PC 2312 - Wave Optics - Notes

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Books: Hecht (Optics ~ £30) (Smith & King, Optics & Photonics)

1. Electromagnetism and Waves
2. Geometric Optics
3. Polarization of Light
4. Interference
5. Diffraction

1. Electromagnetism & Waves

1.1 Maxwell's Equations

Free space (no dielectrics or magnetic fields)

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_o} = 0$$

$$\nabla \cdot \underline{B} = 0$$

$$\nabla x \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla x \underline{B} = \mu_o \underline{j} + \mu_o \varepsilon_o \frac{\partial \underline{E}}{\partial t} = \mu_o \varepsilon_o \frac{\partial \underline{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t}$$

$$\nabla x \left(\nabla x \underline{E} \right) = -\frac{\partial \left(\nabla x \underline{B} \right)}{\partial t}$$

$$\nabla \left(\nabla \cdot \underline{E} \right) - \nabla^2 \underline{E} = -\frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \right)$$

$$\boxed{\nabla^2 \underline{E} = \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2}}$$

c.f. $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2}$

This is the wave equation in free space

Electromagnetic waves (oscillating E and B fields) can propagate in free space.

(You should be able to prove the same for the magnetic field)

$$\nabla^{22} B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}$$

1.2 General solution

$E(x) = f(x - vt) + g(x + vt)$

$v = c$

Physical representation $f(x - vt)$ is any function that retains its' shape and moves in the x direction with a speed v. $g(x + vt)$ moves in the negative direction. x = direction of propagation.

1.3 Particular sinusoidal solution

$E(x,t) = E_o \cos(kx - \omega t + \phi)$

$B(x,t) = B_o \cos(kx - \omega t + \phi)$

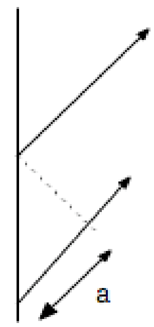
$\left| \underline{E} \right| = E_o e^{i(\underline{k} \cdot \underline{r} - \omega t)}$

NB: $k = \frac{2\pi}{\lambda}$, $\lambda = 2\pi f$.

Where k is the vector along the propagation direction.

$\frac{\omega}{k} = \lambda f = c$

This is useful because any function f, g can be made up as a sum of these.
$f(x - vt) = a_o + \sum a_i \cos(k_i x + \omega_i t) + \sum b_i \sin(k_i x + \omega_i t)$
The phase ϕ tells you how many radians the wave is to the "left" of a cos wave.
$\cos\left(kx - \omega t + \frac{3\pi}{2}\right) = \sin(kx - \omega t)$
Suppose adding 2 waves where wave 1 follows a path distance a longer than wave 2.



Path delay a.
Time delay;
$\frac{a}{c}$
Difference is phase? If $a = n\lambda$, then the phase delay $\phi = 2\pi$. For an arbitrary difference a;
$\phi = \frac{2\pi a}{\lambda}$
Difference Δx in path corresponds to $\Delta\phi = k\Delta x$.
Difference Δt corresponds to;
$\frac{2\pi c}{\lambda}\Delta t = \Delta\phi = \omega\Delta t$

1.4 Nature of Electromagnetic Waves
1.4.1 E and B Transverse
i.e. the oscillation of the E and B fields happens in a plane that is perpendicular to the plane of propagation.
Proof:
Recall $E = f(x - vt)$. Take x as the direction of propagation.
$\frac{\partial E_y}{\partial y} = 0$
$\frac{\partial E_z}{\partial z} = 0$
(E relies on x only)
But for an EM wave propagating in free space, $\nabla . E = 0$.
$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$
Therefore
$\frac{\partial E_x}{\partial x} = 0$.
If the wave were longitudinal;
$\frac{\partial E_z}{\partial z} \neq 0$.
Hence wave is transverse.

1.4.2 E and B perpendicular and in phase

$\nabla \times E = -\frac{\partial B}{\partial t}$

Suppose that;

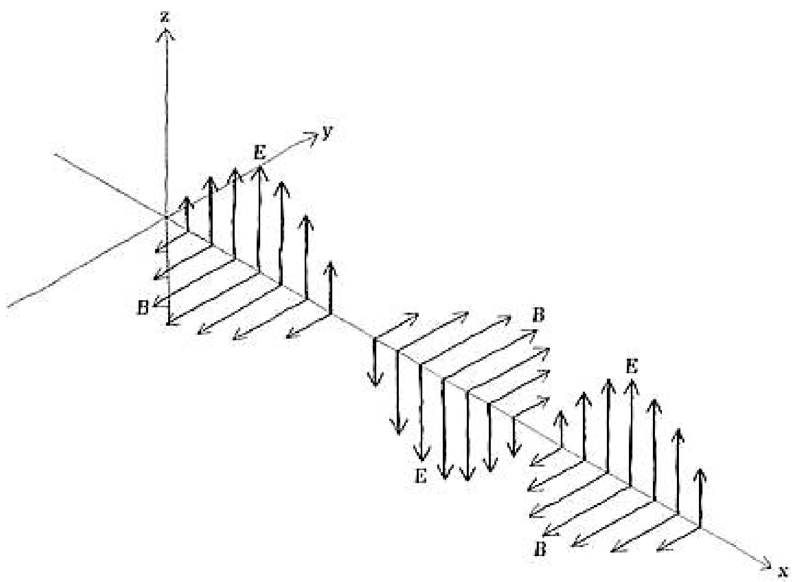
$E = \left(0, E_{oy}\cos(kx - \omega t), 0\right)$.

$$\nabla \times E = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \varepsilon & 0 \end{vmatrix} = \underline{\hat{k}}\frac{\partial \varepsilon}{\partial x} = -\underline{\hat{k}}k\cos(kx - \omega t)E_{oy}dt$$

(Don't confuse $\underline{\hat{k}}$ with $k = \frac{2\pi}{\lambda}$)

$$B = -\int(\nabla \times E)dt = \underline{\hat{k}}\int \sin(kx - \omega t)E_{oy}dt = \underline{\hat{k}}\frac{k}{\omega}\cos(kx - \omega t)E_{oy}dt$$

$$B = \frac{E_{oy}k}{\omega}\cos(kx - \omega t)$$



1.5 Complex notation and phasors

1.5.1 The rules

$e^{i\theta} = \cos\theta + i\sin\theta$

Hence instead of $A\cos(kx - \omega t + \phi)$, write $\mathrm{Re}\left[Ae^{i(kx - \omega t + \phi)}\right]$.

e.g.:

$$\begin{aligned} \cos(x+y) &= \mathrm{Re}\left[e^{i(x+y)}\right] \\ &= \mathrm{Re}\left[e^{ix}e^{iy}\right] \\ &= \mathrm{Re}\left[(\cos x + i\sin x)(\cos y + i\sin y)\right] \\ &= \mathrm{Re}\left[\cos x\cos y - \sin x\sin y + i\sin x\cos y + i\cos x\sin y\right] \\ &= \cos x\cos y - \sin x\sin y \end{aligned}$$

Then, do the math - it's easier.

If you then want to know the amplitude of the wave, look at the resulting formula (Ae^{iz})

If you want to know what the wave really looks like, just take the real part.

$\cos(kx - \omega t) \rightarrow$ (phase shifted) $\cos(kx - \omega t + \theta)$.

For the complex case;
$Ae^{i(kx-\omega t)} \rightarrow Ae^{i(kx-\omega t)}e^{i\varphi}$.
i.e. just multiply by $e^{i\varphi}$.

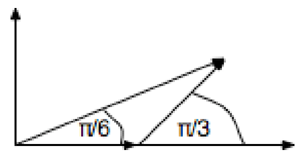
1.5.2 Phasors

$A\cos(kx-\omega t)+A\cos\left(kx-\omega t+\frac{\pi}{3}\right)$

Complex representation:

$Ae^{i(kx-\omega t)}+Ae^{i(kx-\omega t)}e^{i\frac{\pi}{3}}=e^{i(kx-\omega t)}A\left(1+e^{i\frac{\pi}{3}}\right)$

Notice that we can add these in an Argand diagram.



Resulting amplitude: $\sqrt{3}A$ at phase angle $\frac{\pi}{6}$.
→ resulting wave is;

$\mathrm{Re}\left(\sqrt{3}Ae^{i\frac{\pi}{6}}e^{i(kx-\omega t)}\right)=\sqrt{3}A\cos\left(kx-\omega t+\frac{\pi}{6}\right)$

this is the concept of phasors: represent any wave
$A\cos(kx-\omega t+\varphi)$
as a "phasor" with amplitude A, angle φ to the real axis. Then just add for the resultant wave.
What about $\sin(kx-\omega t)$? This is just the same as;
$\cos\left(kx-\omega t+\frac{\pi}{2}\right)$.
i.e. multiplied by $e^{-i\frac{\pi}{2}}$.
All provided that waves have the <u>same frequency.</u>
What about the $kx-\omega t$ bit? This can be regarded as a rotating phasor - because the total $kx-\omega t+\varphi$ changes with time.
Don't need to worry about this when adding the waves - basically, all of the phasors are effectively rotating together.

1.6 Propagation of Light
1) Huygen's Principle
All points on a wave can be considered as point sources for the production of secondary spherical wavelets. After time t the new position of the wave will be a surface tangent to the secondary wavelets.
(See handout for picture)
This is known as the Huygen-Fresnel principle.

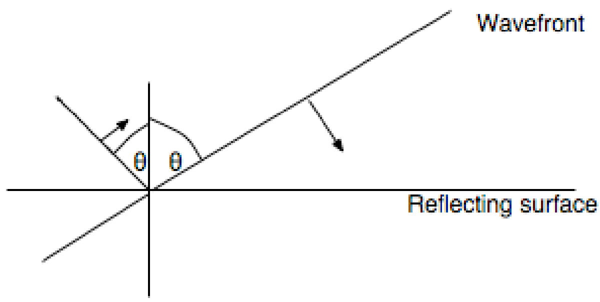
Fresnel & Kirchoff considered the problem of why the wave doesn't go backwards. (Hecht 10.4 and appendix 2). "Obliquity factor" of
$\frac{1}{2}(1+\cos\theta)$
which is 1 in the forward direction, and 0 in the backward direction.
NB: a plane wave can be considered as a spherical wave at an infinite distance. Limiting case, but is often seen.

Rays: pain to keep drawing secondary wavelets. Rays are an artificial construct that describe the propagation of the wave front. They are perpendicular to the wave front. (usually)

2) Fermat's principle.
A light ray going from point A to point B will traverse a path that is stationary with respect to the variations of that path.
aka light will always "choose" the shortest way to get from point a to b.
(time taken is a minimum, maximum or a saddle point)
Subsequent sections will use both for reflection, refraction, focusing, …

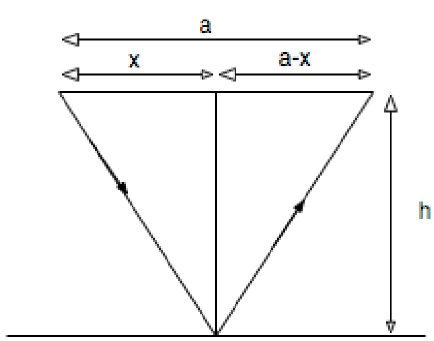
Sidenote:
Geometric optics refers to the case where all of your equipment is much bigger than the wavelength of the light you are considering. Hence you can ignore most of the aspects of the wave nature of life. e.g. a tendency for the light waves to spread out at the edges. This tendency leads to interference and diffraction, which we ignore in section 2.
In geometric optics, use rays (direction of travel of the wavefront. Perpendicular to the wavefront.)
Wave optics is sections 4 and 5. It is basically how waves and wavefronts behave when interacting with systems which are not hugely bigger than the wavelength of light. (Wave aspects of light)
Photon optics is how light behaves when interacting with matter, and low light levels. (Photon aspects of light) This is covered in later courses.
Either way of regarding the light is not 100% correct - it depends on what your experiment is about.

2. Geometric Optics
2.1 Law of Reflection
By wavefronts:



A wavefront comes into contact with a reflecting surface. Each point on the wavefront acts as a point source, carrying on the wave. When in contact with the surface, the secondary wavelet will be in the opposite direction - reflected by the material.
The reflected wavefront will be going out at exactly the same angle from that it came in at.
Angle of incidence = angle of reflection.

By Fermat:
Reflecting surface again. A light ray is going from point A to B, using the reflecting surface.



The travel time of the ray is equal to the path length divided by the speed of light.

$$t = \frac{1}{c}\left[\left(h^2 + x^2\right)^{\frac{1}{2}} + \left(h^2 + (a - x)^2\right)^{\frac{1}{2}}\right]$$

We need $\frac{dt}{dx} = 0$ to find the value of x that gives a stationary travel time.

$$c\frac{dt}{dx} = \frac{1}{2}\left(h^2 + x^2\right)^{-1/2} 2x + \frac{1}{2}\left(h^2 + \left(a - x\right)^2\right)^{-1/2} 2\left(a - x\right)\left(-1\right)$$

Rearranging gives

$$\frac{x}{\left(h^2 + x^2\right)^{\frac{1}{2}}} = \frac{a - x}{\left(h^2 + (a - x)^2\right)^{\frac{1}{2}}}$$

By inspection, $x = a - x$ so $x = \frac{1}{2}a$.

$\tan\theta_i = \tan\theta_r$

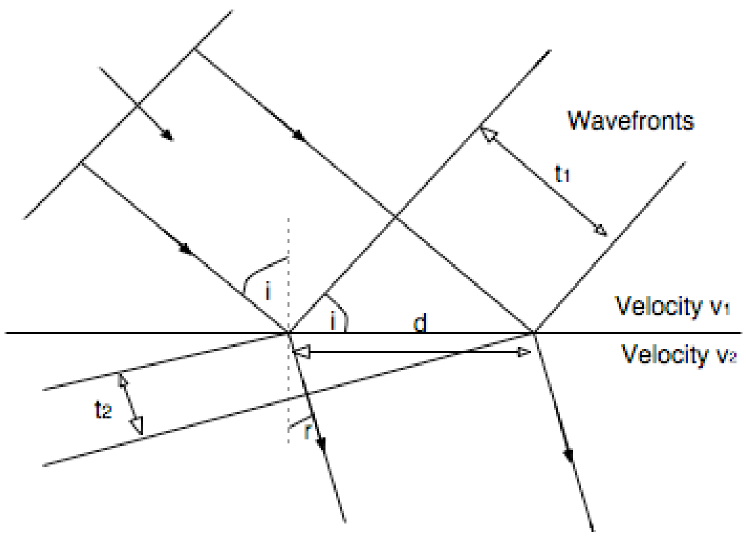
$\theta_i = \theta_r$

2.2 Refraction: Snell's Law

Consider propagation into different media. The light will travel at a different velocity. (In a medium, you have extra separable charges that can "wobble around". The EM light waves starts these wobbling, which then generate their own EM waves. Those will then interact with the original EM wave. Thus you have to add in lots of generated waves - that will give you a phase retardation in the original wave. The wavefronts of the resultant wave tend to travel at a different speed.)

2.2.1 By Huygens

Wavefront picture:



Refractive index;

$n_1 = \frac{c}{v_1}$

$n_2 = \frac{c}{v_2}$

Normally, n is a function of the wavelength.

$$t_1 = \frac{d\sin i}{\frac{c}{n_1}}$$

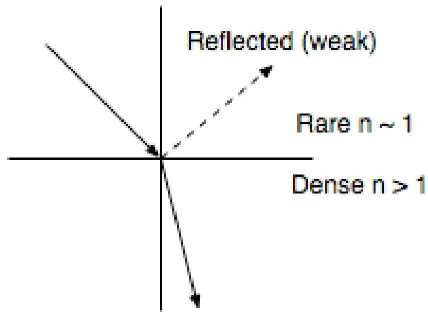
$$t_2 = \frac{d\sin r}{\frac{c}{n_2}}$$

For sanity, $t_1 = t_2$. (the wavefront must be continuous)

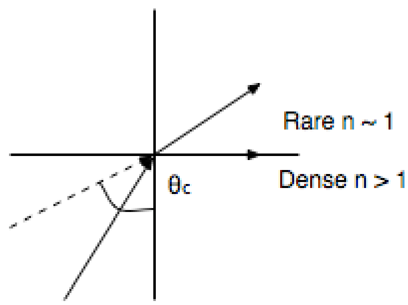
$\boxed{n_1 \sin i = n_2 \sin r}$. This is known as Snell's law.

2.2.2 By Fermat's Principle
See example sheet.

2.2.3 Total Internal Reflection



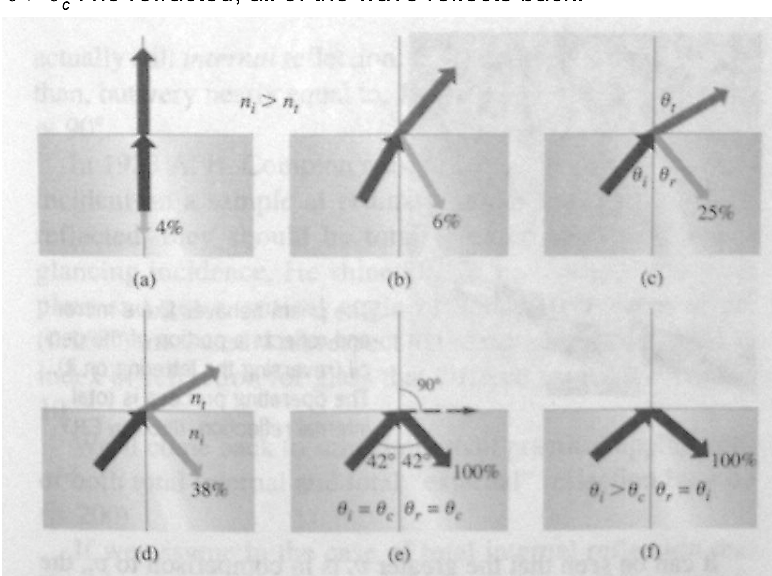
Ray refracts to the normal. (Rare → dense)



Ray refracts away from the normal. (Dense → rare)
At the point where the wave does not refract into the rare medium any more, the light approaches at a critical angle θ_c.
$n_d \sin \theta_c = n_r$

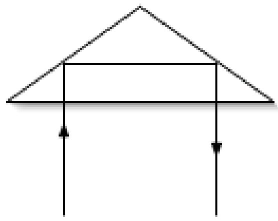
$\theta_c = \sin^{-1}\left(\frac{n_r}{n_d}\right)$

If the wave enters at greater than θ_c, there is no refracted ray. The wave reflects. This is known as total internal reflection.
$\theta > \theta_c$: no refracted, all of the wave reflects back.

