Complex Numbers

Denoted by either i or j

$$i^2=-1$$

$$\sqrt{i}=\frac{1+i}{\sqrt{2}}$$

$$z=x+iy=\underbrace{r\left(\cos\theta+i\sin\theta\right)=re^{i\theta}}_{Euler's\ Formula}$$

$$x=\operatorname{Re}\{z\},\ y=\operatorname{Im}\{z\}$$

$$r=|z|^2=zz*$$

$$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$$

$$z_1-z_2=(x_1-x_2)+i(y_1-y_2)$$

$$z_1z_2=(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)=r_1r_2e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2}=\frac{z_1z_2*}{|z_2|^2}=\frac{r_1e^{i\phi_1}}{r_2e^{i\phi_2}}=\frac{r_1}{r_2}e^{i(\phi_1-\phi_2)}$$

De Moivre's Theorem:

$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n$$

Coordinate Systems

Cartesian Coordinates

$$\underline{\nabla}=\hat{\underline{i}}\frac{\partial}{\partial x}+\hat{\underline{j}}\frac{\partial}{\partial y}+\hat{\underline{k}}\frac{\partial}{\partial z}$$

Laplacian: $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

Gradient: $\underline{\nabla}T=\hat{\underline{i}}\frac{\partial T}{\partial x}+\hat{\underline{j}}\frac{\partial T}{\partial y}+\hat{\underline{k}}\frac{\partial T}{\partial z}$

$$\hat{\underline{n}}=\frac{\underline{\nabla}T}{|\underline{\nabla}T|}$$

$$\underline{\nabla}T\cdot\hat{\underline{u}}=\left.\frac{dT}{ds}\right|_{\hat{\underline{u}}}$$

Divergence:

$$\underline{\nabla}\cdot\underline{T}=\left(\hat{\underline{i}}\frac{\partial}{\partial x}+\hat{\underline{j}}\frac{\partial}{\partial y}+\hat{\underline{k}}\frac{\partial}{\partial z}\right)\cdot\left(T_x\hat{\underline{i}}+T_y\hat{\underline{j}}+T_z\hat{\underline{k}}\right)$$

Curl:

$$\underline{\nabla}\times\underline{T}=\begin{vmatrix}\hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ T_x & T_y & T_z\end{vmatrix}$$

$$=\hat{\underline{i}}\left(\frac{\partial T_z}{\partial y}-\frac{\partial T_y}{\partial z}\right)+\hat{\underline{j}}\left(\frac{\partial T_x}{\partial z}+\frac{\partial T_z}{\partial x}\right)$$

$$+\hat{\underline{k}}\left(\frac{\partial T_y}{\partial x}+\frac{\partial T_x}{\partial y}\right)$$

Cylindrical Polars

$$(x,y,z)=(r\cos\theta,r\sin\theta,z)$$

$$\hat{\underline{r}}\cdot\hat{\underline{i}}=\cos\theta$$

$$\hat{\underline{r}}\cdot\hat{\underline{j}}=\sin\theta$$

$$r^2=x^2+y^2$$

$$\underline{\nabla}f=\frac{\partial f}{\partial r}\hat{\underline{r}}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\underline{\theta}}+\frac{\partial f}{\partial z}\hat{\underline{z}}$$

$$V=\int_0^r dr\int_0^{2\pi}d\theta\int_0^h dz\,r$$

$$\underline{\nabla}\cdot\underline{V}=\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}$$

$$\underline{\nabla}\times\underline{V}=\frac{1}{r}\begin{vmatrix}\hat{\underline{r}} & \hat{\underline{\theta}} & \hat{\underline{z}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ V_r & rV_\theta & V_z\end{vmatrix}$$

$$\nabla^2f=\underline{\nabla}\cdot(\underline{\nabla}f)$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}$$

