j_1, j_2, j_1 - 2j_2, j_2\rangle$
	…	…
	$j_1 - j_2$	$\|j_1, j_2, j_1, -j_2\rangle$

For $m = j_1 + j_2 - 2j_2 - 1$, we do not get any more values of j because that would require the state $|j_1, j_2, j_1, -j_2 - 1\rangle$ which violates $-j_2 \le m_2 \le j_2$.
For $j_1 \ge j_2$:
$j_1 - j_2 \le j \le j_1 + j_2$
For $j_2 \le j_1$:
$j_2 - j_1 \le j \le j_1 + j_2$
In general:
$|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$.

8.2 Examples

a) Total spin of two electrons

Electron 1	Electron 2
$\hat{S}_1^2\lvert s_1,m_1\rangle = \hbar^2 s_1(s_1+1)\lvert s_1,m_1\rangle$ $\hat{S}_{1Z}\lvert s_1,m_1\rangle = \hbar m_1\lvert s_1,m_1\rangle$	$\hat{S}_2^2\lvert s_2,m_2\rangle = \hbar^2 s_2(s_2+1)\lvert s_2,m_2\rangle$ $\hat{S}_{2Z}\lvert s_2,m_2\rangle = \hbar m_2\lvert s_2,m_2\rangle$

$\left\{\hat{S}_1^2,\hat{S}_2^2,\hat{S}_{1Z},\hat{S}_{2Z}\right\}$ compatible observables with eigenfunctions $\lvert s_1,s_2,m_1,m_2\rangle$.

$s_1 = 1/2$, $s_2 = 1/2$.

$$\lvert s_1,s_2,m_1,m_2\rangle \begin{cases} \lvert 1/2,1/2,1/2,1/2\rangle \\ \lvert 1/2,1/2,-1/2,1/2\rangle \\ \lvert 1/2,1/2,1/2,-1/2\rangle \\ \lvert 1/2,1/2,-1/2,-1/2\rangle \end{cases}$$

→ 4 states.

$\underline{\hat{S}} = \underline{\hat{S}_1} + \underline{\hat{S}_2}$.

$\left\{\hat{S}^2,\hat{S}_Z,\hat{S}_1^2,\hat{S}_2^2\right\}$ are compatible observables of eigenfunctions $\lvert s,m_s,s_1,s_2\rangle$.

$-(s_1+s_2) \le m_s \le (s_1+s_2)$

$-1 \le m_s \le 1$

$s = s_1+s_2, s_1+s_2-1,\ldots,\lvert s_1-s_2\rvert$

$s = 1,0$.

$s=0$, $m_s = 0$.

$s=1$, $m_s = -1,0,1$.

$$\lvert s,m_s,s_1,s_2\rangle = \begin{cases} \lvert 0,0,1/2,1/2\rangle \\ \lvert 1,1,1/2,1/2\rangle \\ \lvert 1,0,1/2,1/2\rangle \\ \lvert 1,-1,1/2,1/2\rangle \end{cases}$$

b) Spin-Orbit Coupling

$\underline{L}$: orbital angular momentum of electron.

$\underline{S}$: spin of electron.

$\hat{H}_{S0} = A\underline{\hat{L}}\cdot\underline{\hat{S}}$ (where A is a constant.)

$\lvert \ell,s,m_\ell,m_s\rangle \leftrightarrow \left\{\hat{L}^2,\hat{S}^2,\hat{L}_z,\hat{S}_z\right\}$

$\underline{J} = \underline{L} + \underline{S}$ = total angular momentum.

$\lvert j,m,\ell,s\rangle \leftrightarrow \left\{\hat{J}^2,\hat{J}_z,\hat{L}^2,\hat{S}^2\right\}$

$\left[\hat{H}_{S0},\hat{L}_z\right] \ne 0$, $\left[\hat{H}_{S0},\hat{S}_z\right] \ne 0$.

But $\left[\hat{H}_{S0},\hat{L}^2\right] = \left[\hat{H}_{S0},\hat{S}^2\right] = \left[\hat{H}_{S0},\hat{J}^2\right] = \left[\hat{H}_{S0},\hat{J}_z\right] = 0$.

→ $\left\{\hat{H}_{S0},\hat{J}^2,\hat{J}_z,\hat{L}^2,\hat{S}^2\right\}$ are compatible observables.

→ $|j,m,\ell,s\rangle$ are eigenfunctions of $\hat{H}_{S0}$.

$$\hat{H}_{S0} = A\underline{\hat{L}}\cdot\underline{\hat{S}}$$

$$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\underline{L}\cdot\underline{S}$$

$$\underline{\hat{L}}\cdot\underline{\hat{S}} = \frac{1}{2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right)$$

$$\hat{H}_{S0} = \frac{A}{2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right).$$

Want to find common eigenvalues…

$$\begin{aligned}
\hat{H}_{s0}|j,m,\ell,s\rangle &= \frac{A}{2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right)|j,m,\ell,s\rangle \\
&= \frac{A}{2}\left(\hat{J}^2|j,m,\ell,s\rangle - \hat{L}^2|j,m,\ell,s\rangle - \hat{S}^2|j,m,\ell,s\rangle\right) \\
&= \frac{A}{2}\left(\hbar^2 j(j+1)|j,m,\ell,s\rangle - \hbar^2\ell(\ell+1)|j,m,\ell,s\rangle - \hbar s(s+1)|j,m,\ell,s\rangle\right)
\end{aligned}$$

Therefore:

$$\hat{H}_{s0}|j,m,\ell,s\rangle = \frac{A}{2}\hbar^2\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]|j,m,\ell,s\rangle$$

→ the contributions to the total energy of $\hat{H}_{s0}$ depend on j, ℓ and s (for electron, $S = 1/2$.