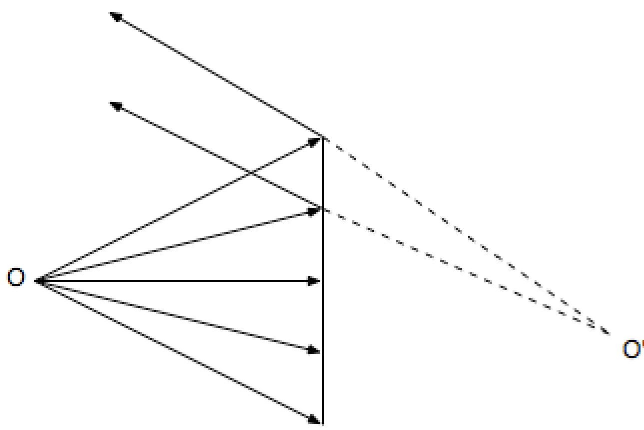


(Hecht p123)

Applications:
- Optical fibres (a core, with cladding around it. Light at a sufficiently narrow angle, the ray will go down the line using multiple TIR)
- Prisms



For a vacuum, $n = 1$.
In most glasses, $n = 1.5$.
Water is about 1.33.
In a diamond, $n = 2.4$.
2.3 Focusing, mirrors and Lenses
2.3.1 Images formed by mirrors
Flat mirror;



Observer sees rays as though coming from O'.
→ formation on an image.
In this case, it is known as a virtual image.
- You cannot form this image on a screen
- Opposite side of device from the outgoing rays.

When you look into a mirror, you see yourself back-to-front. But why not upsidedown? (Look in Young for why)

Curved mirror:
Think about a spherical mirror, with radius of curvature R.
See handout for derivation.

$$\frac{1}{U} + \frac{1}{V} = \frac{2}{R}$$

Comment: if $u = 3$ the rays converge at;

$\frac{1}{V} = \frac{2}{R}$ i.e. $V = \frac{R}{2}$.

Therefore, rays close to the axis coming in parallel to the axis converge to a focal point $f = \frac{R}{2}$.

Note that this is exactly true for parabolic mirrors (via the definition), but approximately true for a spherical mirror for rays close to the axis.

With the image formation, provided the paraxial consideration holds true, all the rays will converge at P', before coming out at separate directions. So for a viewer looking at an angle at the mirror, there is an image formed at P'. This is a real image, i.e. it is on the same side of the optical device as the outgoing rays.

Sign conventions:
We can use the derivation for convex/concave mirrors, as well as for images either side of the mirror - assuming the following sign conventions are followed;
u is positive if the object is on the same side as the incoming rays. (This is almost always true).
v is positive if the image is on the same side as the outgoing rays.
R is positive if the centre of curvature is on the same side as the outgoing rays.

Suppose that instead of a point image, we have a non-point image.
We draw rays from each point on the image. Where the rays converge is where the final image will be formed.
Once more, see the handout.
How big does the image appear?
Similar triangles:

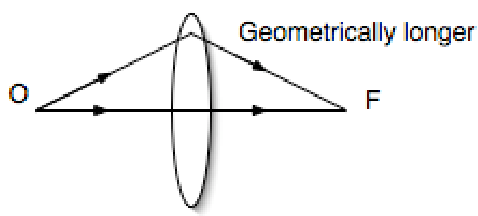
$$\frac{h_V}{h_U} = -\frac{v}{u}$$

The minus sign is due to the image being inverted.

2.3 Focusing and Lenses
Can do this by rays or by Fermat's principle.
Bring an image at 0 to a focus at F. Must arrange for the optical path,

$$\int n \cdot ds$$

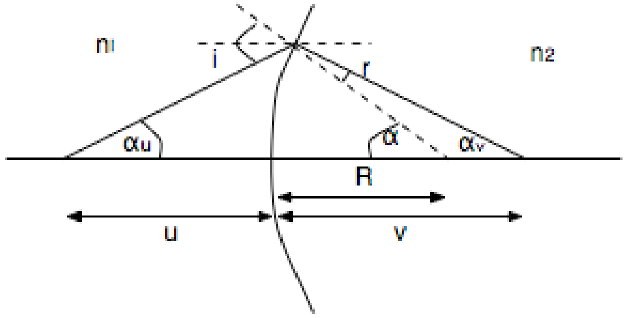
to be equal for all rays.



For the rays going straight through a lens, they spend more time in the lens traveling at a slower speed than those going further outside the lens, but a shorter time within it.
For an ideal focus at F, all the rays should travel for the same amount of time. Thus $\int n \cdot ds$ gives the refractive index over the whole path.
Thus you need to shape your lens to ensure this.
The accuracy needed is that the paths need to be equal to within at least 1 wavelength in light. thus the accuracy typically needed when grinding mirrors etc. is pretty small. (Think: HST was half a wavelength off. This was a big problem.)
This usually only works for a small field of view, or for paraxial rays, or both.

Finally; a convex mirror, Again, see handout.

2.3.3 Lenses
- Refraction at a spherical surface;



$\alpha = r + \alpha_v$
$i = \alpha_u + \alpha$
Assume all the angles are small.

$$\frac{h}{R}(=\tan\alpha) = \sin r + \frac{h}{v}(=\tan\alpha_v)$$

$$\sin i = \frac{h}{u} + \frac{h}{R}$$

(Paraxial approximation - angles are small. All the rays are nearly parallel to the axis)
Snell's Law;
$n_1 \sin i = n_2 \sin r$

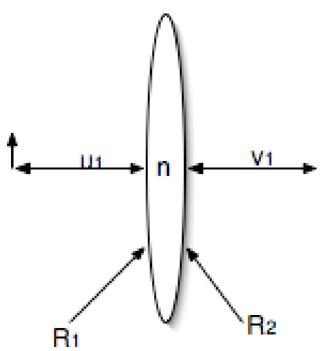
Therefore by combining all the equations;

$$n_1\left(\frac{h}{u} + \frac{h}{R}\right) = n_2\left(\frac{h}{R} - \frac{h}{v}\right)$$

$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_1 - n_2}{R}$$

Comment: this is valid only for paraxial rays. However it is possible to find a shape which gives perfect imaging of on-axis objects. This is known as a Cartesian Oval.

To do lenses, apply this twice for the two surfaces.



Assume air → glass → air. Hence $n_1 = n_3 = 1$.
Object u_1 from the first surface. Image v_1 formed according to;

$$\frac{1}{u_1} + \frac{n}{v_1} = \frac{n-1}{R_1}$$

(R is negative, hence the last part is changed around from $n_1 - n_2$ to $n_2 - n_1$.
This image serves as the object to be re-imaged by the second surface, with object distance $u_2 = -v_1$.
→ second surface (glass → air) is;

$$-\frac{n}{v_1}+\frac{1}{v_2}=\frac{1-n}{R_2}$$

Finally, eliminating v_1;

$$\frac{1}{u_1}+\frac{1}{v_2}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

For the lens as a whole, the object distance is u_1. We are only interested in the final image, not the intermediate ones. The image distance is v_2.

$$\frac{1}{u}+\frac{1}{v}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Suppose that we have an object at $u=\infty$. Parallel rays are coming in. By analogy to mirrors, the image will be formed at the focal point. Therefore, as $v=f$ for this particular case;

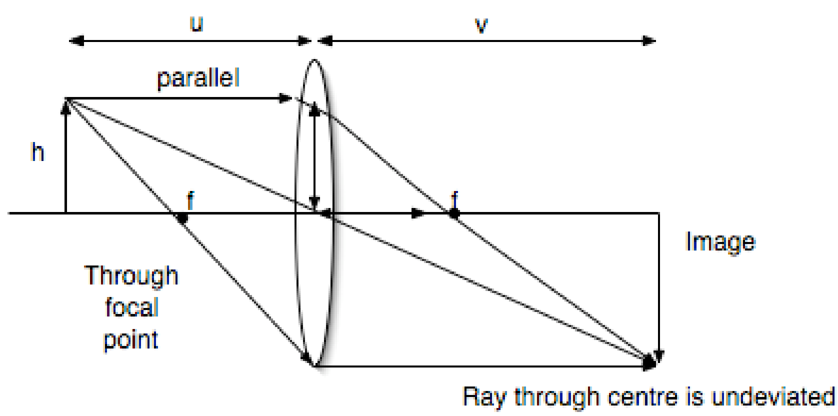
$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

This equation is known as the Lensmaker's Equation. f is known as the focal length.
Therefore;

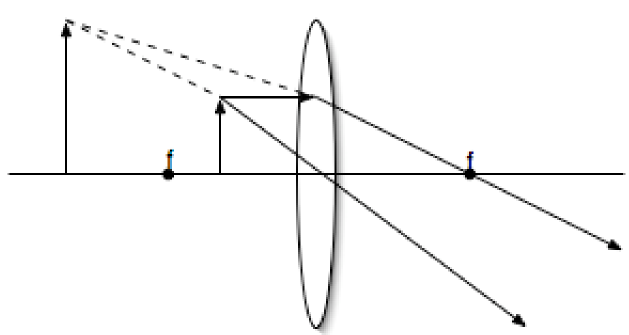
$$\boxed{\frac{1}{u}+\frac{1}{v}=\frac{1}{f}}$$

Sign conventions dictate that the signs for u and v are as for mirrors. f is positive for converging lenses (), and negative for diverging lenses)(.

Typical ray diagrams for lenses;
Converging lenses.

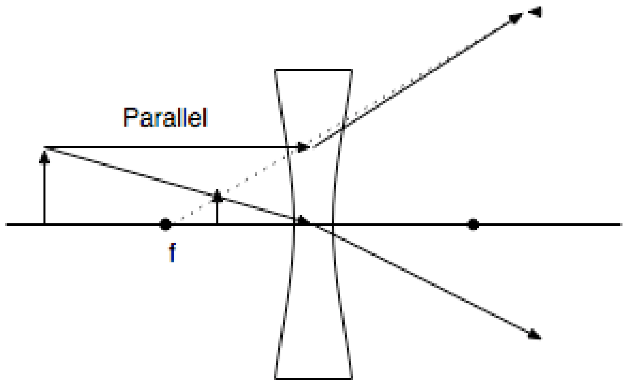


This is a real and inverted image.
Source closer than the focal point:



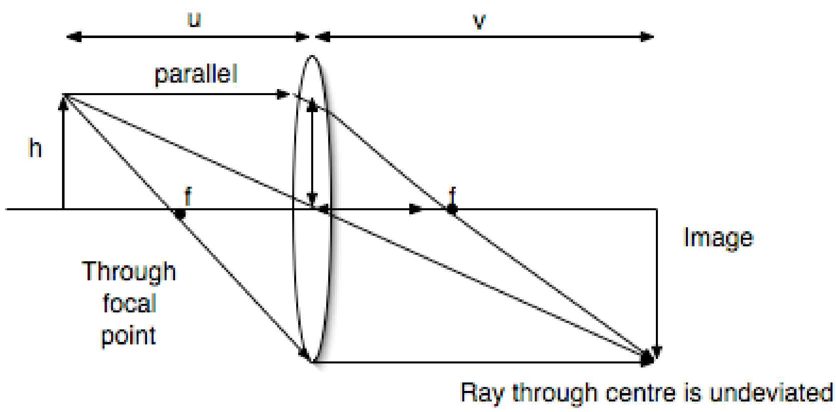
This is a virtual, erect image.

Diverging;



(See tutorial 14)

2.3.4 Lens Magnification
What is the ratio to the height of the image to the height of the object?



From similar triangles:

$$\frac{h_v}{h_u} = -\frac{v}{u}$$

$$\frac{h_v}{f} = -\frac{h_v}{v - f}$$

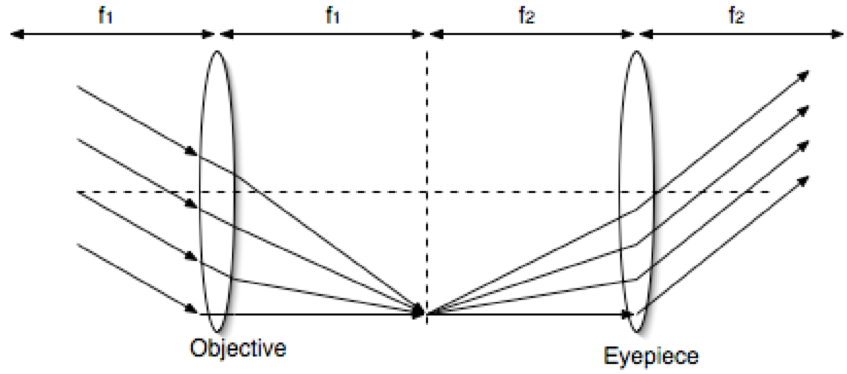
$$\frac{h_v}{h_u} = -\frac{v - f}{f}$$

This is often known as the Newtonian law, as it was derived by Newton in Opticks.
Applications of lenses;
Correction of vision defects.
The ideal eye would be able to take an image from infinity, and be able to converge it to the back of the eye. It should also be able to do the same for the clinically-defined 25cm length.
A myopic eye (short sight) converges too well, and makes the image form in front of the back of the eye. A hyperopic eye (long sight) forms the image too far behind the back of the eye.
The correction of myopia;
- The far point (furthest point at which you can focus) is closer than infinity.

→ arrange for a lens to make the object at infinity appear as though it was at the far point of the eye. This requires a diverging lens with a power that makes the focal point at the far point for the eye.
The correction for hyperopia;
- Near point > 25cm.

→ make an object at 25cm appear as though it was at the near point of the eye, using a converging lens.

Telescopes:
A refracting telescope is designed to increase the angular size of an object.



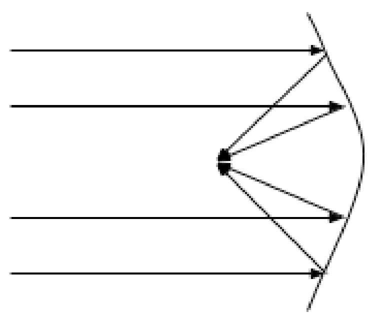
The eye sees the object at infinity, but inverted.

It will be magnified with angular magnification

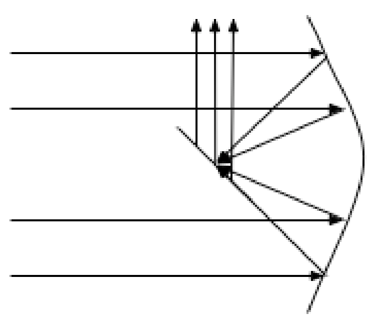
$$= \frac{\textit{angle subtended by eye by final image}}{\textit{angle subtended at objective by object}}$$

$$= -\frac{f_1}{f_2}$$

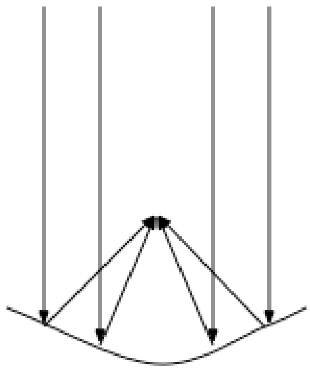
Negative as it is an inverted image.
You cannot shrink f_2 indefinitely - you would get a very thick lens and hence big angles of refraction. The refracting index n depends on the frequency. Thus the focal lengths will depend on the frequency of the light - you'll only get a focus on one wavelength. This is known as chromatic aberration.
The solution is to use a reflecting telescope. Use mirrors to form the image using reflection.
The simplest possible reflecting telescope is essentially a parabolic mirror.



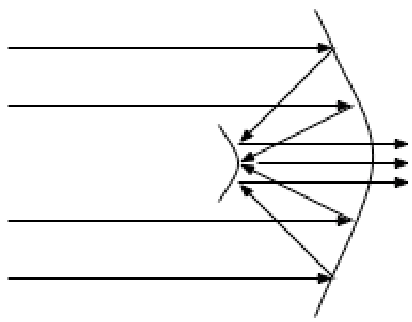
On-axis → single focus.
One possible solution was given by Newton; the Newtonian telescope.



Here there is a mirror just before the focal point.
Spherical mirror (Aracebo)

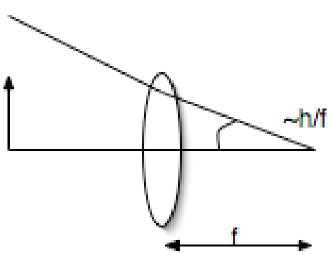


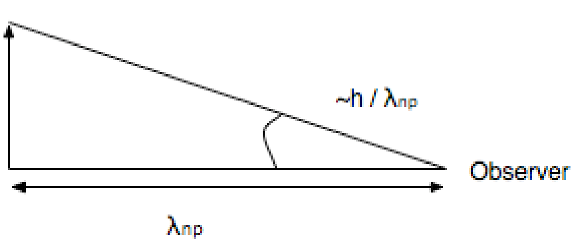
Cassegrain:
Use a parabola as the primary reflector, and a hyperbola as the secondary. The properties of these together focus the rays down to a point at the bottom.



Compromise between optical quality and off-axis performance.
- Change the shapes of the primary and secondary mirrors slightly.

Magnifier:
You are still trying to get an increase in the angle subtended, but the object is no longer an infinity. To do this, use a single converging lens.

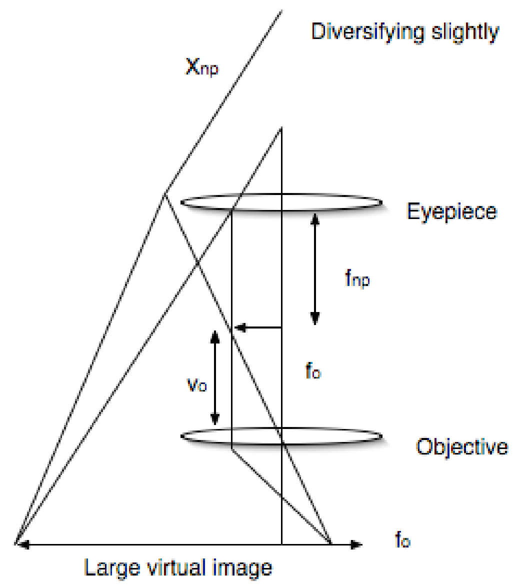




Angular magnification of approximately;

$$\frac{\lambda_{np}}{f}$$

This is the basis of the microscope.



$\lambda_{np} \equiv X_{np}$

Magnification;

$\frac{v_o}{f_o} \frac{\lambda}{f_\theta}$

2.4 Gravitational Lenses

- Mass deflects light (in particular, the gravitational field associated with the mass)

This deflection is very small - predicted by Einstein in 1916.

$a = \frac{4GM}{bc^2}$

If the distance from the gravitational source to the photon is b.

a is in radians. It will be fairly small, due to the small gravitational constant and the large speed of light. Hence you don't see this in everyday life.