Spherical Polars

$$(x,y,z)=(r\sin\theta\cos\phi,r\sin\theta\sin\phi,r\cos\theta)$$

$$r^2=x^2+y^2+z^2$$

$$\hat{\underline{r}}\cdot\hat{\underline{i}}=\sin\theta\cos\phi$$

$$\hat{\underline{r}}\cdot\hat{\underline{j}}=\sin\theta\sin\phi$$

$$\hat{\underline{r}}\cdot\hat{\underline{k}}=\cos\theta$$

$$\hat{\underline{r}}=\frac{\underline{r}}{|\underline{r}|}$$

$$V=\int_0^r dr\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\,r^2\sin\theta$$

$$\underline{\nabla}f=\frac{\partial f}{\partial r}\hat{\underline{r}}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\underline{\theta}}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\underline{\phi}}$$

$$\underline{\nabla}\cdot\underline{V}=\frac{1}{r^2}\frac{\partial(r^2V_r)}{\partial r}+\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta V_\theta)}{d\theta}+\frac{\partial V_\phi}{\partial\phi}\right)$$

$$\underline{\nabla}\times\underline{V}=\frac{1}{r\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ V_r & rV_\theta & r\sin\theta V_\phi\end{vmatrix}$$

$$\nabla^2 f = \underline{\nabla}\cdot(\underline{\nabla} f)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Conservative Fields

$$\oint_C \underline{F}\cdot \underline{dl} = 0$$
$$\underline{\nabla}\times\underline{F} = 0$$
$$F = -\underline{\nabla}V$$
$$\underline{\nabla}\times(\underline{\nabla}V) = 0$$

Continuity Equation

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{V}) = 0$$

Incompressible fluid: $\frac{\partial\rho}{\partial t} = 0$

Data Analysis

Propagation of errors:

$$\sigma_A{}^2 = \left(\frac{\partial f}{\partial x}\right)^2\sigma_x{}^2 + \left(\frac{\partial f}{\partial y}\right)\sigma_y{}^2$$

Combination in quadrature

If $A = x + y - z$, $\sigma_A{}^2 = \sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2$

If $A = \frac{xy^n}{z}$,

$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + n^2\left(\frac{\sigma_y}{y}\right)^2 + \left(\frac{\sigma_z}{z}\right)^2$$

Least Squares Fit

Chisquared $\chi^2 = \sum_{i=1}^{N}\left(\frac{\delta_i}{\sigma_i}\right)^2\Bigg|_{\min}$

where δ_i is the distance from the data point to the corresponding fit point.

Degrees of freedom: $N_{dof} = N - p$

where N is the number of data points and p is the number of fitted parameters.

For a good fit, $\chi^2 \approx N_{dof}$

Repeated measurements

Mean: $\langle x\rangle = \frac{1}{N}\sum_{i=1}^{N} x_i$

Error of mean: $\sigma_{\langle x\rangle} = \frac{1}{\sqrt{N}}\sigma$

Best estimate of σ from the spread of points:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \langle x\rangle\right)^2}$$

Weighted mean: $X_m = \frac{w_a X_a + w_b X_b}{w_a + w_b}$

where $w_i = \frac{1}{\sigma_i{}^2}$, and $w_m = w_a + w_b$.

Delta functions

Dirac delta:

$$\delta(x - x_0) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$
$$\int_{-\infty}^{\infty}\delta(x - x_0)\,dx = 1$$

Kroenecker Delta:

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

Differentiation and Integration

Differentiation

$$\frac{df(x)}{dx} = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$
$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$$

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^{kx}	ke^{kx}
a^x	$a^x \log a$
$\sin x$	$\cos x$
$\sin^2 x$	$2\sin x\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2 + x^2}$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sec x$	$\tan x\sec x = \frac{\sin x}{\cos^2 x}$
$\mathrm{cosec}\, x$	$-\cot x\,\mathrm{cosec}\, x$ $= -\frac{\cos x}{\sin^2 x}$

$\cot^{-1}\left(\frac{x}{a}\right)$	$-\frac{a}{a^2+x^2}$
$\sec^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)$	$-\frac{a}{x\sqrt{x^2-a^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2+x^2}}$
$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$
$\tanh^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2-x^2}$

Chain rule: $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$

Product Rule: $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Quotient Rule: $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} + u\frac{dv}{dx}}{v^2}$

