7. Magnetic Moment and Spin

(Gasiorowicz Chapter 10)

Imagine an electron orbiting at a radius r, velocity v. Classically:

$$i = -\frac{ev}{2\pi r}$$

The magnetic moment for the current loop is:

$$\underline{m} = i\underline{A}$$

$\underline{A}$ is perpendicular to the loop. $|\underline{A}| = \pi r^2$.

$$|\underline{m}| = \frac{evr}{2}$$

The angular momentum of the electron is:

$$\underline{\ell} = \underline{r} \times \underline{p}$$

where

$\underline{p} = m_e \underline{v}$.

Then we have:

$$\underline{m} = -\frac{e}{2m_e}\underline{\ell}$$

In QM, we then have:

$$\underline{\hat{M}} = -\frac{e}{2m_e}\underline{\hat{L}}$$

$$\hat{M}_z = -\frac{e}{2m_e}\hat{L}_z$$

$$\underline{\hat{M}} = -\frac{\mu_B}{\hbar}\underline{\hat{L}}$$

$U_{mag} = -\underline{B} \cdot \underline{m}$. In quantum mechanics, the energy of interaction between $\underline{\hat{M}}$ and an external magnetic field is:

$$\hat{H}_m = -\underline{\hat{M}} \cdot \underline{B}$$

If $\underline{B}$ is in the "Z" direction.

$$\hat{H}_m = \frac{eB}{2m_e}\hat{L}_z$$

The eigenfunctions of $\hat{L}_z$ are also eigenfunctions of $\hat{H}_m$.

So,

$$\hat{H}_m |\ell, m_\ell\rangle = \frac{eB}{2m_e}\hat{L}_z |\ell, m_\ell\rangle = \frac{eB}{2m_e} m_\ell \hbar |\ell, m_\ell\rangle$$

(subs in Bohr magnetron)

$$\hat{H}_m |\ell, m_\ell\rangle = M_B B m_\ell |\ell, m_\ell\rangle$$

where $M_B = \frac{e\hbar}{2m_e}$ (i.e. Bohr magnetron)

If the Hamilton of the system does not depend on m_ℓ, then add in $\hat{H}_m$ produces a splitting of the energy into $(2\ell + 1)$ different levels. (Zeeman effect)

$$\hat{H}_m = -\underline{M} \cdot \underline{B}$$

for B in the Z-direction.

7.2 The Stern-Gerlach Experiment

(magnetic energy) $\hat{H}_m = -\underline{M} \cdot \underline{B}$

$\underline{F} = -\nabla \hat{H}_m = \nabla(\underline{M} \cdot \underline{B})$

In the case where B is in the "Z" direction, and uniform in the "x" and "y".

Experimental setup: source of neutral atoms → beam of neutral atoms → through magnet. The beam will diverge depending on the B field.

B varies mainly in the Z-direction.

$$\hat{F}_z = \hat{M}_z \frac{\partial B}{\partial Z}$$

and therefore,

$$\hat{F}_z |\ell, m_\ell\rangle = \frac{\partial B}{\partial Z} \hat{M}_z |\ell, m_\ell\rangle$$

$$\hat{F}_z |\ell, m_\ell\rangle \propto m_\ell |\ell, m_\ell\rangle$$

→ the deflection of the atoms is different depending on the value of m_ℓ.

So we expect to see $(2\ell + 1)$ fringes because ℓ is an integer we expect to see an odd number of fringes.

With neutral silver atoms, only two fringes are found.

$$\hat{F}_z \propto \hat{M}_z \frac{\partial B}{\partial z}$$

→ it was proposed that the fringes are due to the interaction of the magnetic field and a magnetic moment due not to the orbital angular momentum, but to an intrinsic angular momentum or spin.

The EVEN number of fringes implies that the spin is half-integer (for electrons).

7.3 Spin

We propose that the spin is an angular momentum, i.e.

$$\underline{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$$

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$

$$[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_y$$

$$[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

and of course,

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

$$[\hat{S}, \hat{S}_x] = [\hat{S}, \hat{S}_y] = [\hat{S}, \hat{S}_z] = 0$$

Common orthonormal eigenfunctions of $\hat{S}^2$ and $\hat{S}_z$:

$$|s, m_s\rangle$$

$$\hat{S}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$

$$\hat{S}_Z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

For an electron, $S = \frac{1}{2}$, $m = -\frac{1}{2}, \frac{1}{2}$

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = |+\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = |-\rangle$$

$$\hat{S}^2|+\rangle = \frac{3}{4}\hbar^2|+\rangle$$

$$\hat{S}^2|-\rangle = \frac{3}{4}\hbar^2|-\rangle$$

$$\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$$

$$\hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle$$

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y$$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y$$

$$\hat{S}_+|-\rangle = \hbar|+\rangle$$

$$\hat{S}_-|+\rangle = \hbar|-\rangle$$

Associated to the spin, there is a magnetic moment

$$\underline{\hat{M}}_s = -\frac{e}{m_e} = -\frac{2\mu_B}{\hbar}\underline{S}$$

The factor 2 is due to the relativistic effects.

The state space corresponding to spin is independent of the 3D state space. The three components of spin commute with all the components of $\underline{\hat{R}}$ (position), $\underline{\hat{P}}$ (momentum) and $\underline{\hat{L}}$ (angular momentum).

Example:

Which of the following are sets of compatible observables?

$\hat{S}^2, \hat{L}^2, \hat{S}_z\hat{L}_z$ *yes*

$\hat{S}^2, \hat{L}_y, \hat{S}_z, \hat{L}_z$ *no* $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$

$\hat{X}, \hat{P}_y, \hat{S}^2, \hat{S}_x$ *yes*

$\hat{Y}, \hat{P}_y, \hat{S}^2, \hat{S}_x$ *no* $\left[\hat{Y}, \hat{P}_y\right] = i\hbar$

$\hat{Y}, \hat{L}_x, \hat{S}^2, \hat{S}_z$ *no* $\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y$

$\hat{S}^2, \hat{L}^2, \hat{S}_z\hat{L}_z$ are a set of compatible observables and we can find a set of common eigenfunctions: $|s, m_s, \ell, m_\ell\rangle$.