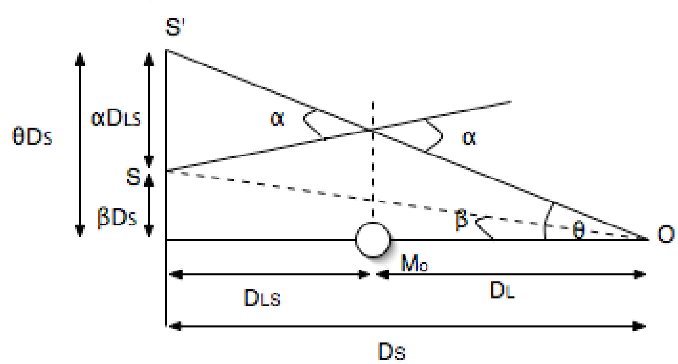
A very large mass is needed: use a galaxy ~ $10^{41} kg$.

The difference from a lens is;

1) the physical mechanism

2) "shape" of the lens is very different - not anything like nice focusing properties. - hence instead of a single point image, you get multiple distorted images.

2.4.1 Lens Equation

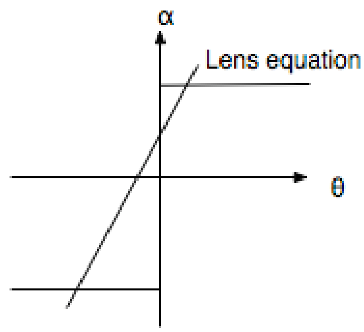


$i \, D_s = a D_{ls} + b D_s$

$i - b = a \frac{D_{ls}}{D_s}$

2.4.2 Image Formation

For real galaxies, a is roughly constant with $b - \; i D_l$.
Two equations must be satisfied at once.



Multiple images are found, where the equation lines meet. You will get 3 or 5 images in a gravitational lens system involving only one focusing object.
If b is bigger, then only one image will form.
In the completely symmetric case, you will get an Einstein ring.

$$i = \sqrt{\frac{4GMD_{ls}}{c^2 D_l D_s}}$$

An alternative method is using Fermat.

3. Polarization

3.1 Polarization: the Pictures

3.1.1 Plane Polarized

The wave that came straight out of Maxwell's equations is one with an oscillating electric field, and an oscillating magnetic field at right-angles to the electric field. The direction of propagation is $\underline{E} x \underline{B}$.
The E vector in this case is always in the same plane, varying in magnitude at any particular point. Hence the name "plane polarized". The plane of polarization is the plane containing the E vectors and the direction of motion k.
Normally forget the $\underline{B}$ fields (it can be calculated from the E field.)

3.1.2 Unpolarized

Most light sources don't produce plane polarized light - in general they produce unpolarized light. It has random and rapid change of polarization plane, with a timescale $\sim 10^{-8} s$. (Atoms are disturbed by particle collision, etc.)
Result: any observer integrates over a lot of these changes effectively sees equal amounts of E in every direction.
It is also possible to see partially polarized light.

3.2.1 Circular Polarization

This is where there is an E-vector of constant magnitude, but the plane of polarization rotates. The envelope that is traced out is a helical envelope (The shape is a helix). The E vector varies in direction, but has a constant magnitude.
Right-hand circular if the E vectors are rotating anticlockwise as seen by an observer looking backwards along the ray.
The more general state is known as elliptical polarization.
Plane rotates, but the magnitude of the E-vectors is not constant.
In other words. the tips of the E-vectors trace a squashed helix - elliptical in projection.

3.2 Polarization: the Maths

3.2.1 Linear Polarization

Assume that the wave is traveling in the z-direction.

$$\underline{E} = \hat{\underline{i}} E_{ox} \cos(kz - \omega t) + \hat{\underline{j}} E_{oy} (kz - \omega t)$$

This is the equation of a traveling wave going along the z-axis.
The E vector is at some angle in the x-y plane, but is perpendicular to the z-axis.
Hence E_{ox} and E_{oy}
The oscillations in the x-y plane are in phase.
No phase shift (or 180 degree phase shift) between the x and y components.
Aside: A 180 degree phase shift is the same as multiplying by $e^{i\pi}$, or -1.
The angle to the x-axis is given by;

$$\tan^{-1}\frac{E_{oy}}{E_{ox}}$$

The amplitude of the wave is; $\sqrt{E_{ox}^{2} + E_{oy}^{2}}$.

3.2.2 Circular Polarization

Phase shift of 90 degrees (or -90) between the oscillation in the x and y directions.

$$\underline{E} = \underline{i} E_o \cos(kz - \omega t) + \underline{j} E_o \sin(kz - \omega t)$$

$$\underline{E} = \underline{i} E_o \cos(kz - \omega t) + \underline{j} E_o \cos\left(kz - \omega t - \frac{\pi}{2}\right)$$

The other hand is;

$$\underline{E} = \underline{i} E_o \cos(kz - \omega t) - \underline{j} E_o \sin(kz - \omega t)$$

The scalar amplitude is constant.
The amplitude of this wave is;

$$|\underline{E}| = \sqrt{E_o^2 \cos^2(kz - \omega t) + E_o^2 \sin^2(kz - \omega t)} = E_o$$

Other hand of polarization has a -90 degree phase shift.
Add circularly polarized waves of the opposite hand.

$$\underline{E_1} = \hat{\underline{i}} + \hat{\underline{j}}, \; \underline{E_2} = \hat{\underline{i}} - \hat{\underline{j}}$$

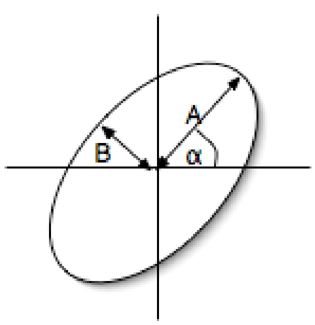
$$\underline{E} = \underline{E_1} + \underline{E_2} = 2\hat{\underline{i}} E_o \cos(kz - \omega t)$$

This is then linear polarization.

3.2.3 Elliptical Polarization

$$\underline{E} = \underline{i} E_{ox} \cos(kz - \omega t) + \underline{j} E_{oy} \cos(kz - \omega t + \varepsilon)$$

At any point, the E-vector rotates around an ellipse.



$$\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\varepsilon}{E_{0x}^{2} - E_{0y}^{2}}$$

This equation will not be needed in the exam.

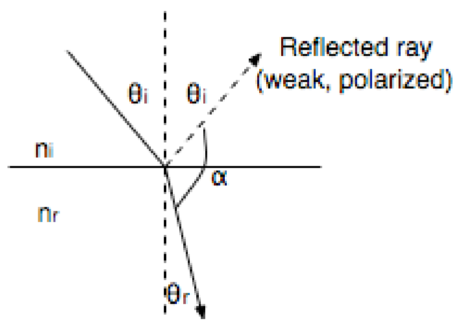
$\frac{A}{B}$ is a function of E_{ox}, E_{oy} and ε.

3.2.4 Unpolarized Light
Can represent unpolarized light as the sum of two orthogonally linearly polarized waves of equal amplitude but whose phase difference varies rapidly and randomly.

3.2.5 Stokes Parameters
These are four parameters that between them specify the state of polarization.
$I = \langle E_{ox}^2 \rangle + \langle E_{oy}^2 \rangle$
This is related to the intensity.
$Q = \langle E_{ox}^2 \rangle - \langle E_{oy}^2 \rangle$
This is related to linear polarization along the x-y axis.
$U = \langle 2E_{ox} E_{oy} \cos\phi \rangle$
This tells you the linear polarization at 45 degrees to the axis.
$V = \langle 2E_{ox} E_{oy} \sin\phi \rangle$
This is the amount of circular polarization.
In general, for unpolarized light, $I = 1$, but $Q = 0$ as the average of E_{ox}^2 is the same as that for E_{oy}^2.
These parameters can be put into vector form.

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} 2I_0 \\ 2I_1 - 2I_0 \\ 2I_2 - 2I_0 \\ 2I_3 - 2I_0 \end{pmatrix}$$

3.3 Generating Polarized Light
3.3.1 Reflection



$n_i \sin\theta_i = n_r \sin\theta_r$
If $\alpha = 90°$, the reflected ray is completely polarized.
Therefore;
$\theta_i + \theta_r = 90°$
$\sin\theta_r = \cos\theta_i$
Putting this into Snell's law;
$n_i \sin\theta_i = n_r \cos\theta_i$
$\tan\theta_i = \frac{n_r}{n_i}$
This is known as Brewster's Angle.

If an EM wave is passing through a medium, the atoms will oscillate and produce with EM waves, interacting with the original waves. Hence why the speed of light is not c in media (phase retardation). This also explains $\alpha = 90°$, as the dipoles in the atom have a non-uniform radiation pattern, and in particular they don't radiate along their length (the way they are oscillating) - dipole radiation.

The reflected ray is generated using the dipoles. If the waves coming in are polarized, then the atomic oscillations will also be polarized in the same way when the waves "hit" them. Hence radiation given out at the angle of refraction.
→ completely polarized ray.
The refracted ray will be slightly polarized, as it will loose some of the possible angles due to the dipole radiation.

General case (don't learn);
The reflection coefficient for the waves parallel and perpendicular to the wave are as follows;

$$R_{par} = \frac{\tan^2(\theta_i - \theta_r)}{\tan(\theta_i + \theta_r)}$$

$$P_{perp} = \frac{\sin^2(\theta_i - \theta_r)}{\sin(\theta_i + \theta_r)}$$

Obviously, if $\theta_i + \theta_r = 90°$, $R_{par} = 0$.

Generally, for $\theta_i < 60°$ both components are small and the reflected ray is rather weak.
e.g. for air → glass (n~1.5)

$\theta_i = 5°$

$\theta_r = 3.3°$

$R_{par} = 0.0396$

$R_{perp} = 0.0405$

Near the Brewster angle, $\theta_i = 60°$;

$R_{par} = 0.02$

$R_{perp} = 0.18$

3.3.2 Scattering
Pass an unpolarized EM wave into a medium containing dipoles. Then pass in an EM wave (polarized) oscillating in the y direction. This will kick the dipole, along the direction of the E wave. As before, the dipole won't radiate along its' axis. It will radiate in a plane in the x-z axis, with nothing in the y-axis.
Suppose instead we put in something polarized in the x-direction. Here, it will radiate in the y-z plane. Nothing in the x direction.
Remember that unpolarized light can be considered as the sum of polarized components, but with a rapidly varying phase shift and amplitude.
So with unpolarized light coming in, we can consider the output as a combination of the above results.
There will be lots of radiation (unpolarized) in the forward direction. To one side (at $90°$), polarized.

In an active galactic nucleus, we have a light source. In some types of galaxy, we have an obscuring material so we can't see the light source directly. Outside this, we have dust, electrons etc. The light source will excite the EM material. Hence looking at the light source, we will see light that has been scattered at $90°$, and we will see this light to be polarized.

3.3.3 Dichroism
This is the "brute force approach" - get some unpolarized light, and selectively absorb one component.
e.g. using a piece of Polaroid.
Some of the intensity of the input wave is obviously lost. Overall, the intensity is reduced by;

$$I = \frac{I_o}{2}$$

If polarized light is put through the Polaroid, then the output intensity for the input light absorbed at θ is $I_o \cos^2\theta$. This is Malus's Law.
Hence at $\theta = 0$, you get I_o. At $\theta = 90$, 0.

Polaroid etc. use strings of conducting molecules. These have a preferred axis along which it can be oscillated. Hence it absorbs the light at the correct direction, while ignoring that at the wrong angle.
This will originally done with quinine derivatives.

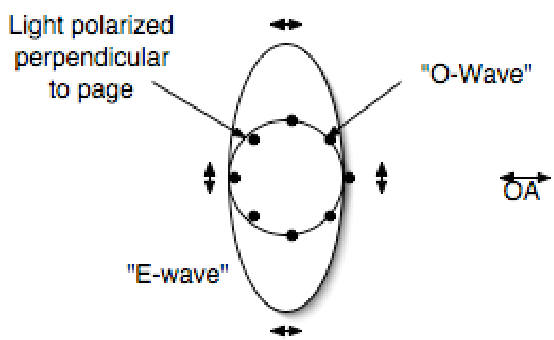
Typical applications include sunglasses (to cut out reflected glare). (horizontal or vertical?)

3.3.4 Birefringence
There are two types of crystal - some are isotropic, where n (and hence v) is independent of polarization and propagation direction, and others are birefringent, which is the opposite of the isotropic conditions.
The simplest birefringent crystals are known as "uniaxial" crystals, e.g. quartz and calcite. There exists a set of axis $\left(x, y, z\right)$ such that $v_z = v_x \neq v_y$ where v_x is the velocity of the wave with the E-vector oscillation in the x-direction (propagating perpendicular).
Obviously in this case the y-direction is special. This is known as the optic axis.
Suppose you have a wave propagating along the optic axis, the only way the electric field can oscillate is in the x-z plane. Hence all components of polarization travel at the same speed.
When the wave propagates perpendicular to the optic axis, This will separate out the oscillations along the direction of the x-axis from those of the z-axis. Any portion of the wave which is polarized along the y-axis travels at a different speed from that polarized along the x-axis.
e.g. in calcite, $v_y > v_x = v_z$.
The consequence of this is that phase shifts will occur between the different polarizations. This allows us to manipulate polarized light.
Say a circularly polarized light is traveling along the z-axis.

$$E_o \underline{i} \cos\left(kz - \omega t\right) + E_o \underline{j} \cos\left(kz - \omega t + \frac{\pi}{2}\right)$$

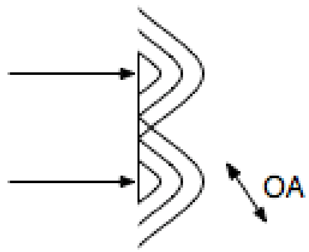
Using birefringence, we can arrange to turn the $\pi/2$ into 0, where the circulised polarized light becomes linear.
Thus, you can use calcite to change the polarization of light and phase angle very specifically.

<u>Consequence 1</u>
Unpolarized source in calcite (unpolarized - emits all polarizations in all directions) produces two sets of orthogonally polarized wavefronts.

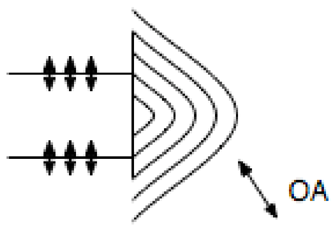


Remember that anything propagating along the optic axis has the same speed no matter the polarization.
Part of the wavefront propagates perpendicular to the optic axis: polarization parallel to the optic axis travels faster - ends up with a non-circular wavefront.

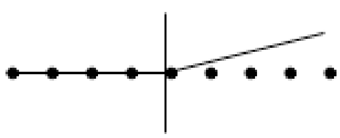
Consequence 2
Unpolarized light on calcite.



Nothing odd - polarization is parallel to the optic axis.



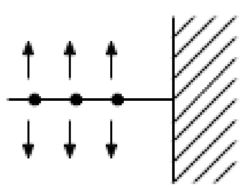
Vector combination of perpendicular to OA and parallel to OA.
You end up with non-spherical wavefronts (because of the velocity differences between along the OA and up-down)
Effectively (rays):



Deviation of the different polarizations of light - can separate out different polarizations of light.

3.4 Retardation plates
These manipulate polarization.

3.4.1 General Principle
Cut BC so that OA is parallel to the surface.



Extraordinary ray: $v_{\parallel} = \frac{c}{n_e}$ Ordinary ray: $v_{\perp} = \frac{c}{n_0}$
(In calcite, parallel propagates faster that perpendicular)
Effect is to introduce a phase shift between light of orthogonal polarizations - wavelength dependant.
Block of thickness d;
Path difference $= dn_o - dn_e$

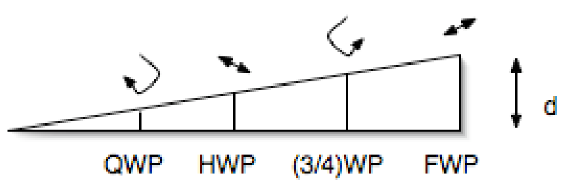
Phase difference: $\Delta\phi = \frac{2\pi}{\lambda}\left(n_0 - n_e\right)$ where $\left(n_0 - n_c\right)$ is the phase difference.

3.4.2 Quarter Wave Plates
Thickness is chosen such that
$$\Delta\varphi = \frac{\pi}{2}$$
Plane-polarized light enters at 45 degrees to the optic axis.
→ exiting light has the two components of polarization of equal magnitude but with an induced phase shift $\Delta\varphi$ (as up-down has gone faster than left-right) of $\pi/2$.
$$E_o\underline{i}\cos(kz-\omega t)+E_o\underline{j}\cos\left[kx-\omega t-\frac{\pi}{2}\right]$$
→ circular polarization.
Circularly polarization in → plane polarized out.
- Plane polarized at 0 degrees or 90 degrees - unchanged.
20, 30, etc. degrees plane polarized will become elliptically polarized light.
Elliptical into QWP - elliptical out.
Occasionally (if axis ellipse along optic axis) → linear out.
Unpolarized in QWP → unpolarized out.

3.4.3 Half-Wave Plate
Delay of π.
$$E_o\underline{i}\cos(kz-\omega t)+E_o\underline{j}\cos(kx-\omega t-\pi)$$
Plane polarized in will be rotated by 2θ, but will still be plane polarized.
- Circular in → circular out, of opposite hand.
$$E_o\underline{i}\cos(kz-\omega t)+E_o\underline{j}\cos\left[kx-\omega t+\frac{\pi}{2}\right]$$

3.4.4 Full-Wave Plate
This has a shift of 2π. This is useful as only one wavelength is shifted by this much, other wavelengths have different phase shifts.
Examples of usage;
Horizontal polaroid → full wave plate set at λ_o → Vertical polaroid.
Light at wavelength λ_o will not pass through the system.
Light at different wavelengths will have elliptically polarized light after the FWP, hence you will get some light coming out of the system for these frequencies as some of the light coming out of the FWP will be horizontally polarized.

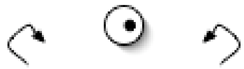
3.4.5 Quartz wedge
If you have a wedge of quartz, and you put linear polarized light going in at 45 degrees.
Depending on the thickness of the wedge, you will get different phases of polarization out.



The angle of the wedge can be calculated by the previous formula.
Thickness d is given by;
$$\frac{2\pi}{\lambda}d(n_o-n_e)=\frac{2\pi}{\lambda}\Delta x=2\pi$$
Hence you can calculate the apex angle of the wedge;
$$\tan\alpha=\frac{d}{l}$$

3.5 Propagation effects
3.5.1 Faraday Effect

Consider left hand circular and right hand circular light propagating in a region with a magnetic field.