Stationary point: $\frac{dx}{dy} = 0$

$$\Delta = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

$\Delta > 0$, $\frac{\partial^2 f}{\partial x^2} > 0$, minimum

$\Delta > 0$, $\frac{\partial^2 f}{\partial x^2} < 0$, maximum

$\Delta < 0$ saddlepoint

$\Delta = 0$ tests with higher derivatives needed

Point of inflexion: $\frac{d^2x}{dy^2} = 0$, $\frac{d^2x}{dy^2}$ changes sign on either side

Integration

$$\int \frac{df}{dx}dx = f(x) + c$$

$$\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\delta x = \int_{aa}^{b} f(x)dx = [f(x)]^b$$

Divergence Theorem: $\int_S \underline{V} \cdot \underline{dA} = \int_V (\underline{\nabla} \cdot \underline{V}) dV$

Stoke's Theorem: $\oint_C \underline{V} \cdot \underline{dl} = \int_S (\underline{\nabla} \times \underline{V}) \cdot \underline{dA}$

Integration by Parts:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx + c$$

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv(x)}{dx} - f(u(x))\frac{du(x)}{dx}$$

$$M = \int_0^a dx\left(\int_0^b dy\,\rho(x,y)\right) = \int_0^b dy\left(\int_0^a dx\,\rho(x,y)\right)$$
$$= \int_0^a \int_0^b \rho(x,y)\,dy\,dx = \int_0^b \int_0^a \rho(x,y)\,dx\,dy$$
$$= \iint_A dx\,dy\,\rho(x,y) = \int_A dx\,dy\,\rho(x,y)$$

$$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x+a)(x-a)} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$ via $x = a\sin u$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ via $x = a\tan u$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sin^{-1} h\left(\frac{x}{a}\right) + c$ via $x = a\sinh u$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cos^{-1} h\left(\frac{x}{a}\right) + c$ via $x = a\cosh u$

To solve $\int \sin^n ax \cos^n axdx$;

If $n+m$ is odd,

$\int \sin^n ax \cos^n axdx = \int \sin^{n-1} ax \cos^m ax \sin ax$

and substitute $u = \cos ax$ (n odd)

$\int \sin^n ax \cos^n axdx = \int \sin^n ax \cos^{n-1} ax \cos axdx$

and substitute $u = \sin ax$ (m odd)

If $n+m$ is even; use either.

Differential Equations

First Order Ordinary Differential Equations

Separable: $\frac{dy}{dx} = f(x,y) = g(x)h(y)$

Take x,y to opposite sides, integrate to get solution

Linear: $\frac{dy}{dx} + P(x)y = Q(x)$

Use integrating factor

$I(x) = \exp\left(\int P(x)dx\right)$

Multiply by $I(x)$, then choose suitable $I(x)$, e.g. $\frac{d(I(x)y)}{dx} = I\frac{dy}{dx} + IP(x)y$

→ $\frac{d(I(x)y)}{dx} = I(x)Q(x)$

Then as for separable case

Bernoulli's Equations

$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (n \neq 0,1)$

Divide through by y^n, substitute $u = y^{1-n}$

$\left(\frac{du}{dx} = (1-n)y^{-n}\frac{dn}{dx}\right)$

→ $\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$

Then use integrating factor $I(x) = \exp\left(\int(-n)P(x)dx\right)$, and proceed as above

Second Order ODE with constant coefficients

$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x)$

Set $f(x) = 0$ and solve → complementary function $y_c(x)$. The find solution for $f(x) \neq 0$ → particular integral $y_p(x)$.

Then $y_T(x) = y_c(x) + y_p(x)$

Trial solution: $y = Ae^{\lambda x}$

Eigenvectors and Eigenvalues

$\underline{\underline{A}}\underline{X} = \lambda\underline{X}$

$\underline{\underline{A}} \equiv \hat{A}$ is an operator, λ is an eigenvalue and $\underline{X}$ is an eigenvector.

$\det\left(\underline{\underline{A}} - \lambda\underline{\underline{\Im}}\right) = 0$

$\underline{\underline{\Im}}$ is the unitary matrix (see later)

$\left(\underline{\underline{A}} - \lambda_n\underline{\underline{\Im}}\right)\underline{X}_n = 0$

eigenvector $\underline{X}_n = \begin{pmatrix} a \\ b \end{pmatrix}$, a,b anything.

