

1. Functions and Graphs

1.1 Functions

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

$$y = L(x)$$

y = dependant variable

x = independent variable.

If y has a unique value for a certain value of x \rightarrow y is 'single-valued'.

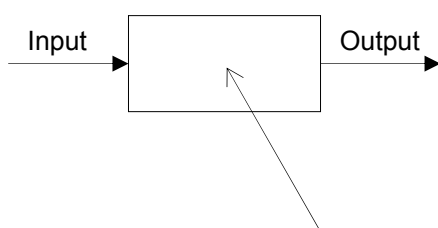
We can use a formula to describe the function.

e.g. $y=mx+c=f(x)$

$$ax+by+c$$

$$p=qf+\Phi$$

$$I=V/R \rightarrow I=f(V,R)$$



Some functions can only have a certain range of values that the dependant variable can take on.

$y=f(x)$ y=dependant.

e.g. $E(t) = A \cos \omega t$

$$-1 \leq \cos \omega t \leq 1$$

$$-A \leq E(t) \leq A$$

Sometimes an independent variable can only have a certain domain.

e.g. $f(x)=\ln(x)$

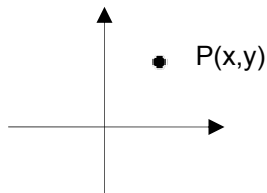
$$x>0$$

Other useful properties:

The zero's of a function – where the dependant variable = 0 (i.e. $f(x)=0$)

The intercept is when the independent variable = 0 (i.e. $f(0)$)

1.2 Cartesian Coordinate System & Graphs



2D:

y-axis - Ordinate

x-axis – Abscissa

Things to help when drawing a graph:

1. Intercepts and Zeros

2. Asymptotes – does the function tend to a specific value?

3. Infinities
4. Gradient (1st derivative)
5. Gradient (2nd derivative)
6. Symmetry
7. Range & Domain

e.g. $p = \cos \theta$

$$i) \quad p(0) = 1, p(\theta) = 0, \theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$$

$$ii) \quad \frac{dp}{d\theta} = -\sin \theta$$

When $\theta=0$, $-\sin \theta=0$

When $\theta=\pi/2$, $-\sin \theta=-1$

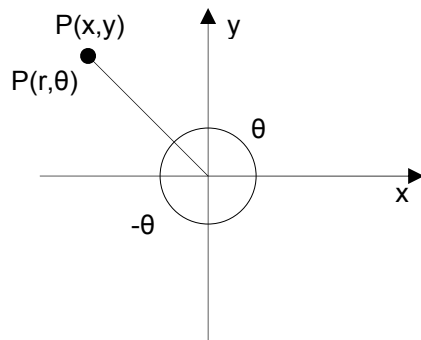
$$iv) \quad \frac{d^2 p}{d\theta^2} = -\cos \theta$$

vi) Symmetric about p-axis

If $f(x) = f(-x) \rightarrow$ Even function, e.g. $\cos \theta$ is even.

If $f(x) = -f(-x) \rightarrow$ Odd function, e.g. $\sin \theta$ is odd.

1.3 Polar Functions



$P(x,y) \rightarrow y=f(x)$

$P(r,\theta) \rightarrow r=f(\theta)$

$$\overline{OP} = r = \sqrt{x^2 + y^2} > 0$$

θ is the polar angle.

Relationships between x , y , r , θ .

$$x = r \cos \theta - 1.1$$

$$y = r \sin \theta - 1.2$$

$$r = \sqrt{x^2 + y^2} - 1.3$$

N.B. θ is not unique – we can go around the circle again to get to the point.

\rightarrow Polar angle = $\theta + n2\pi$ $n = \text{integer}$.

e.g. $(-3,4)$

$$r=5$$

$$\sin \theta = 4/5 = 0.8$$

$$\cos \theta = -3/5 = -0.6$$

$$\arcsin(0.8) = 53.13^\circ$$

$$\arccos(-0.6) = 126.87^\circ$$

$$\sin(126.87) = 0.8$$

$$(-3,4) \rightarrow (5, 126.87^\circ)$$

e.g. Equation of a circle

$$x^2 + y^2 = r^2$$

$$y = f(x) = \pm\sqrt{r^2 - x^2}$$

Polar coordinates $r=c$ (where c = radius of circle)

e.g. tracks on a CD

$$r = b - \frac{h}{2\pi}\theta$$

1.4 Specific Functions

1.4.1 Trigonometry Functions

$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

$$\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$

$$\sec \theta \equiv \frac{1}{\cos \theta}$$

$$\operatorname{cosec} \theta \equiv \frac{1}{\sin \theta}$$

$$\cot \theta \equiv \frac{1}{\tan \theta}$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

Nomenclature: We will use $\arcsin(x)$ and not $\sin^{-1}(x)$.