One hand of the circular polarization is with the natural electron circulation, while the other is against it. There will be different refractive indexes for LCP and RCP → there will also be different velocities.
Consider plane polarized light as equal to a superposition of both LCP and RCP light.
Inducing phase shifts between left and right circular - you will end up rotating the plane of linear polarization.
$\theta = VBd$
where θ is the angle the linear polarization is rotated by, V a constant, B the B-field, and d the thickness.
In astronomy, you can work out the properties of the foreground material of a radio source.
Typically, B is around $10^{-11}T$, but d is very large. Hence you can get quite large rotation.

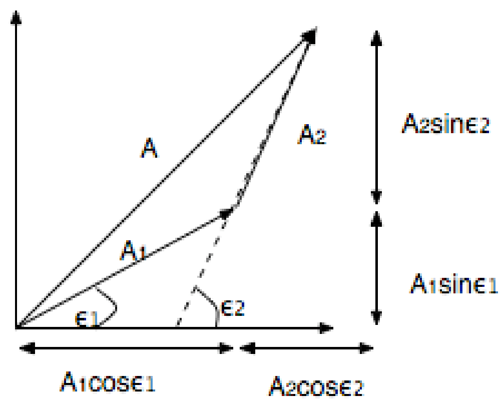
3.5.2 Kerr Effect
This is where instead of a magnetic field, you use an electric field. The molecules have a dipole moment, hence they line up along the electric field. You then have an optic axis, where all of the molecules are aligned up in a preferred direction.
Hence;
$n_e - n_o = \lambda_0 kE^2$
(Do not learn this)

4. Interference
4.1 Principles
4.1.1 Definition
Interaction of a finite number of light waves giving a resultant amplitude E_{tot} governed by the principle of superposition and intensity by $E^*_{tot}E_{tot}$.

4.1.2 Principle of Superposition
The maths:
$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$
Suppose we have the following solutions;
$$\psi_1 = f(x - vt)$$
$$\psi_2 = g(x - vt)$$
then $\psi_1 + \psi_2$ is also a solution.
$$\frac{\partial^2(\psi_1+\psi_2)}{\partial x^2} = \frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_2}{\partial x^2} = \frac{1}{v^2}\left(\frac{\partial^2\psi_1}{\partial t^2} + \frac{\partial^2\psi_2}{\partial t^2}\right) = \frac{1}{v^2}\frac{\partial^2(\psi_1+\psi_2)}{\partial t^2}$$
Generally;
$\sum c_i\psi_i$ is a solution.

The physics;
Suppose that ψ_1 is a solution with velocity v_1, and ψ_2 is a solution traveling at v_2, then the resultant at any time and place is the sum of the disturbances as they would appear if the other were absent.

4.1.3 How to do the addition
(Phasors)
Suppose you have two waves;
$A_1\cos(kx - \omega t + \phi_1)$
$A_2\cos(kx - \omega t + \phi_2)$
Use an Argand diagram;



Hence;

$$|A|^2 = (A_1 \cos\phi_1 + A_2 \cos\phi_2)^2 + (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2$$

$$|A|^2 = |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos\phi_1 \cos\phi_2 + 2|A_1||A_2|\sin\phi_1 \sin\phi_2$$

$$|A|^2 = |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos(\phi_1 - \phi_2)$$

The last part of this is called the interference term.
Aside;

$$\tan\phi_A = \frac{A_2 \sin\phi_2 + A_1 \sin\phi_1}{A_2 \cos\phi_2 + A_1 \cos\phi_1}$$

If you have two waves with a constant phase relation $(\phi_1 - \phi_2 = const)$ you need to add the amplitudes of the waves. They are known as coherent waves.
Intensity$=|A|^2$.
If you don't, i.e. over time $<\cos(\phi_1 - \phi_2)> = 0$ then you can get away with adding the intensities. These are incoherent.

4.1.4 Conditions for interference
- Constant phase relation
Any light used must have a restricted range of frequencies. (strictly speaking, 0).
If you have a finite range Δf present, then the different frequencies will get out of step.

$$\Delta t \sim \frac{1}{\Delta f}$$

In practice, you never get perfect temporal coherence in a wave train. There are a number of reasons for this;
- There is a limit set to any emission process that causes natural broadening - the uncertainty principle.
- Collision broadening
- Doppler broadening - atoms / molecules are in thermal motion. Different line of sight velocities.

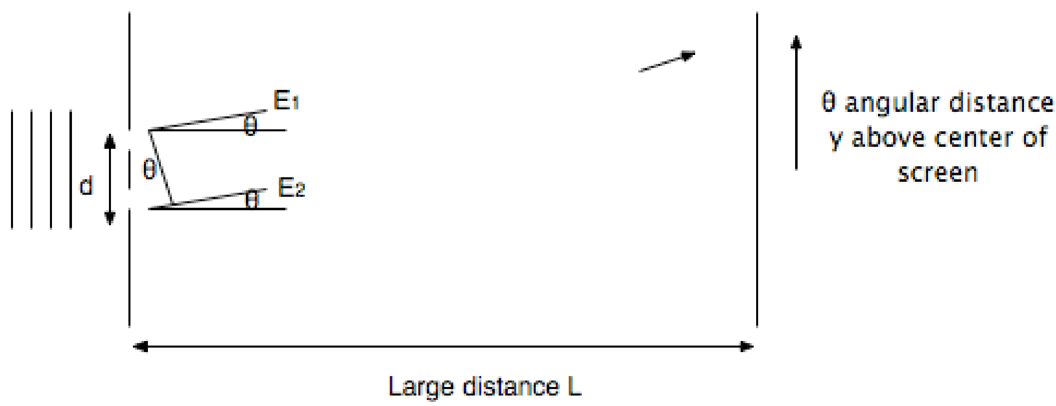
- Spatial coherence;
If you have a Young's Split setup, with light coming in, then the light between the two slits will have a constant phase relation. If there are random corrugations in the wave front, you will not get a constant phase relation between the two slits.
Here you can define a coherence length across the wave front for which the light is spacially coherent.
How far across the wavefront do you need to go before the wave is not coherent.

- Temporal coherence
How far down the wavefront do you have to go before the waves get out of step?
e.g. range of frequencies. Waves get out of step. $<\cos(\phi_1 - \phi_2)> = 0$

Types of interference
- Division of wavefront
Select 2 parts of a wavefront and use them as interfering waves (as sources of secondary wavelets).
(use slits. Secondary wavelets will be those two generated by the parts that pass through the slits. → interference pattern if coherent. This is Young's Slits)
- Division of amplitude
Select one part of the wavefront, split it in two, delay one part with respect to the other, then re-add them.
e.g. Michelson inferometer
(think: Aether experiment, Michelson-morley)

In both cases, you need to have coherent light.
→ Use a laser.

4.2 Interference by division of wavefronts
4.2.1 Young's Slits



The approximation is that the distance to the screen is large enough that the rays arrive at it roughly in parallel. This gives us Fraunhofer diffraction.
Path difference is $d\sin\theta$.
Phase difference is;
$kd\sin\theta = \frac{2\pi}{\lambda} d\sin\theta$
If this phase difference is 0, 2π , 4π , ... you will get constructive interference as the waves will reinforce each other.
If π , 3π , ... you will get destructive interference, and the waves will cancel.
- Intensity distribution;
First wave;
$E_1 = E_o e^{i(kx-\omega t+\phi)}$
Second wave;
$E_2 = E_o e^{i(kx-\omega t+\phi)} e^{i\frac{2\pi}{\lambda} d\sin\theta}$
This is because in complex notation a phase shift of ϕ is the same as multiplying by $e^{i\phi}$.
Total intensity $= E^* E$ where $E = E_1 + E_2$.
$E_1 + E_2 = E_o e^{i(kx-\omega t+\phi)}\left(1 + e^{i\frac{2\pi}{\lambda} d\sin\theta}\right)$

$E^* E = E_o^2 e^{-i(kx-\omega t+\phi)} e^{i(kx-\omega t+\phi)}\left(1 + e^{i\frac{2\pi}{\lambda} d\sin\theta}\right)\left(1 + e^{-i\frac{2\pi}{\lambda} d\sin\theta}\right)$

$E^* E = E_o^2\left(1 + e^{i\frac{2\pi}{\lambda} d\sin\theta}\right)\left(1 + e^{-i\frac{2\pi}{\lambda} d\sin\theta}\right)$

$E^* E = E_o^2\left(2 + e^{i\frac{2\pi}{\lambda} d\sin\theta} + e^{-i\frac{2\pi}{\lambda} d\sin\theta}\right)$
But;
$e^{i\phi} + e^{-i\phi} = 2\cos\phi$

So;

$$E^*E = E_o^2\left(2 + 2\cos\left(\frac{2\pi d\sin\theta}{\lambda}\right)\right)$$

also;

$$1 + \cos z = 2\cos^2\frac{z}{2}$$

So;

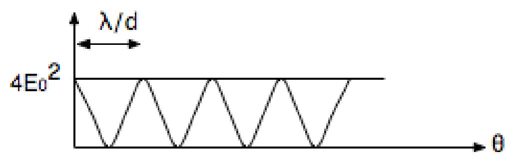
$$E^*E = 4E_o^2\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)$$

If θ is small, then;

$$\sin\theta \approx \tan\theta = \frac{y}{L}$$

So;

$$Intensity = 4E_o^2\cos^2\left(\frac{\pi yd}{\lambda L}\right)$$



The first maximum will happen when;

$$\frac{\pi yd}{\lambda L} = \pi$$

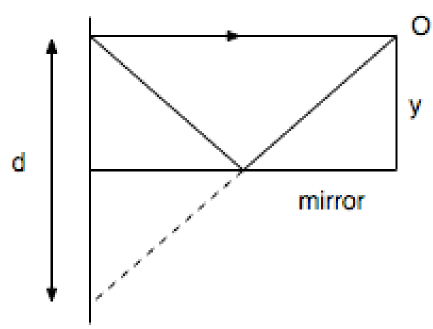
So;

$$y = \frac{\lambda L}{d}$$

So the separation of the interference patterns on the screen is $\lambda L/d$.

4.2.2 Lloyd's Mirror

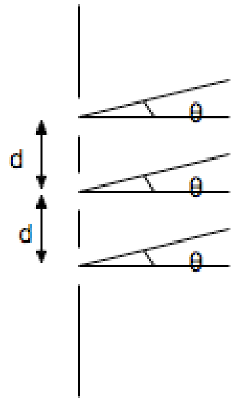
Suppose that you have a single source of light observed by an observer a height y above the ground. Suppose there is also a mirror, and light bounces off the mirror to the observer.



Is this exactly equivalent to young's slits? It is almost - there is only one problem.
The light reflecting off an optically denser surface has a phase change of π.
The reason is complicated - it is essentially the same as the Frenel coefficients.
→ the mathematics is almost the same, but instead of $\cos^2$ write $\sin^2$.
So;

$$Intensity = 4E_o^2\sin^2\left(\frac{\pi yd}{\lambda L}\right)$$

4.2.3 Three slits



Here, The first slit has $E_o e^{i(kx-\omega t)}e^{-i\frac{2\pi}{\lambda}d\sin\theta}$, second $E_o e^{i(kx-\omega t)}$, third $E_o e^{i(kx-\omega t)}e^{i\frac{2\pi}{\lambda}d\sin\theta}$.
Putting all this together, the resultant wave is;

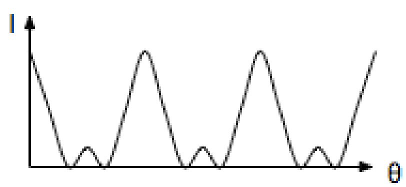
$$E_o e^{i(kx-\omega t+\phi)}\left[1+e^{i\frac{2\pi}{\lambda}d\sin\theta}+e^{-i\frac{2\pi}{\lambda}d\sin\theta}\right]$$

$$=E_o e^{i(kx-\omega t+\phi)}\left(1+2\cos\left(\frac{2\pi\, d\sin\theta}{\lambda}\right)\right)$$

We are interested in the intensity distribution.

$$E^*E=E_o^2\left(1+2\cos\left(\frac{2\pi\, d\sin\theta}{\lambda}\right)\right)^2$$

$$E^*E=E_o^2\left(4\cos^2\left(\frac{\pi\, d\sin\theta}{\lambda}\right)-1\right)^2=E_o^2\left(4\cos^2\left(\frac{\pi\, dy}{\lambda L}\right)-1\right)^2$$



→ narrower main peak, but small subsidiary peaks.
Decreasing the separation of the slits will increase the width of the fringes. Increasing the wavelength will also increase the width.

4.2.3 n Slits
Generally as the number of slits increases, the central peak gets narrower and more subsidiary, small peaks appear.
Total amplitude is;

$$E_o\left(1+e^{iz}+e^{i2z}+...+e^{inz}\right)e^{i(kx-\omega t)}$$

Where

$$z=\frac{2\pi}{\lambda}d\sin\theta$$

Side-note (do not learn this);

$$\left(1+x+x^2+...+x^{N-1}\right)=\frac{x^N-1}{x-1}$$

So;

$$A = E_o e^{i(kx-\omega t)} \frac{e^{Ni\phi}-1}{e^{i\phi}-1}$$

$$= E_o e^{i(kx-\omega t)} \frac{e^{\frac{1}{2}Ni\phi}\left(e^{\frac{1}{2}Ni\phi} - e^{-\frac{1}{2}Ni\phi}\right)}{e^{\frac{1}{2}i\phi}\left(e^{\frac{1}{2}i\phi} - e^{-\frac{1}{2}i\phi}\right)}$$

$$= E_o e^{i(kx-\omega t)} \frac{e^{\frac{1}{2}Ni\phi}\sin\left(\frac{N\phi}{2}\right)}{e^{\frac{1}{2}i\phi}\sin\left(\frac{\phi}{2}\right)}$$

This is the total amplitude. The same thing can be done for the intensity using;

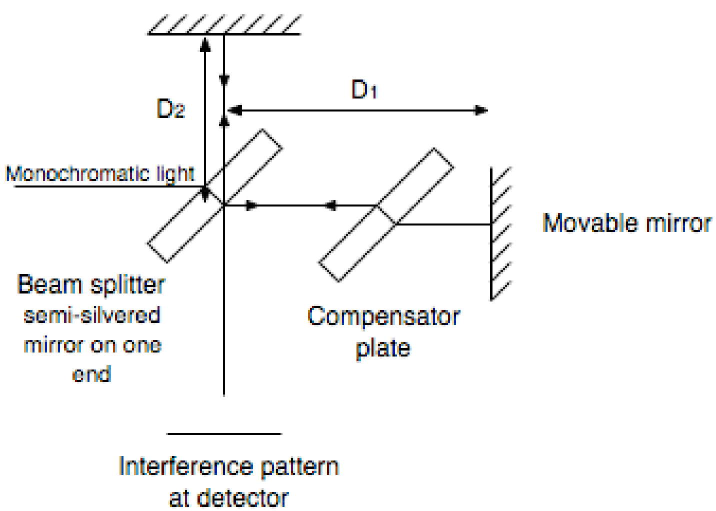
$$I = A^* A = E_o^2 \frac{\sin^2\left(\frac{N\phi}{2}\right)}{\sin^2\left(\frac{\phi}{2}\right)}$$

where $\phi = \dfrac{2\pi d \sin\theta}{\lambda}$

You should be able to prove that if $N = 2$, it reduces to the earlier formula for Young's slits → $\cos^2$.

The method used so far was Division of wavefronts. Now, look at division of amplitude - i.e. where we take one wave, and split it into two, allow one to go along a different path, then interfere them.

4.3 Michelson Interferometer.



4.3.1 Basic principle

The compensator plate is exactly the same thickness as the beam splitting plate. It is used to compensate for the extra path length in glass of the other beam.

The path difference $\Delta d = 2(d_1 - d_2)$.

The output amplitude is;

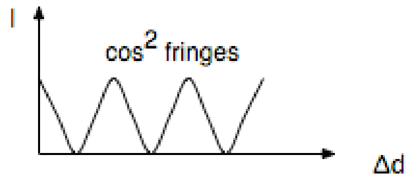
$$E_{output} = E_o e^{i(kx-\omega t)} + E_o e^{i(kx-\omega t)} e^{\frac{2\pi}{\lambda} i\Delta d}$$

We have exactly the same maths as the young's slit experiment, despite having a different setup.

So;

$$I = 4E_o^2 \cos^2\left(\frac{\pi \Delta d}{\lambda}\right)$$

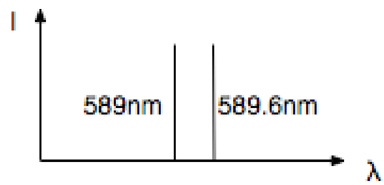
As the movable mirror is moved, you change Δd.



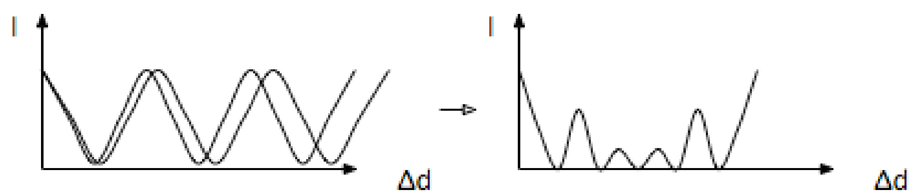
Fringes will be λ apart in units of Δd.
So they will be $\lambda/2$ apart in units of mirror motion.
This is an extremely sensitive way of measuring motion. It allows you to measure the motion to the order of λ.

4.3.2 Michelson Interferometer as spectrometer: (i) sodium doublet

Use it where the input light is a sodium doublet, rather than a monochromatic source. These are two closely spaced lines.



$\Delta\lambda$ is the separation between 2 lines.
What will happen to the intensity as the mirrors move?
Each spectral line will produce a separate fringe system. These will be mutually incoherent.



→ Δd_{min}, fringe patterns get out of step.
Δd_{min} is given by requirement of ½ oscillation difference between λ_1 and λ_2.

$$\frac{\Delta d_{min}}{\lambda_1} = \frac{\Delta d_{min}}{\lambda_2} + \frac{1}{2}$$

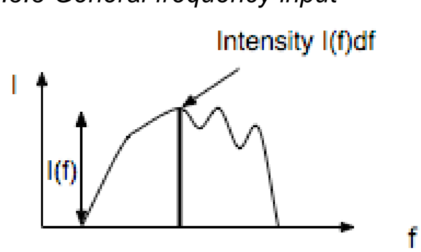
$$\Delta d_{min}\frac{(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2} = \frac{1}{2}$$

$$\Delta d_{min} = \frac{\lambda_1 \lambda_2}{2(\lambda_2 - \lambda_1)}$$

$$\Delta d_{min} = \frac{\lambda^2}{2\Delta\lambda}$$

Hence, we can work out $\Delta\lambda$.