Unit eigenvectors $\hat{\underline{X}}_n = \frac{\underline{X}_n}{\det \underline{X}_n}$

Expansions

Binomial expansion:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots + b^n$$

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \ldots + x^n = \sum_{i=0}^{n}\frac{n!}{i!(n-1)!}$$

Taylor expansion:

About $x = 0$ (Maclaurin expansion);

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0)$$

About $x = x_0$:

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots$$

About $x = x_0, y = y_0$;

$$f(x,y) = f(x_0,y_0) + h\frac{\partial f(x_0,y_0)}{\partial x} + k\frac{\partial f(x_0,y_0)}{\partial y}$$

$$+\frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2} + \ldots$$

Exponentials & Logarithms

$\ln \equiv \log_e$

$x^a x^b = x^{a+b}$

$(x^a)^b = x^{ab}$

$\ln(xy) = \ln x + \ln y$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$\ln(x^a) = a\ln x$

$e^{\ln x} = x$

$\ln 1 = 0$

$\ln e = 1$

Fourier Transforms

Standard Notation

$$f(x) = a_o + \sum_{n=1}^{\infty}(a_n\cos kx + b_n\sin kx)$$

$$a_o = \frac{1}{2L}\int_{-L}^{+L}f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{+L}f(x)\cos kx\,dx, \; n = 1,2,3,\ldots$$

$a_n = 0$ for an odd function, i.e. not symmetric on the y-axis.

$$b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin kx\,dx\ ,\ n = 1,2,3,...$$

$b_n = 0$ for an even function, i.e. symmetric on the y-axis.

Complex Notation

Kronecker Delta: $\delta_{nm} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$

$$\int_a^b U_n * U_n\,dx = I_m \delta_{nm}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L} = \sum_n \underbrace{C_n U_n(x)}_{Basis\ functions}$$

$$C_n = \frac{1}{2\pi}\int_{-L}^{L} e^{-in\pi x/L} f(x)\,dx = \frac{1}{I_n}\int_a^b U_n * (x) f(x)\,dx$$

Parseval, normalization:

$$\int_{-L}^{L} |f(x)|^2\,dx = \int_{-L}^{L} f(x) * f(x)\,dx = 2L\sum_{n=-\infty}^{+\infty} |C_n|^2$$

Dirichlet's Conditions

A set of $\cos kx$ and $\sin kx$ is complete – any sensible function can be built from them.

$f(x)$ must be single-valued, and have a finite number of finite discontinuities.

$\int_{-L}^{+L} |f(x)|\,dx$ must be finite.

Integral Transforms

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx}\,dk$$

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx$$

Gives the amount of each wave needed in the Fourier series.

For any function $f(x)$ and its fourier transform $g(k)$, $\Delta x \Delta k \geq 1/2$

Legendre Series, $(-1 \leq x \leq 1)$

Any 'sensible' function on $-1 \leq x \leq 1$ can be expanded using Legendre polynomials

$$f(x) = \sum_{L=0}^{\infty} c_L P_L(x)$$

$$c_L = \frac{2L+1}{2}\int_{-1}^{1} f(x) P_L(x)\,dx$$

Orthogonality

$$\int_{-L}^{+L} U * (x) V(x)\,dx = 0$$

$$\int_a^b U(x) V(x)\,dx = 0$$

$a \leq x \leq b\ ,\ a, b \in \mathbb{R}$

$$\int_{-L}^{+L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} = 0\ ,\ m \neq n$$

$$\int_{-L}^{+L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = 0\ ,\ \forall m, n$$

Solutions

$$f(x) = A\cos kx + B\cos kx \quad (-k^2)$$

$$f(x) = A_n e^{n\pi x/L} + B_n e^{n\pi x/L} \quad (k^2)$$

Convolution theory

$$f(x) = \int_{-\infty}^{\infty} f_1(x - x') f_2(x')\,dx$$

$$G(k) = 2\pi g_1(k) g_2(k)$$

Fourier transform of convolution of 2 functions is the product of their individual transforms $(\times 2\pi)$

Newton-Raphson Method

(Numerically solving equations)

- Make initial guess at solution $\rightarrow x_0$
- Find the gradient at $x_0 \rightarrow f'(x_0)$
- Find where it cuts the x-axis

  $$\rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
- Repeat as needed

Polynomials

All just combinations of $\cos x$, $\sin x$ and x^{-1}.

Can solve using power series $y = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Legendre Polynomials $P_L(x)$

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + L(L+1)y = 0$$

$$\int_{-1}^{1} P_L(x) P_m(x)\,dx = 0$$

$$P_0 = 1\ ,\ P_1 = x\ ,\ P_2 = \frac{3x^2 - 1}{2}\ ,\ \ldots$$

Finite at $x = \pm 1$

Eigenvalues: $L(L+1)$, $L = 0,1,2,...$

(waves with axial symmetry, i.e. no ϕ dependence)

Associated Legendre Polynomials

$$\left(1-x^2\right)y''-2xy'+\left(L\left(L+1\right)-\frac{m^2}{1-x^2}\right)y=0$$

$$P_\ell^m\left(x\right)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{d_x^{\ m}}P_\ell\left(x\right)$$

$$P_\ell^m\left(\cos\theta\right)=\frac{\left(-1\right)^m}{2^\ell\ell!}\left(r\cos^2\theta\right)^{\frac{m}{2}}\frac{d^{\ell+m}\left(\cos^2\theta-1\right)}{\left(d\cos\theta\right)^{\ell+m}}$$

Finite at $x=\pm1$

Eigenvalues: $L\geq|m|$, $m=...,-1,0,1,...$

(waves with ϕ dependence)

Hermite Polynomials $H_n\left(x\right)$

$y''-2xy'+2ny=0$, $n=0,1,2,...$

$\sim x^n$ as $x\to\infty$

(quantum harmonic oscillator)

Laguerre Polynomials $L_n\left(x\right)$

$xy''+\left(1-x\right)y'+ny=0$, $n=0,1,2,...$

$\sim x^n$ as $x\to\infty$, finite at $x=0$

(hydrogen atom radial wave functions)

Bessel Functions $J_m\left(x\right)$

$x^2y''+xy'+\left(x^2-m^2\right)y=0$

Finite at $x=0$

(free waves in plane or cylindrical polars)

Spherical Bessel Functions $J_L\left(x\right)$

$x^2y''+2xy'+\left(x^2-L\left(L+1\right)\right)y=0$

Finite at $x=0$

(Free waves in spherical polar cords)

Quadratics

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Series

Arithmetic

$$S_n=a+\left(a+d\right)+...+\left(a+\left(N-1\right)d\right)$$

$$=\sum_{n=1}^{N-1}\left(a+nd\right)=\frac{N}{2}\left[a+\left(a+d\left(N-1\right)\right)\right]$$

$$\sum_{n=1}^{N-1}\left(n\right)=\frac{1}{2}\left(N\left(N-1\right)\right)$$

Geometric

$$\sum_{n=0}^{N-1}ax^n=a\left(\frac{1-x^n}{1-x}\right)$$

$$\sum_{n=0}^{\infty}ax^n=\lim_{N\to\infty}\left(a\left(\frac{1-x^N}{1-x}\right)\right)$$

Trigonometry

$$\cos\phi=\frac{x}{r}=\frac{1}{2}\left(e^{i\phi}+e^{-i\phi}\right)=\frac{1}{\sec\phi}$$

$\cos\left(-x\right)=-\cos\left(-x\right)$ (even)

$\cos\left(n\pi\right)=\left(-1\right)^n$

$\cos x\approx x-\frac{x^2}{2!}+\frac{x^4}{4!}-...$ (Maclaurin)

$$\sin\phi=\frac{y}{r}=\frac{1}{2i}\left(e^{i\phi}-e^{-i\phi}\right)=\frac{1}{\operatorname{cosec}\phi}$$

$\sin\left(x+2\pi\right)=\sin\left(x\right)$

$\sin\left(-x\right)=-\sin\left(x\right)$ (odd)

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-...$ (Maclaurin)