4.3.3 General frequency input



Each bit of the frequency spectrum produces its' own $\cos^2$ fringes with;

$$dI = 4I(f)\cos^2\left[\frac{\pi f\Delta d}{c}\right]df$$

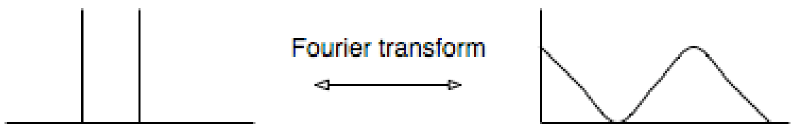
$$= 2I(f)\left[1+\cos\left[\frac{2\pi f\Delta d}{c}\right]\right]df$$

(Each bit is incoherent with each other). Now, add them all (intensities).

$$I(\Delta d) = \int_0^\infty 2I(f)\left[1+\cos\frac{2\pi f\Delta d}{c}\right]df$$

$$I(\Delta d) = const. + \int_0^\infty I(f)\cos\frac{2\pi f\Delta d}{c}df$$

This is a type of Fourier transform. The frequency power spectrum of the input light is a Fourier transform of the fringe contrast as a function of delay.



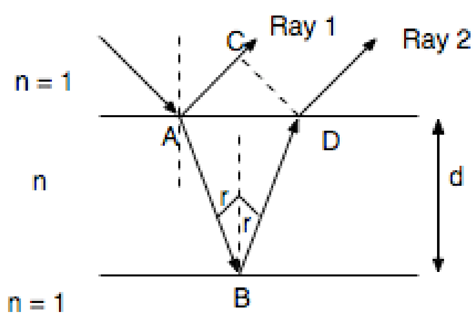
4.3.4 White light fringes

If you put white light in, you only get fringes if $\Delta d \approx 0$ as white light consists of a large range of frequencies, so each of these frequencies will give you a different interference pattern that will only be in phase if $\Delta d = 0$, and will quickly come out of phase and hence the intensity will go to 0. These fringes will disappear if;

$$\Delta d \approx \frac{\lambda^2}{2\Delta\lambda}$$

this also means that you can work out filter bandwidths.

4.4 Thin Films



Difference in optical path Ray 1 has traveled from A to C while Ray 2 has traveled from A to B.

Now $AB = 2d\tan r$

→ $AC = 2d\tan r\sin i$

$$Difference = \frac{2dn}{\cos r} - 2d\tan r\sin i$$

$$= 2d\left(\frac{n}{\cos r} - \tan r\cdot n\cdot\sin r\right)$$

$$= 2d\left(\frac{n - n\sin^2 r}{\cos r}\right)$$

$$= 2dn\cos r$$

This is useful for later - Fabry-Perot etalon.

If $2dn\cos r = m\lambda$ → destructive interference.

If $2dn\cos r = \left(m + \frac{1}{2}\right)\lambda$ → constructive interference.
(division of amplitude)
Outcome: coloured bright / dark bands at different wavelengths.

4.4.2 Use in Anti-Reflection coatings
Recall from section 3 that the reflection coefficients

$$R_{\perp} = \frac{\sin^2(\theta_i - \theta_t)}{\sin^2(\theta_i + \theta_t)} \quad R_{\parallel} = \frac{\tan^2(\theta_i - \theta_t)}{\tan^2(\theta_i + \theta_t)}$$ (don't learn these!)

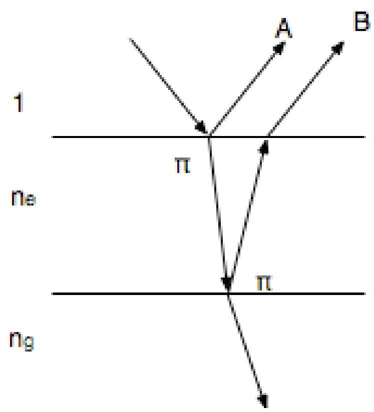
Near normal incidence, both become

$$R = \frac{(\theta_i - \theta_t)^2}{(\theta_i + \theta_t)^2} = \frac{(n-1)^2}{(n+1)^2}$$

Remember Snell's law → $\theta_i = n\theta_t$.
e.g. glass $n \approx 1.5$
Reflection coefficient $R \approx 0.04$
If air → glass, R becomes $n_i \to n_t$.

$$R = \frac{\left(\frac{n_t}{n_i} - 1\right)^2}{\left(\frac{n_t}{n_i} + 1\right)^2}$$

Optical systems loose light on reflection. The trick is to coat glass so that reflected waves cancel.



For A, B are π out of phase and of equal amlitude.

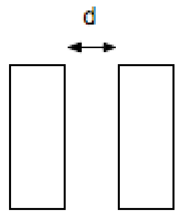
$$2dn_c\cos(r) = \left(m + \frac{1}{2}\right)\lambda$$ → in this case destructive.

Also want equal reflection coefficients at both interfaces.

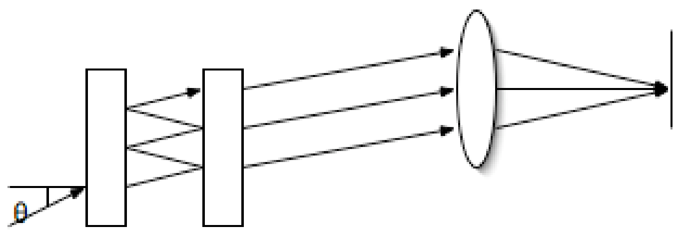
$$\left(\frac{n_c - 1}{n_c + 1}\right)^2 = \frac{\left(\frac{n_g}{n_c} - 1\right)^2}{\left(\frac{n_g}{n_c} + 1\right)^2} \to n_c = \sqrt{n_g}$$

→ choose thickness and refractive index of coating → no reflection.
This is why camera lenses are purple.

4.5 Fabry-Perot Etalon
4.5.1 Basic Principle



Glass plates:
Accurate separation and very reflective coatings. Intensity coefficient ρ , transmission coefficient σ



Phase shift between rays? - same geometric argument as thin films.
Path difference $= 2d\cos\theta$

Phase difference $\delta = \frac{2\pi}{\lambda} 2d\cos\theta$

Amplitude difference between successive rays $\rho\left(= \underbrace{\sqrt{\rho}\times\sqrt{\rho}}_{2\,reflections} = \rho \right)$

4.3.2 Intensity Distribution

$$E = E_o e^{i(kx-\omega t)}\left(\sigma + \sigma\rho e^{i\delta} + \sigma\rho^2 e^{i2\delta} + \ldots\right)$$

But binomial expansion $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

So;

$$E = E_o \sigma e^{i(kx-\omega t)} \frac{1}{1-\rho e^{i\delta}}$$

$$I = E^* E = E_o^{\,2}\sigma^2 \frac{1}{1-\rho e^{i\delta}} \frac{1}{1-\rho e^{-i\delta}}$$

$$= \frac{E_o^{\,2}\sigma^2}{1+\rho^2 - 2\rho\left(\frac{e^{i\delta}+e^{-i\delta}}{2}\right)} = \frac{E_o^{\,2}\sigma^2}{1+\rho^2-2\rho\cos\delta}$$

$$\left(\cos\delta = 1 - 2\sin^2\left(\frac{\delta}{2}\right)\right)$$

$$I = \frac{E_o^{\,2}\sigma^2}{1+\rho^2-2\rho\left(1-2\sin^2\frac{\delta}{2}\right)} = \frac{E_o^{\,2}\sigma^2}{(1-\rho)^2\left(1+\frac{4\rho}{(1-\rho)^2}\sin^2\left(\frac{\delta}{2}\right)\right)}$$

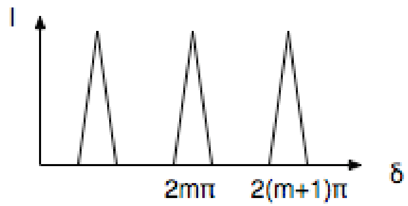
(reminder: $\delta = \frac{2\pi}{\lambda} 2d\cos\theta$)

Vary λ, θ → vary δ → peaks in intensity.

Peaks if $\delta = 0, 2\pi, \ldots, 2m\pi$

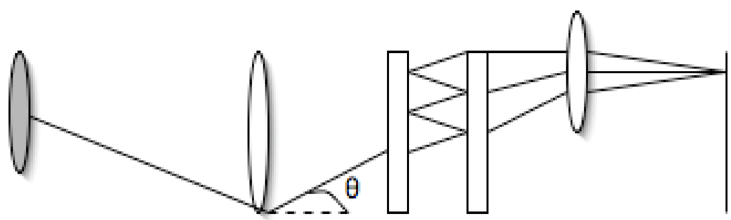
In that case, $\sin^2 \frac{\delta}{2} = 0$ → $\frac{E_o^{\,2}\sigma^2}{(1-\rho)^2}$ → large!

Peaks very steep and sharp.



4.5.3 Variation with θ

Extended monochromatic source



$$\delta = \frac{2\pi}{\lambda} 2d\cos\theta$$

Peaks at values of θ such that $\delta = 2m\pi$. Suppose θ is small; $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

→ $\delta = \frac{2\pi}{\lambda} 2d\left(1 - \frac{\theta^2}{2}\right) = 2m\pi$

(θ in radians)

$$\boxed{\frac{\theta^2}{2} = 1 - \frac{m\lambda}{2d}}$$

Only see fringes for certain θ → will see concentric rings on the screen. Radius will depend on λ e.g. if you use the Sodium doublet → each ring will be double.

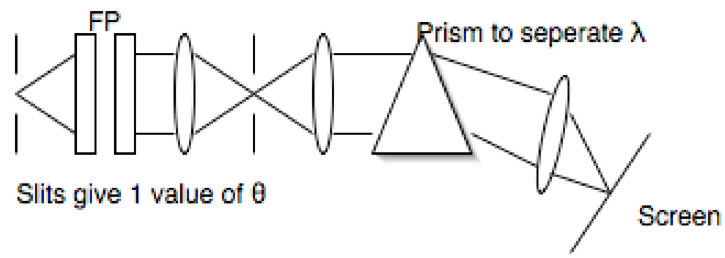
→ high resolution spectrometer (can distinguish to 0.01nm), provided that reflectivity $\rho \approx 0.9$ or more.

4.5.4 White light (Edser-Butler fringes)

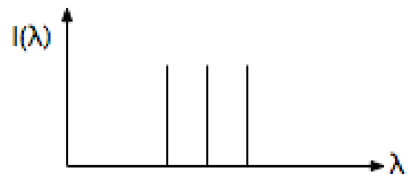
Recall that $\delta = \frac{2\pi}{\lambda} 2d\cos\theta = 2m\pi$.

θ constant ≈ 0 , range of λ .

$$\delta = \frac{2\pi}{\lambda} 2d = 2m\pi$$



Only for certain λ will fringes appear.

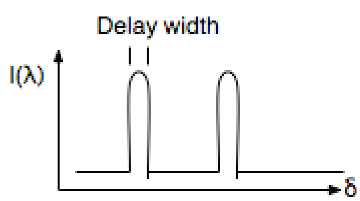


4.5.5 Important parameters of the FPE

First calculate $\delta\lambda$ (not $\delta \times \lambda$ but $\delta\lambda$ the width in λ of each interference pattern). This controls the spectral sharpness.

Recall (4.5.1)

$$I = \frac{E^2\sigma^2}{(1-p)^2\left(1+\frac{4\rho}{(1-\rho)^2}\sin^2\frac{\delta}{2}\right)}$$



Calculate the width in units of δ.

Peak falls to half the peak value when

$$\frac{4\rho}{(1-p)^2}\sin^2\frac{\delta_{1/2}}{2} = 1$$

Assume $\delta_{1/2}$ is small.

$$\delta_{1/2}{}^2 = 4\sin^{-1}\left(\frac{(1-\rho)^2}{4\rho}\right) \approx \frac{(1-\rho)^2}{\rho}$$

$$\rightarrow \delta_{1/2} = \sqrt{\frac{(1-\rho^2)}{\rho}}$$

Width of peak is $2\delta_{1/2} \rightarrow \left(\frac{1-\rho}{\sqrt{\rho}}\right)$ (1)

Note that delay width of peak is given by $\delta_{+1/2} - \delta_{-1/2}$.

$$= \frac{2\pi}{\lambda + \frac{\delta\lambda}{2}} 2d\cos\theta - \frac{2\pi}{\lambda - \frac{\lambda}{2}} 2d\cos\theta$$

At peak $2d\cos\theta = m\lambda$ (interference condition)

$$\rightarrow \text{delay width} = 2\pi m\lambda\left(\frac{1}{\lambda + \frac{\delta\lambda}{2}} - \frac{1}{\lambda - \frac{\delta\lambda}{2}}\right) = 2\pi m\lambda\left(\frac{1}{\lambda}\left(\left(1 + \frac{\delta\lambda}{2\lambda}\right)^{-1} - \left(1 - \frac{\delta\lambda}{2\lambda}\right)^{-1}\right)\right)$$

Using binomial expansion,

$$\text{delay width} = 2\pi m\left(1 - \frac{\delta\lambda}{2\lambda} + ... - 1 - \frac{\delta\lambda}{2\lambda}...\right) = \frac{2\pi m\delta\lambda}{\lambda} \quad (2)$$

Equate (1) and (2).

$$\boxed{\delta\lambda = \frac{\lambda(1-\rho)}{\pi m\sqrt{\rho}}}$$

3 important parameters;

4.5.5.1 Resolving Power

$$R \equiv \frac{\lambda}{\delta\lambda} = \frac{\pi m\sqrt{\rho}}{1-\rho}$$

Typically $\rho \approx 0.95$.

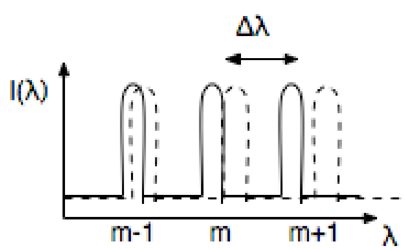
$m = \frac{2d}{\lambda}$ from interference condition assuming θ small.

For dynamics, d is the plate separation, λ is the wavelength.

$m \approx 10^4$ (d $\approx$ few mm)

Hence $R \approx 10^5$ → high == good.

4.5.5.2 Free Spectral Range $\Delta\lambda$



Cannot observe a very large spectral range without ambiguity (mth order fringe of one wavelength overlaps $m+1th$ order of another).

→ free spectral range is the range of wavelength that can be observed at once.

$$\Delta\lambda = \frac{2d}{m} - \frac{2d}{m-1} \approx \frac{2d}{m^2} = \frac{\lambda}{m}$$

assuming m large.

Typically $\approx \frac{500nm}{10^4} \approx 0.05nm$ (terrible!)

4.5.5.3 Finesse

$F = \frac{\Delta\lambda}{\delta\lambda}$ → how many spectral lines can fit onto one FSR.

$F = \frac{\lambda}{m\delta\lambda} \approx \frac{\pi\sqrt{\rho}}{1-\rho}$

$\rho = 0.95$ → $F = 60$

5. Diffraction

5.1 The Maths (& principles)

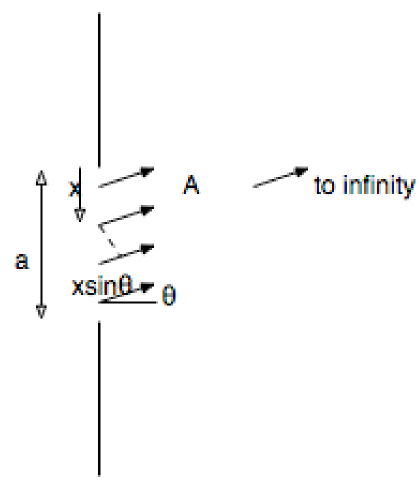
5.1.1 Types of Diffraction & Definitions

Interference is the adding together a finite number of waves. Diffraction is interference in the limit where the number of waves goes to infinity. To do this, instead of a narrow slit, use a wide slit.

Fraunhofer diffraction is the case where the phase variation is linear across the aperture, i.e. the phase difference between each of the waves passing through the aperture is proportional to the distance along the aperture. The observer in this case is a large distance away from the aperture, hence this is also called far-field diffraction.

The opposite is near-field, or Fresnel diffraction. Here the observer is much closer to the aperture. The waves are not linear, as they are all heading to the observer in a spherical pattern rather than traveling pretty much parallel to each other. This leads to spherical terms in the maths, which is complicated. We will deal with this later in section 5.5.

5.1.2 Wide Slit



A has representation

$E_o e^{i(kx-\omega t)}$

$E_o = E_{tot}\frac{dx}{a}$

Any other part a distance x down the slit from a has phase difference

$x\sin\theta\frac{2\pi}{\lambda}$

$E_{tot}\frac{dx}{a}e^{i(kx-\omega t)}e^{i\left(\frac{2\pi}{\lambda}x\sin\theta\right)}$

Superposition principle tells you that for coherent light you just add the amplitudes together. In this case, we have an infinite number of waves each with a very small amplitude and different phase differences. Use integration.

$$A(\theta) = \int_{-a/2}^{a/2} E_{tot} \frac{dx}{a} e^{i(kx-\omega t)} e^{\frac{2\pi}{\lambda} x \sin\theta}$$

$$= \frac{E_{tot} e^{i(kx-\omega t)}}{a} \left[\frac{e^{i\frac{2\pi}{\lambda} x \sin\theta}}{\frac{2\pi}{\lambda} i \sin\theta} \right]_{x=-a/2}^{x=a/2}$$

$$= \frac{E_{tot} e^{i(kx-\omega t)}}{a\left(\frac{\pi \sin\theta}{\lambda}\right)} \left[\frac{e^{\frac{i\pi a \sin\theta}{\lambda}} - e^{\frac{-i\pi a \sin\theta}{\lambda}}}{2i} \right]$$

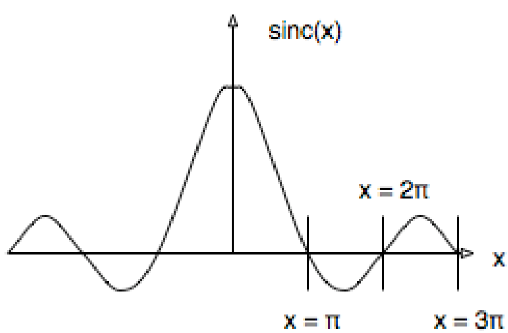
$$= \frac{E_{tot} e^{i(kx-\omega t)}}{a\left(\frac{\pi \sin\theta}{\lambda}\right)} \sin\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

$$= E_{tot} e^{i(kx-\omega t)} \operatorname{sinc}\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

Remember that $\operatorname{sinc} x = \frac{\sin x}{x}$.

$$\text{Intensity} = I_{obs} = A_{obs} * A_{obs} = E_{tot}{}^2 \operatorname{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

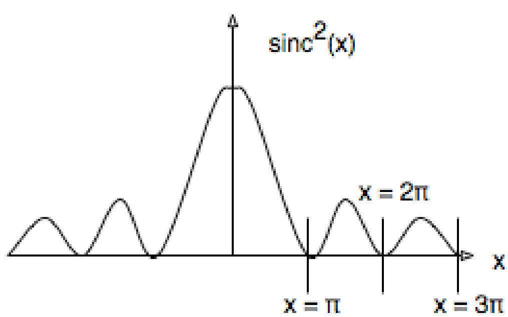
Properties of the sinc function;



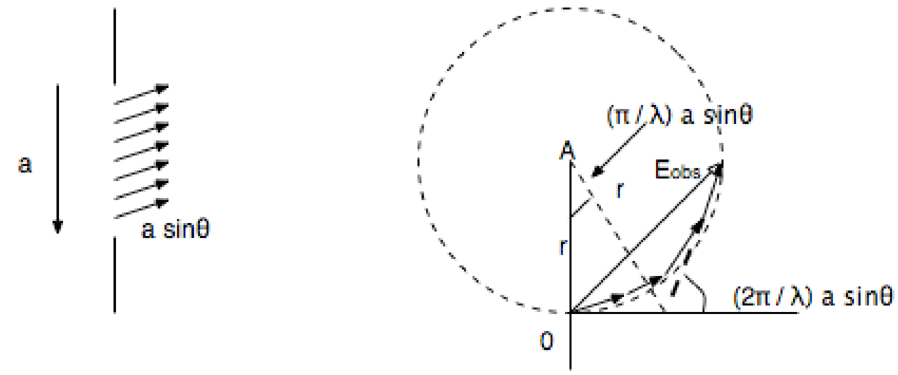
$x \to 0$

$\sin x \to x$

As a result, the central peak is twice as wide as the others.



The function is symmetric. The second bump contains about 5% of the peak bump's intensity. Subsequent bumps are weaker.
As the aperture a gets bigger, sinc^2 function will get narrower. This is rather suggestive of the idea of Fourier transform.
This can also be done a different way - through using phasors.
Suppose that $\theta = 0$. All the phasors add together along the same axis. Hence you observe E_{tot}
Suppose that $\theta \neq 0$. Take the first part of the wavefront, then add the second on.



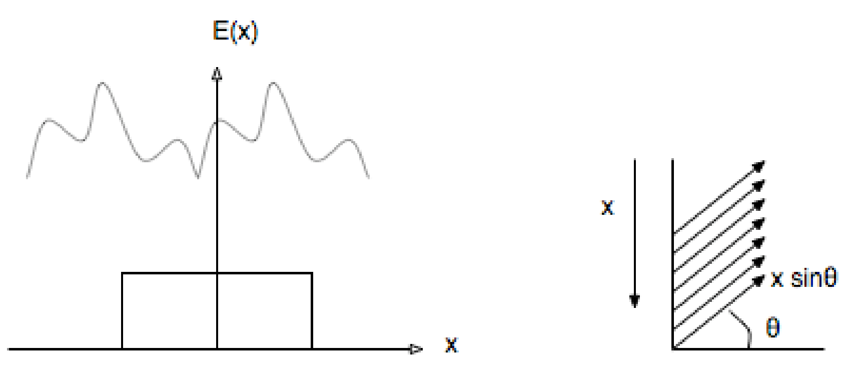
Angle $OAP = \frac{2\pi}{\lambda} a \sin\theta$.

→ $\frac{1}{2} E_{obs} = r \sin\left(\frac{\pi}{\lambda} a \sin\theta\right)$

E_{tot} = the length of the arc O → P $= 2\pi r \, x \frac{\frac{2\pi}{\lambda} a \sin\theta}{2\pi} = \frac{2\pi r a \sin\theta}{\lambda}$

$$E_{obs} = E_{tot} \frac{\sin\left(\frac{\pi a \sin\theta}{\lambda}\right)}{\frac{\pi a \sin\theta}{\lambda}} = E_{tot} \,\text{sinc}\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

So why the Fourier transform?
Apply method to an arbitrary slit with throughput variation;



General ray with respect to any reference point has a phase shift of $x \sin\theta . \frac{2\pi}{\lambda}$. This can be represented as $E(x) dx . e^{i(kx-\omega t)} e^{\frac{2\pi}{\lambda} i x \sin\theta}$

Total field seen at θ, $A(\theta)=e^{i(kx-\omega t)}\int E(x)e^{\frac{2\pi}{\lambda}ix\sin\theta}dx$

This looks much like a Fourier transform. The integral part is the amplitude.

i.e. the amplitude of a diffracted wave as a function of θ is the Fourier transform of the aperture function $E(x)$, provided we have far-field diffraction.

5.1.3 Reminder of Fourier Methods

- Orthogonal basis functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$, where n is an integer, can be used to make up any function in the range $-L\to L$.

$$f(x)=a_o+\sum_n a_n\sin\left(\frac{n\pi x}{L}\right)+\sum_n b_n\cos\left(\frac{n\pi x}{L}\right)$$

where $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ etc.

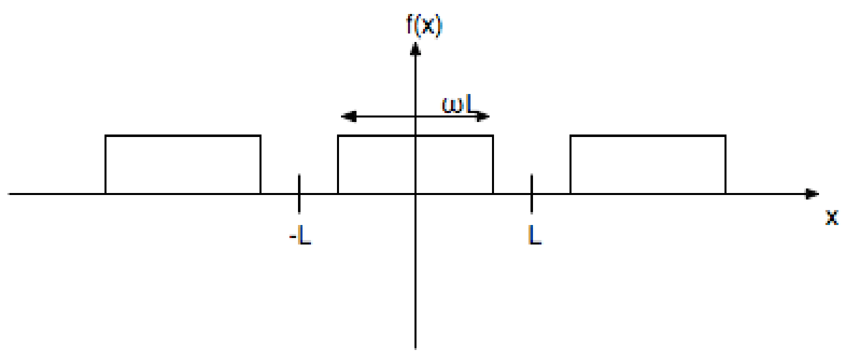
- Also works for any periodic function with a $2L$ repeat.

$\sin$ and $\cos$ are functions of $e^{i\frac{n\pi x}{L}}$, $e^{-i\frac{n\pi x}{L}}$.

→ $e^{i\frac{n\pi x}{L}}$, $e^{-i\frac{n\pi x}{L}}$ are also a set of orthogonal basis functions.
→ periodic functions.

$$a_n=\int_{-L}^{L}f(x)e^{-i\frac{n\pi x}{L}}dx$$

- Suppose we have a repeating top-hat function.



Here $f(x)=\sum_{-\infty}^{\infty}a_n e^{i\frac{n\pi x}{L}}$. The height of the peaks is H.

$$a_n=\frac{1}{L}\int_{-\omega L/2}^{\omega L/2}He^{-i\frac{n\pi x}{L}}dx=\frac{2H}{L}\left(\frac{e^{\frac{i\pi\omega}{2}}-e^{\frac{-i\pi\omega}{2}}}{\frac{2in\pi}{L}}\right)=\frac{2H}{n\pi}\sin\left(\frac{n\omega\pi}{2}\right)$$

Now consider the case where $L\to\infty$ but leave ωL finite and constant.
Then harmonics get closer in frequency as ω gets smaller until sum → integral.

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}a(k)e^{ikx}dk \text{ and } a(k)=\int_{-\infty}^{\infty}f(x)e^{ikx}dx$$

Fourier transform pairs $f(x)$ ← FT → $a(k)$.

5.1.4 Fourier transforms already seen

- $E(x) \leftrightarrow A(\theta)$. Width $\Delta\theta \propto \frac{1}{\delta x}$.

But also;

If you have a pulse (non-periodic) $f(t)$ and want to know the power as a function of frequencies which it contains $f(\omega)$

Then $F(\omega) \leftrightarrow f(t)$

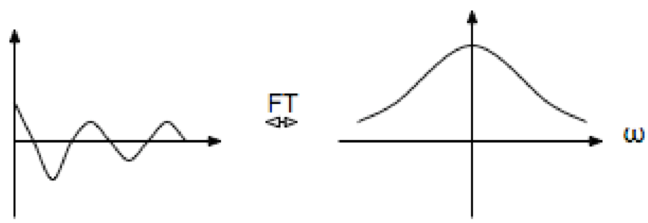
e.g. top-hat pulse from $-\frac{\Delta t}{2}$ to $\frac{\Delta t}{2}$, $F(\omega) = \int_{-\Delta t/2}^{\Delta t/2} He^{-i\omega t} = H\Delta t \,\text{sinc}\left(\frac{\omega \Delta t}{2}\right)$

We saw this when we talked about mechanisms of line broadening. (temp. coherence)

- Michelson interferometer

Input power (λ) is a fourier transform of the amplitude of the fringes as a function of Δ.

$\Delta f \propto \frac{1}{\Delta\Delta}$



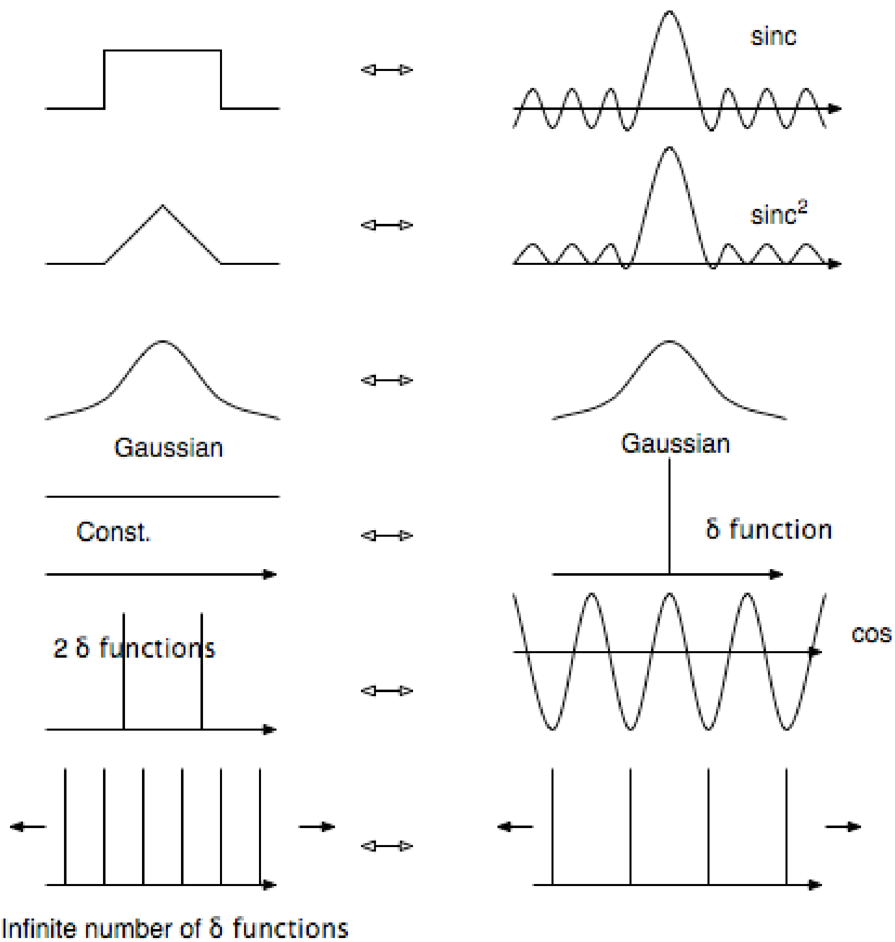
5.1.5 Examples of fourier transforms

- δ function $\delta(x_o) = \infty$ for $x = x_o$ and 0 elsewhere.

Area $\infty x 0 = 1$

$\int_{-\infty}^{\infty} \delta(x_o)\, dx = 1$

$\int_{-\infty}^{\infty} \delta(x - x_o) f(x)\, dx = f(x_o)$

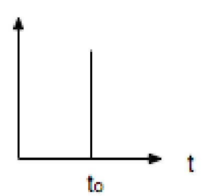


Young's slits;

$$F(k) = \int_{-\infty}^{\infty} \delta\left(x - \frac{d}{2}\right) e^{-ikx} dx + \int_{-\infty}^{\infty} \delta\left(x + \frac{d}{2}\right) e^{-ikx} dx$$

$$= e^{-ikd/2} + e^{ikd/2} = 2\cos\left(\frac{kd}{2}\right)$$

Example; what is the fourier transform of a delta function offset from 0.

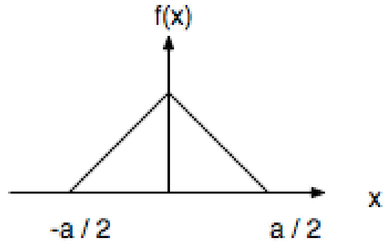


What range of frequencies?

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t - t_o) e^{i\omega t} dt = e^{i\omega t_o}$$

$e^{i\omega t_o}$ is something with a constant unit amplitude in ω. Phase shift $\propto \omega$. All frequencies appear at a constant amplitude, but different phases so that they all coincide at t_o instead of 0.

5.1.6 Tapered aperture (wigwam)



$f(x) = \frac{1}{a}(a - |x|)$ between $-a/2 \to a/2$, or $f(x) = 0$ otherwise.

$A(\theta) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$ where $u = \frac{2\pi \sin\theta}{\lambda}$

$A(\theta) = \frac{1}{a}\int_{-a/2}^{0}(a + x)e^{iuk}dk + \frac{1}{a}\int_{0}^{a/2}(a - x)e^{iuk}dk$

$aA(\theta) = \left[(a + x)\frac{e^{iux}}{iu}\right] + \ldots$

$= \frac{2}{u^2} - \frac{\left(e^{iua} + e^{-iua}\right)}{u^2}$

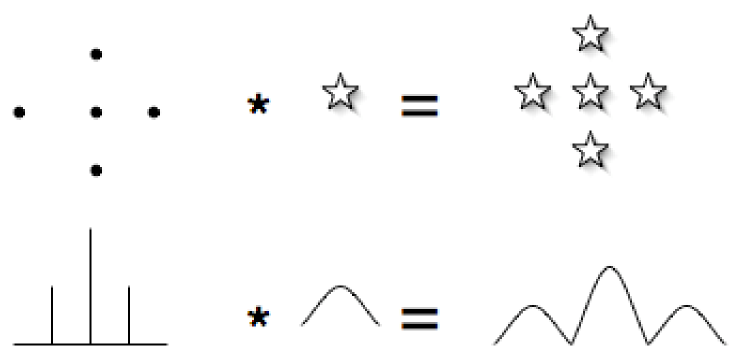
$= \frac{2 - 2\cos ua}{u^2} = \frac{4}{u^2}\sin^2\left(\frac{ua}{2}\right)$

→ $A(\theta) = \frac{4a}{u^2a^2}\sin^2\left(\frac{ua}{2}\right) = a\,\mathrm{sinc}^2\left(\frac{ua}{2}\right) = a\,\mathrm{sinc}^2\left(\frac{\pi \sin\theta a}{\lambda}\right)$

$I(\theta) = a^2\,\mathrm{sinc}^4\left(\frac{\pi a \sin\theta}{\lambda}\right)$

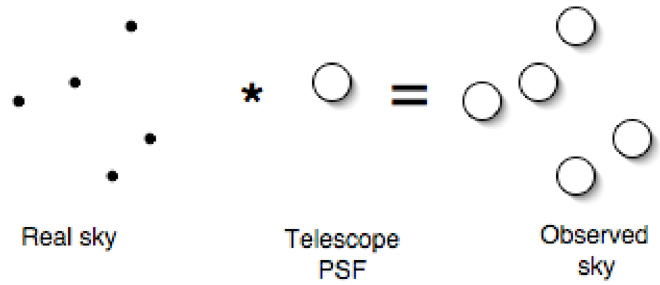
There is an easier way to do this…

5.2 Convolution
5.2.1 Principle

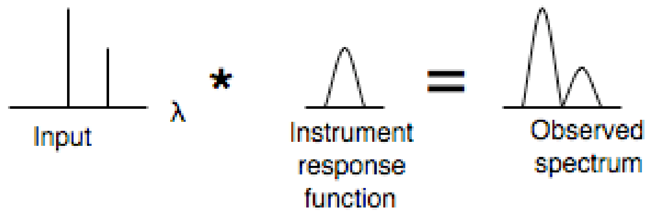


Amplitudes of the Gaussians at the end have a size proportional to the height of the delta function.

5.2.2 Applications
Stars in the sky (akin to delta functions) convoluted with a telescope (Point Spread Function "PSF" i.e. blurring) → stars become a lot bigger and blurred.



If the PSF is too big, you cannot separate the stars out.
PSF's can also be used in wavelength, e.g. Etalon.

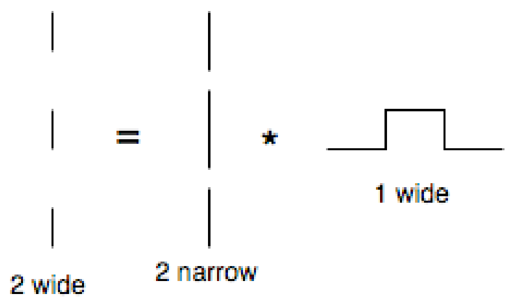


$$\delta\lambda = \frac{\sqrt{\rho}}{1-\rho}$$

If $\delta\lambda$ is too big, you cannot separate out the separate lines.

If you have a circular aperture telescope, then the width of the PSF (the resolution of the telescope) is $\frac{1.22\lambda}{d}$ radians where λ is the observed wavelength, and d the diameter.
e.g.;
Eye, $\lambda = 500nm$ and $d = 1mm$ → a few arc-minutes.
Lovell 76m telescope → 10 arc-minutes (wavelength is much longer)
HST; 2.5m at optical → 50 milli-arc-seconds.
MERLIN radio array $d \sim 220km$ → 50 milli arc-seconds.