$\left\langle\sin^2kx\right\rangle=1/2$

$$\tan\phi=\frac{y}{x}=\frac{\sin\phi}{\cos\phi}=\frac{1}{\cot\phi}$$

$\sin\left(x\pm y\right)=\sin x\cos y\pm\cos x\sin y$

$\cos\left(x\pm y\right)=\cos x\cos y\mp\sin x\sin y$

$\sin^2\phi+\cos^2\phi=1$

$\cos\phi+\sin\phi=1+\tan\phi$

$\sin2\phi=2\sin\phi\cos\phi$

$\cos2\phi=\cos^2\phi-\sin^2\phi=2\cos^2\phi-1=1-2\sin^2\phi$

$$\operatorname{sinc}\left(\alpha\right)=\frac{\sin\left(\alpha\right)}{\alpha}$$

L'Hopital's Rule: $\frac{\sin0}{0}=1$ (from Maclaurin)

$$\sinh\left(\alpha\right)=\frac{e^\alpha-e^{-\alpha}}{2}=\sin\left(i\alpha\right)$$

$$\cosh\left(\alpha\right)=\frac{e^\alpha+e^{-\alpha}}{2}=\cos\left(i\alpha\right)$$

$$\tanh\left(\alpha\right)=\frac{\sinh\left(\alpha\right)}{\cosh\left(\alpha\right)}$$

Vector Identities

Scalar Product:

$$\underline{A}\cdot\underline{B}=\underline{B}\cdot\underline{A}=|\underline{A}||\underline{B}|\cos\theta$$

$$=A_xB_x+A_yB_y+A_zB_z$$

Vector Product:

$$\underline{A} \times \underline{B} = -\underline{B} \times \underline{A} = |\underline{A}||\underline{B}|\sin\theta\hat{\underline{n}}$$

$$= \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Scalar Triple Product:

$$(\underline{A} \times \underline{B}) \cdot \underline{C} = |\underline{A} \times \underline{B}||\underline{C}|\cos\varphi$$

$$= \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

where φ is the angle between $|\underline{A} \times \underline{B}|$ and $|\underline{C}|$

Vector Triple Product:

$$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$

$$\underline{\nabla} \cdot (\phi\underline{V}) = \phi\underline{\nabla} \cdot \underline{V} + \underline{V} \cdot \underline{\nabla}\phi$$

$$\underline{\nabla} \times (\phi\underline{V}) = \phi\underline{\nabla} \times \underline{V} - \underline{V} \times \underline{\nabla}\phi$$

$$\underline{\nabla} \cdot (\underline{\nabla}T) = \nabla^2 T$$

$$\underline{\nabla} \times (\underline{\nabla}T) = 0$$

$\underline{\nabla} \cdot (\underline{\nabla} \cdot \underline{V})$ cannot be simplified

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{V}) = 0$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{V}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{V}) - \nabla^2 V$$

$$\underline{\nabla} \cdot (\underline{V} \times \underline{W}) = (\underline{\nabla} \times \underline{V}) \cdot \underline{W} - (\underline{\nabla} \times \underline{W}) \cdot \underline{V}$$

$$\underline{\nabla} \times (\underline{V} \times \underline{W}) = (\underline{\nabla} \cdot \underline{W})\underline{V} + (\underline{W} \cdot \underline{\nabla})\underline{V} - (\underline{\nabla} \cdot \underline{V})\underline{W} - (\underline{V} \cdot \underline{\nabla})\underline{W}$$

Matrices

$\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{C}}$ → $c_{ij} = a_{ij} + b_{ij}$

$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}}$ → $c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$

Identity matrix $\underline{\underline{\Im}} = \Im_{ij} = \delta_{ij}$

Transpose $A_{ij}^{T} = \tilde{A}_{ij} = A_{ji}$

If $\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{B}}$, $\underline{\underline{C}}^{T} = (\underline{\underline{A}}\,\underline{\underline{B}})^{T} = \underline{\underline{B}}^{T}\underline{\underline{A}}^{T}$

Inverse: $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{\Im}} = \delta_{ij}$

Complex conjugate: $A^* = \bar{A}$

Hermetian conjugate: $\underline{\underline{A}}^{\dagger} = (\underline{\underline{A}}^*)^{T}$

Trace: $Tr(\underline{\underline{A}}) = \sum_{i=1}^{N} a_{ii}$

Rotation matrix:

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$