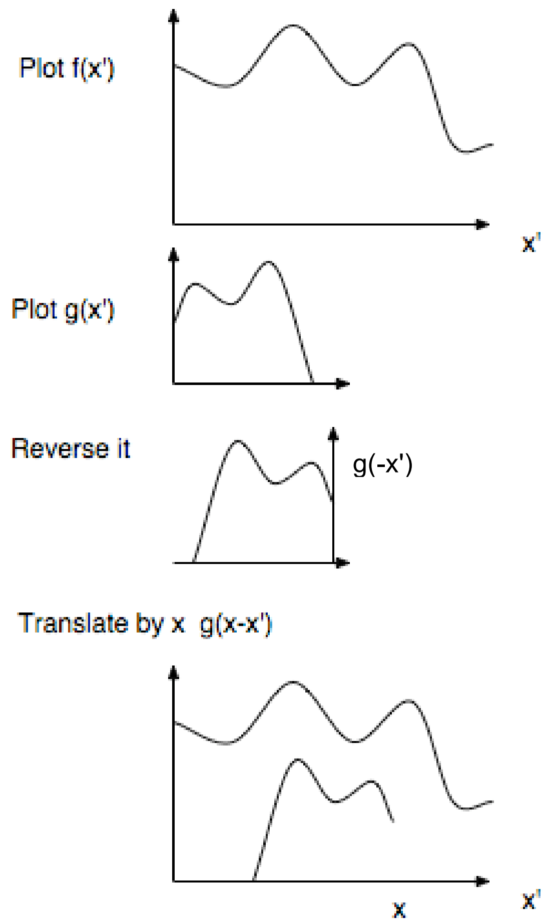
Can consider aperture functions as convolutions of simpler ones. e.g.;



5.2.3 Formal definition of convolution

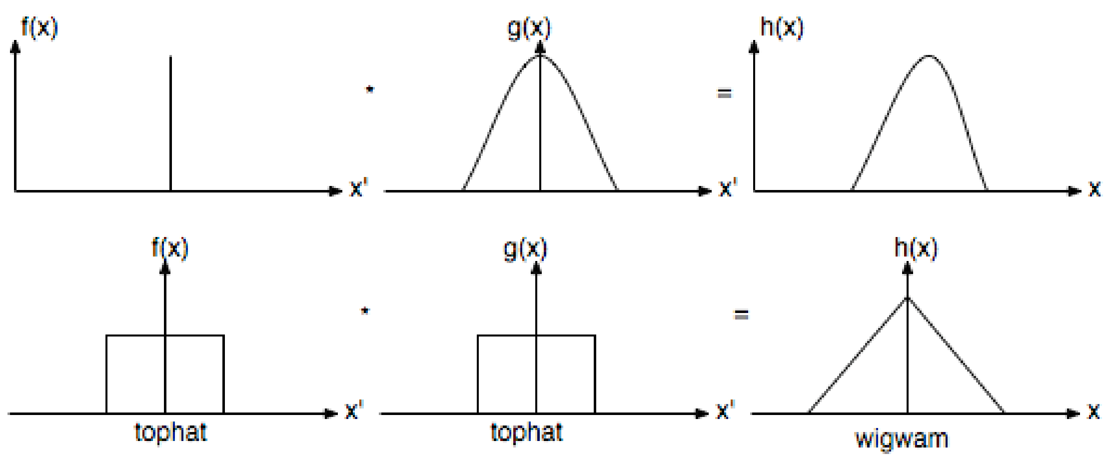
Convolution of two functions $f(x)$, $g(x)$ defined as $h(x) = \int_{-\infty}^{\infty} f(x')g(x-x')dx'$.

Geometric interpretation;



Take the area under $f(x')g(x-x')$ → this is the value of $h(x)$. Then move onto the next value of x.

Suppose;



5.2.4 Convolution Theorem

The fourier transform of the convolution of two functions $f(x)$ and $g(x)$ is the product of the individual fourier transforms $F(u)$ and $G(u)$.

i.e. $f(x)*g(x)=h(x)$ → $F(u)G(u)=H(u)$

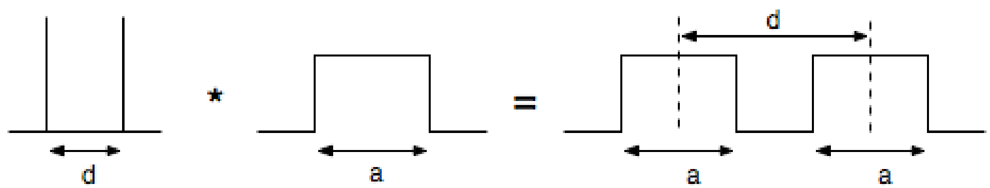
($f(x)$ ←FT→ $F(u)$ etc)

Proof;

$$x''=x-x'$$
$$dx''=dx$$
$$h(x)=\int_{-\infty}^{\infty} f(x')g(x-x')dx'$$
$$H(u)=\int_{-\infty}^{\infty} h(x)e^{iux}dx=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')g(x-x')dx'dxe^{iux}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')g(x'')dx'dx''e^{iu(x'+x'')}$$
$$=\int_{-\infty}^{\infty} f(x')e^{iux'}dx'\int_{-\infty}^{\infty} g(x'')e^{iux''}dx''$$
$$=F(u)G(u)$$

5.2.5 Application; two wide slits

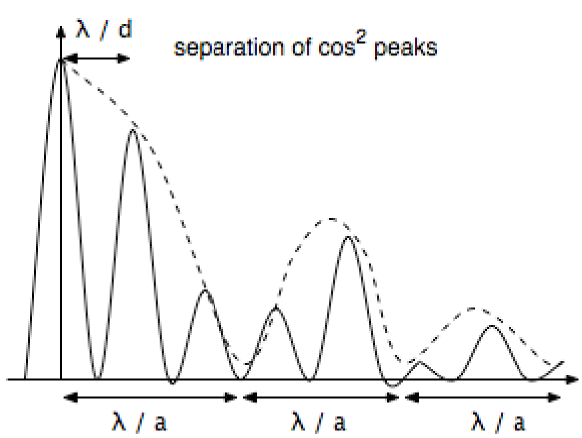
Consider two wide slits as convolution of two narrow slits with a single wide slit.



CT implies amplitude diffraction pattern is product of two narrow slit pattern with a wide slit pattern.

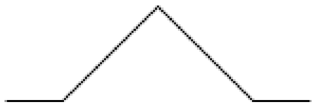
i.e. $A_{obs}=A_{wide}(\theta)xA_{\perp\perp}(\theta)$

$$I_{obs}=I_{wide}(\theta)xI_{\perp\perp}(\theta)$$
$$=I_o\,\text{sinc}^2\left(\frac{\pi a\sin\theta}{\lambda}\right)\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)$$

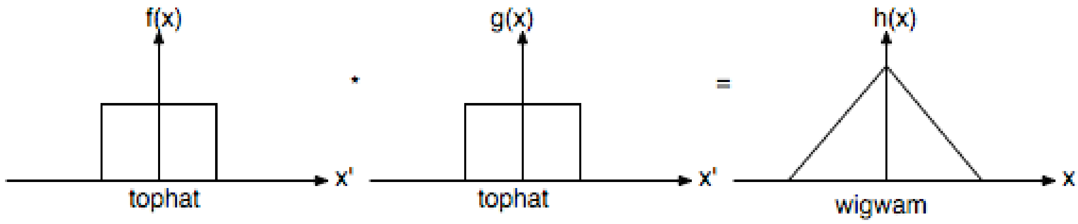


e.g. if $d=3a$, then the third, sixth and ninth $\cos^2$ fringes are absent. → diffraction grating; N wide slits.

5.2.6 Application; tapered slit



→ amplitude sinc^2, intensity sinc^4.



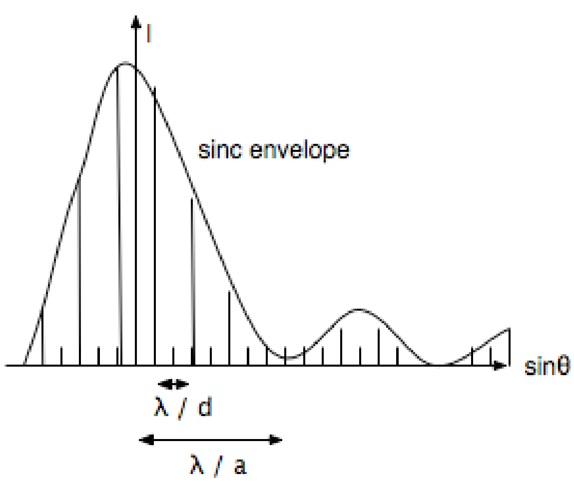
The two tophats each give sinc → the wigwam is just $\text{sinc} * \text{sinc} = \text{sinc}^2$.

5.3 Diffraction Grating
5.3.1 Intensity Pattern
The diffraction grating is a series of slits, which may be quite wide.
Suppose for a start that it consists of N infinitely narrow slits.
From §4.2.4;

Intensity = $E_o^{\,2}\dfrac{\sin^2\left(\dfrac{N\phi}{2}\right)}{\sin^2\dfrac{\phi}{2}}$ where $\phi = \dfrac{2\pi}{\lambda} d\sin\theta$ = phase difference between successive slots.

If the individual slits are not infinitely narrow, we have N wide slits. This can be considered as a convolution of N narrow slits and 1 wide slit.
We know the amplitude of the narrow slits, and that of the wide slit. So we can multiply the two intensity patterns to get the diffraction pattern.

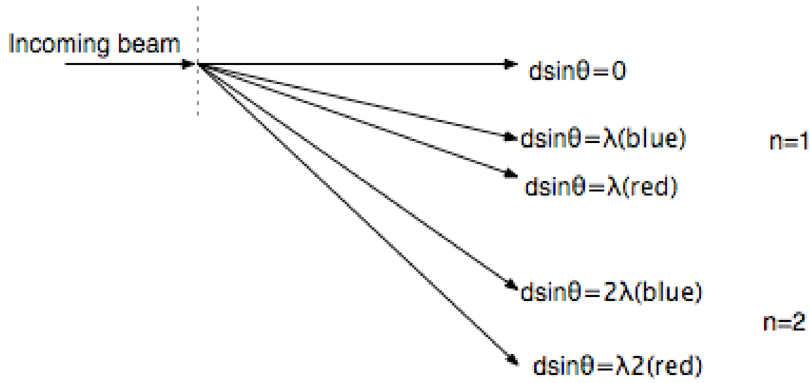
$$I = I_o\,\text{sinc}^2\left(\frac{\pi a\sin\theta}{\lambda}\right)\frac{\sin^2\left(\dfrac{N\pi d\sin\theta}{\lambda}\right)}{\sin^2\left(\dfrac{\pi d\sin\theta}{\lambda}\right)}$$



Separation of peaks such that $d\sin\theta = n\lambda$. "Missing orders" whenever $\dfrac{m\lambda}{d} = \dfrac{\lambda}{a}$ where m is an integer.

5.3.2 Comments on pattern
If N is larger, the maxima will get sharper / narrower.
Grating pattern maxima spacing is inversely proportional to the line spacing on the grating.
Amplitude of maxima falls off if the width of the lines on the grating is not 0.

5.3.3 Diffraction grating as spectrometer
The separation of the maxima will be different for different wavelengths.



Second order is twice as wide, hence you get better spectral resolution.

5.3.3.1 Dispersion

$\frac{d\theta}{d\lambda}$ is the rate of change of angle with λ.

$d\sin\theta = m\lambda$

$d\cos\theta d\theta = md\lambda$

$\frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}$

→ θ small, $\frac{d\theta}{d\lambda} \approx \frac{m}{d}$

5.3.3.2 Free Spectral Range
Don't want overlap of first order with second order fringes.
Overlap happens if $\lambda_{red}m = \lambda_{blue}(m+1)$.

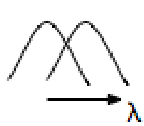
→ $m\Delta\lambda \approx \lambda$ where $\Delta\lambda = \lambda_{red} - \lambda_{blue}$ and $\lambda = \frac{\lambda_{red} - \lambda_{blue}}{2}$

So $\Delta\lambda = \frac{\lambda}{m}$

Typically $m \sim 2{,}3$

$\Delta\lambda \approx 100 - 200nm$

5.3.3.3 Resolving Power



Just resolved if the 1st minimum of one overlaps the peak of the other.
How wide are the diffraction peaks?

$$I = I_o \frac{\sin^2\left(\frac{N\pi d \sin\theta}{\lambda}\right)}{\sin^2\left(\frac{\pi d \sin\theta}{\lambda}\right)}$$

(forgetting the slit width)
Width of the peaks is determined by the top part of this equation, as the bottom one varies slowly.

First minimum is $\frac{\pi}{N}$ away in units of $\frac{\pi d \sin\theta}{\lambda}$. → $\sin\theta = \frac{\lambda}{Nd} + \frac{m\lambda}{d}$

At peak $d\sin\theta = m\lambda$. 1st minimum $Nd\sin\theta = (Nm+1)\lambda$.

Suppose that you have two lines λ and $\lambda + \delta\lambda$. The peak of one wants to land on the minimum of the next.

$$\frac{m\lambda}{d} = \sin\theta = \frac{(Nm+1)(\lambda+\delta\lambda)}{Nd}$$

→ $Nm\lambda = Nm\lambda + Nm\delta\lambda + \lambda + \delta\lambda$

$\delta\lambda$ can be neglected, as it is small compared to the other factors.

$$\frac{\delta\lambda}{\lambda} = \frac{1}{Nm} = \frac{1}{resolving\ power}$$

$\frac{\lambda}{\delta\lambda}$ is typically about $\frac{1}{Nm} \approx 10^4$.

5.3.4 Comparison with FPE

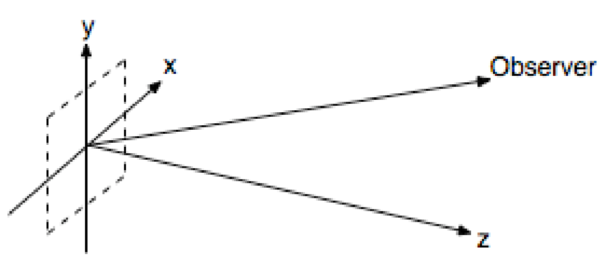
	Diffraction Grating	Fabry-Perot Etalon
Resolving power $\frac{\lambda}{\delta\lambda}$	10^4	$\frac{\pi m\sqrt{\rho}}{1-\rho} \approx 10^6$
Free Spectral Range	$\sim 100nm$	$0.01nm$
Finesse $\frac{\Delta\lambda}{\delta\lambda}$	$\frac{\lambda}{m}\frac{Nm}{\lambda} = N \sim 10^4 - 10^5$	~ 100

The major disadvantage is that the diffraction grating has a much lower resolving power. It does however have a wider spectral range, which is generally much more important.

5.4 2D Diffraction

So far we have had 1D apertures, and a path delay given by $x\sin\theta$.
In general, if you have a 2D aperture;
$\theta =$ angle to y-z plane
$\phi =$ angle to x-z plane.

Distances x, y across aperture, path delay $x\sin\theta + y\sin\phi$.



$$\left|E(\theta,\phi)\right| = \iint E(x,y) e^{i\frac{2\pi}{\lambda}(x\sin\theta + y\sin\phi)} dxdy$$

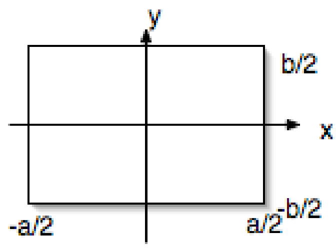
We usually define $u = \frac{2\pi}{\lambda}\sin\theta$ and $v = \frac{2\pi}{\lambda}\sin\phi$

So;

$E(u,v) = \iint E(x,y)e^{i(ux+vy)}dxdy$

This integral is easy or impossible depending on the limits. (i.e. the shape of the aperture)

5.4.1 Rectangle aperture (easy case)



Phase difference $= \frac{2\pi}{\lambda}x\sin\theta + \frac{2\pi}{\lambda}y\sin\phi = ux + vy$

$A(\theta,\phi) = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} E(x,y)e^{i(ux+vy)}dxdy$

Limits are independent → easy.

If constant illumination;

$E_o^{\,2}\int_{-a/2}^{a/2} e^{iux}dx\int_{-b/2}^{b/2} e^{ivy}dy$

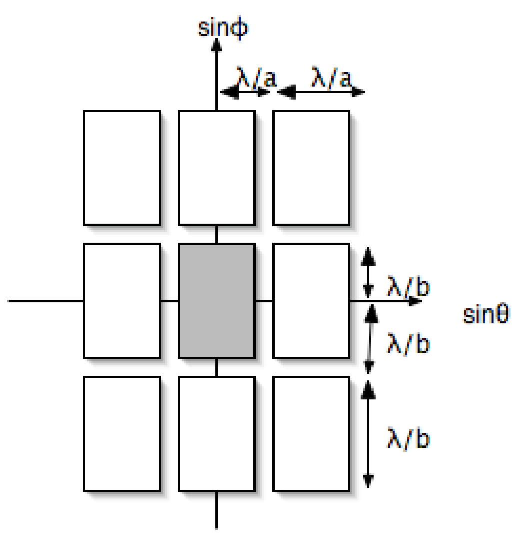
Already done;

(wide slit)

$= E_o^{\,2}ab\,\mathrm{sinc}\left(\frac{ua}{2}\right)\mathrm{sinc}\left(\frac{vb}{2}\right)$

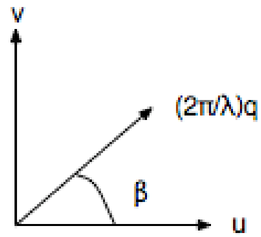
The intensity is proportional to $\mathrm{sinc}^2\left(\frac{ua}{2}\right)\mathrm{sinc}^2\left(\frac{vb}{2}\right)$.

$u \propto \sin\theta$ and $v \propto \sin\phi$.



= PSF of rectangular telescope.

5.4.2 Circular aperture
This is difficult as x-limits depend on y and vice-versa.
→ polar coordinates.
Aperture (x,y) → $x = p\cos\alpha$ and $y = p\sin\alpha$.
These parameterise distances across observer plane.



$$u = \frac{2\pi}{\lambda} q \cos\beta$$

$$v = \frac{2\pi}{\lambda} q \sin\beta$$

$$A(q,\beta) = \iint f(x,y) e^{i(ux+vy)} dxdy$$

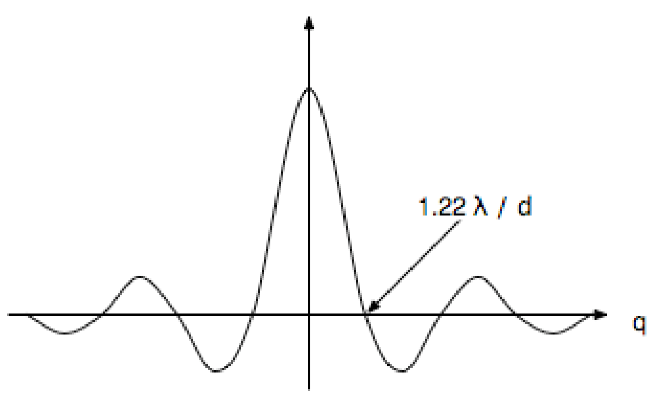
$$= \int_0^{a/2} \int_0^{2\pi} e^{i\frac{2\pi}{\lambda}(qp\cos\beta\cos\alpha + qp\sin\beta\sin\alpha)} dp.pd\alpha$$

$$= \int_0^{a/2} \int_0^{2\pi} e^{ikqp\cos(\alpha-\beta)} pdpd\alpha$$

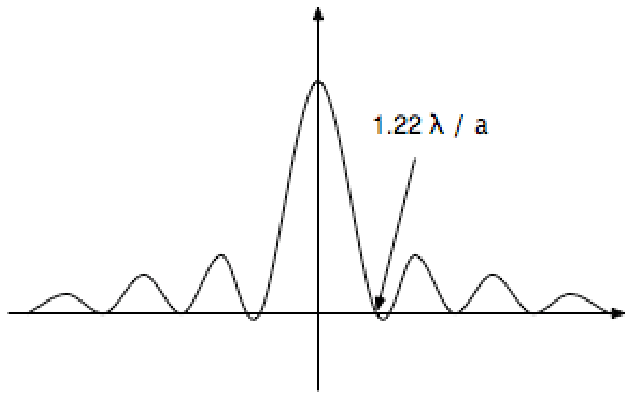
Integral turns out to involve a Bessel function J_1.

$$A(q,\beta) \propto \left[\frac{2J_1\left(\frac{2\pi aq}{\lambda}\right)}{\frac{2\pi aq}{\lambda}} \right]$$

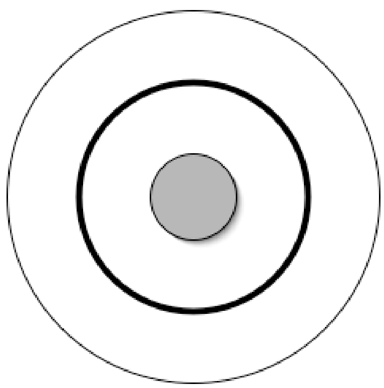
(do not learn this…)



q is the angular distance from the center.
→ intensity pattern



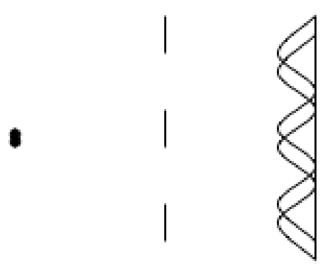
in 2D:



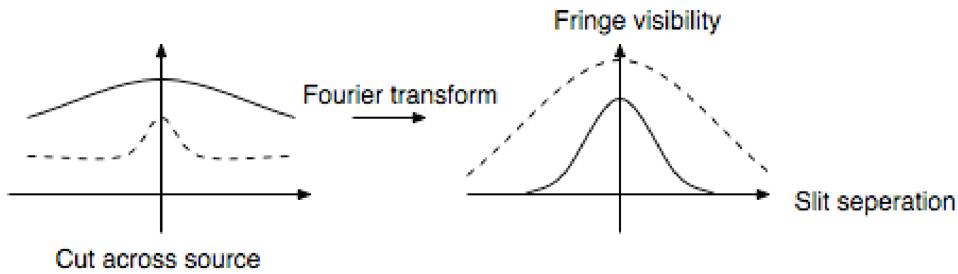
This is known as the Airy function = PSF of a telescope.
→ resolution of telescope; "Rayleigh resolution" if the peak of one star lands on the first Airy minimum of another.

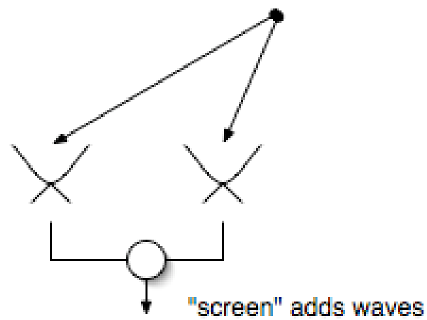
→ $1.22\frac{\lambda}{a}$ is the resolution in RADIANS.

5.5 Radio interferometry
For a single telescope → 10 arcminutes.



Each bit of the source can be considered as a separate radiator. It will the shift the fringe pattern a little. If the source is big enough, then the fringes will wash out.
If the slits are wide apart, then in general phase differences get bigger. → smaller source will still wash out the fringes.



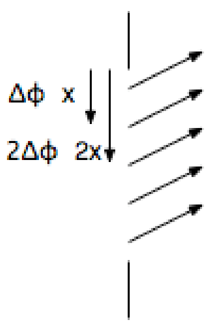


Fringe visibility (telescope separation) ← FT→ map of sky.

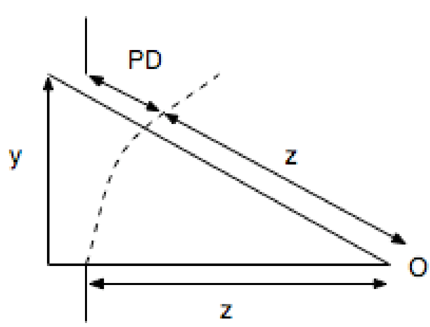
5.6 Fresnel diffraction

5.6.1 Fraunhofer & Fresnel

- Fraunhofer - assumed we were talking about far-field diffraction, so the phase varies linearly across the aperture.



If near-field, you can't assume this.



$PD = \sqrt{y^2 + z^2} - z$

If $y << z$, $PD = z\sqrt{1+\frac{y^2}{z^2}} - z \approx z\left(1+\frac{y^2}{2z^2}\right) - z \approx \frac{y^2}{2z}$

(through binomial)

→ phase difference between those two is $\frac{2\pi}{\lambda}\frac{y^2}{2z} = \frac{\pi y^2}{\lambda z}$

→ varies quadratically across the aperture.
When do we have to worry about this?

→ path difference i.e. Fraunhofer diffraction only if $z >> \frac{y^2}{\lambda}$. This is known as the Rayleigh distance.

Examples;

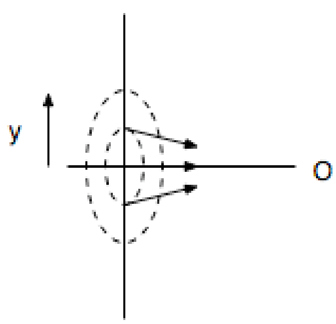
$y = 1mm$, $\lambda \sim 500nm$ → $z = 2m$.

$y = 100m$, $\lambda = 0.1m$ → $10^5 m$

5.6.2 Fresnel diffraction

Problems;
- Quadratic phase variation (i.e. non-linear.)
- Amplitudes vary due to different propagation distances.
- Obliquity factor $\frac{1}{2}(1+\cos\theta)$ (detailed explanation is beyond this course, and is covered in an appendix of Hecht).

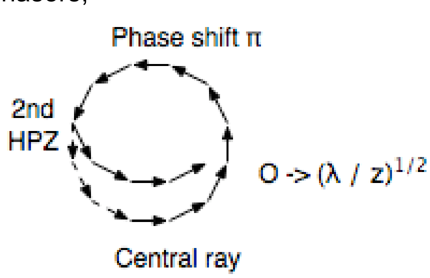
5.6.3 Circular apertures, on axis



- Close to the axis, waves reinforce each other until the phase difference $y^2/2z$ from the central wave becomes $\lambda/2$, which corresponds to π phase shift.

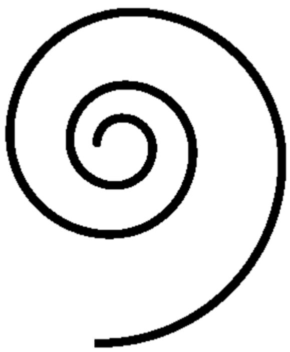
i.e. $0 < y < \sqrt{\lambda z}$

Phasors;



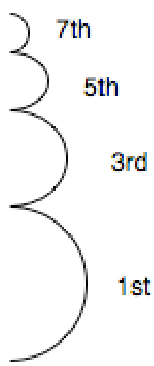
This is known as the first Half-Period Zone, known as a Fresnel HPZ.

As zones further out;
- Zones get bigger; more emission.
BUT
- Amplitudes will get smaller.
BUT
- Obliquity factor will lower amplitude.

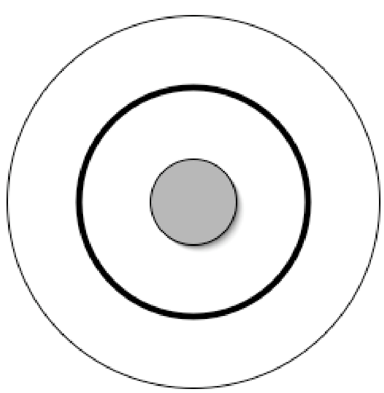


HPZ every time the spiral crosses a central x axis.

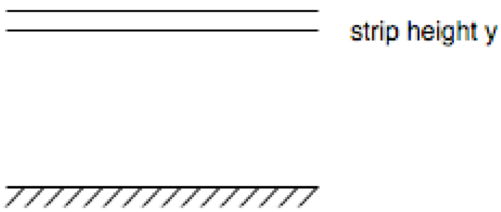
HPZ1 gives twice the undestructed amplitude.
- Block 1st HPZ. → amplitude = unobstructed.
- Block alternate zones.



- Very bright spot on axis. (Fresnel lens). Destructive part is gone, hence only left with the constructive.



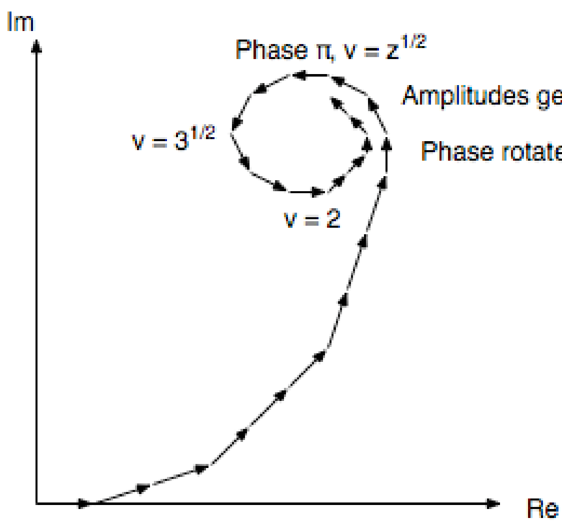
5.6.4 Straight edges and slits



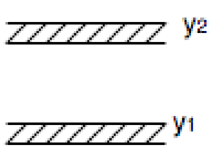
Phase differences; at a height of y, phase difference $\frac{2\pi}{\lambda}\frac{y^2}{2z} = \frac{\pi y^2}{\lambda z}$.

Let $v = y\sqrt{\frac{2}{\lambda z}}$.

At height y, phase difference $= \frac{1}{2}\pi v^2$.



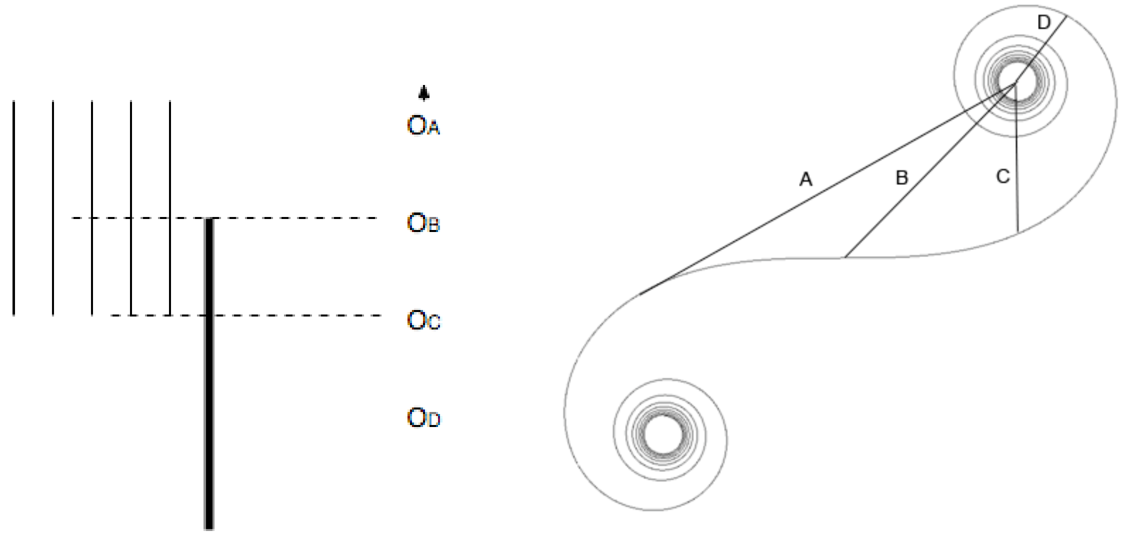
Add phasors. → Cornu spiral.
Hence can produce resultant amplitudes for waves from height $y_1 \to y_2$.

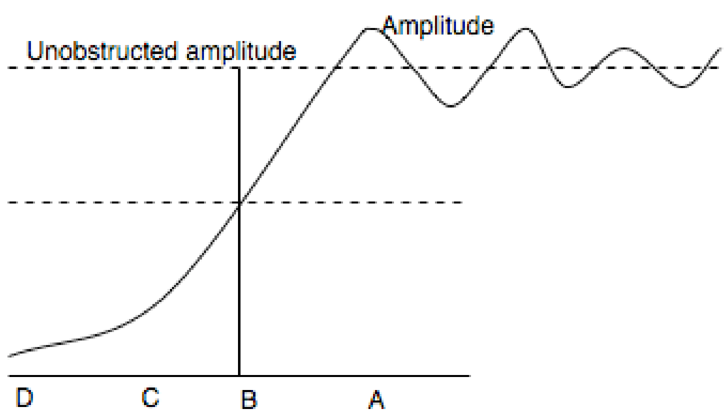


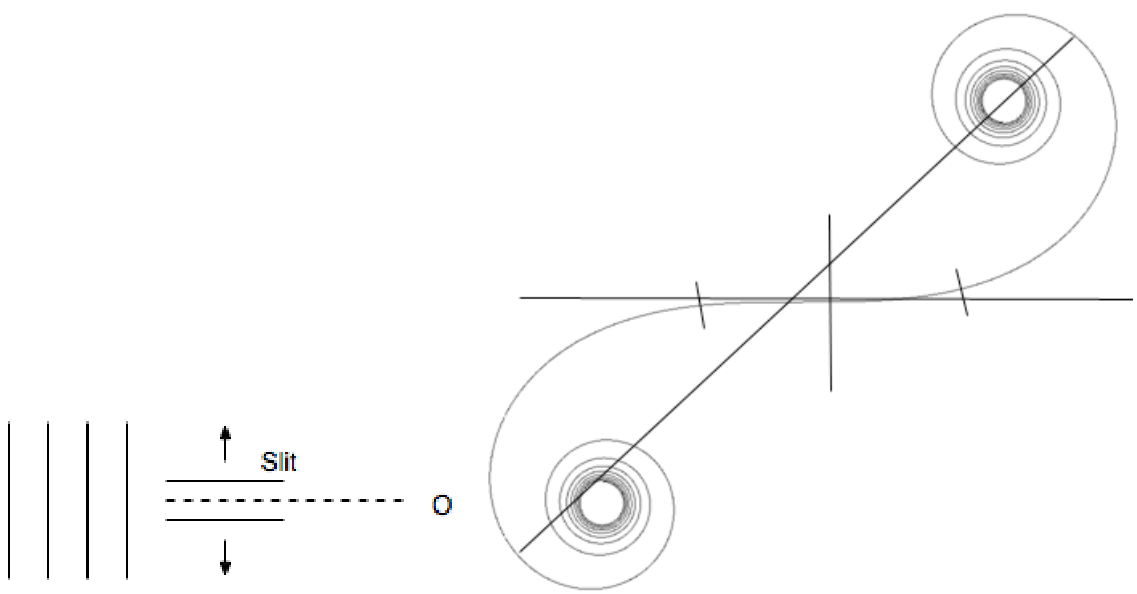
→ v_1, v_2 correspond to y_1, y_2.

e.g. $v_1 = 1$, $v_2 = 2$.

Consider the case where we have a semi-infinite obstruction, and we position ourselves at several different points.

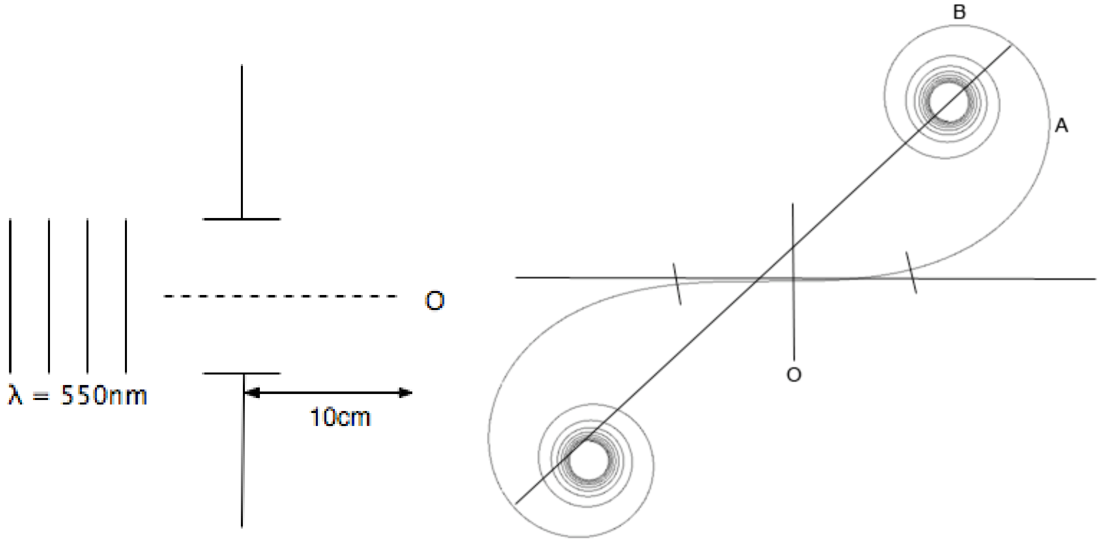






Slit slowly gets wider; amplitude follows the Cornu spiral.

Exam question;



<u>Estimate</u> the slit width at which the light intensity is at a maximum.

At A, phase shift $\frac{1}{2}\pi v^2 = \frac{\pi}{2}$. At B, phase shift $= \pi$.

Where are the points A and B?

$$v = y\sqrt{\frac{2}{\lambda z}}$$

A occurs at the point where the phase shift is $\frac{\pi y^2}{\lambda z} = \frac{\pi}{2}$ i.e. $y = \sqrt{\frac{\lambda z}{2}}$.

Put in numbers $y = \sqrt{\frac{550x10^{-9}x0.1}{2}} = 0.166mm$

At B, phase shift is π;

$$\frac{\pi y^2}{\lambda z} = \pi$$

$$y = \sqrt{\lambda z} = 0.234mm$$

The maximum point is around half the distance between A and B. So use the half-way point between the above calculations. half-width of the slit will be around $0.2mm$, so the slit width is $0.4mm$.