Trig Identities

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin^2 x + \cos^2 x = 1$$

Useful ones are:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

Harmonic functions

Take $f(u) = a \cos u + b \sin u$ (a, b constants)

$$\text{Let } c = \sqrt{a^2 + b^2}$$

$$\rightarrow f(u) = c \left[\frac{a}{c} \cos u + \frac{b}{c} \sin u \right]$$

Let Φ so that $\cos \Phi = a/c$

$$\sin \Phi = b/c$$

$$\rightarrow f(u) = c [\cos \Phi \cos u - \sin \Phi \sin u]$$

$$\rightarrow f(u) = c \cdot \cos(u + \Phi)$$

With harmonic functions, it is normal to put a constant in front of u .

$$\rightarrow f(u) = \cos(ku + \Phi)$$

From this, we can define frequency, period, amplitude and phase.

e.g.:

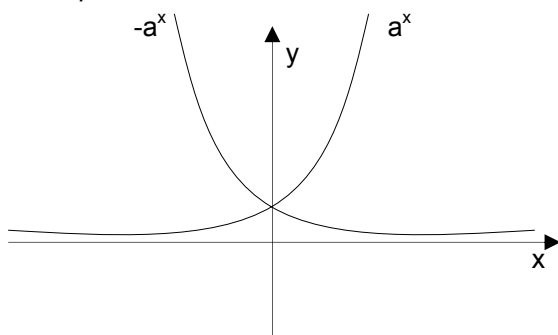
$$\text{Frequency}(v) = \frac{1}{k}$$

$$\text{Period} = \frac{1}{v}$$

$$\text{Amplitude} = c$$

$$\text{Phase} = \Phi$$

1.4.2 Exponential Functions



$y = a^x$ where a is a positive constant.

This is an exponential function of base a .

When $x=0$, $a^x=1$

When $x \rightarrow \infty$, $a^x \rightarrow \infty$

When $x \rightarrow -\infty$, $a^x \rightarrow 0$

Special case when $a=e$ (2.7182....)

$y=f(x)=e^x$ [or $\exp(x)$ or $\exp x$]

This has the properties that when $x=0$, $g=1$.

\rightarrow Implies that $\frac{dy}{dx} \equiv \frac{de^x}{dx} = e^x - 1.6$

$$\frac{de^{kx}}{dx} = ke^x - 1.7$$

$\ln \equiv \log_e$

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^a) = a \ln x$$

$$e^{\ln x} = x$$

$$\ln 1 = 0$$

$$\ln e = 1$$

2. Complex Numbers

2.1 Introduction

Consider the quadratic equation $x^2 - 2x + 2 = 0$. What are the solutions?

$a=1, b=-2, c=2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = 1 \pm \sqrt{-1}$$

There is no ordinary number whose square is -1 . We call $\sqrt{-1}$ an 'imaginary' number, and use the symbol i (not to be confused with \hat{i} or i for current. Also, engineers use j rather than i .)

Our solutions can now be written $x = 1 \pm i$

$x_2 - iy_2$ is called the complex conjugate of $x_2 + iy_2$. It is denoted by z^* .

To find the complex conjugate, replace all i 's with $-i$'s.

NB: you can always equate the real or imaginary parts on either side of an equation.

2.2 Standard Form

The standard form of a complex number is:

$$z = x + iy$$

(x and y are real numbers).

The real part of z is $x \rightarrow x = \text{Re}\{z\}$

The imaginary part of z is $y \rightarrow y = \text{Im}\{z\}$

(Think of z as having components x and $y \rightarrow (x, y)$)

If a complex number is not in standard form, then it cannot be used.

2.3 Manipulation of complex numbers

2.3.1 Addition

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z + z^* = 2x = 2\text{Re}\{z\}$$

2.3.2 Subtraction

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z - z^* = 2iy = 2i\text{Im}\{z\}$$

2.3.3 Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$z_1 z_2 = x_1 x_2 + x_1 i y_2 + i y_1 x_2 + i^2 y_1 y_2$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Note that $i^2 = -1$

2.3.4 Division

Note that we cannot divide straight away, as $\frac{1}{x_2 + iy_2}$ is not in standard form.

$$\frac{z_1}{z_2} \equiv z_1 \frac{1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$\frac{1}{z_2} = \frac{1}{x_2 + iy_2} = \frac{(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_2 - iy_2)}{x_2^2 + y_2^2} = \left(\frac{x_2}{x_2^2 + y_2^2} \right) - i \left(\frac{y_2}{x_2^2 + y_2^2} \right)$$

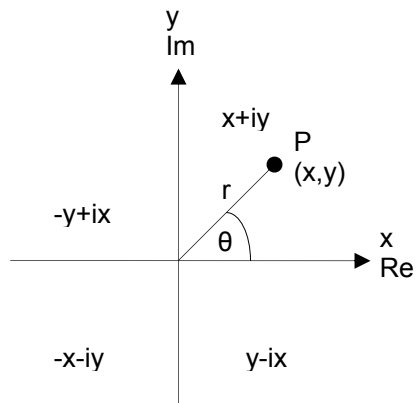
As this is now in standard form, we can get the desired answer.

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2}$$

2.4 Argand Diagrams

Take $z = x + iy$

We can consider this as having two components $\rightarrow (x, y) \rightarrow$ we can plot the number as Cartesian-like coordinates.



The length of the line OP is $r = \sqrt{x^2 + y^2}$.

r is known as the modulus of z (also known as $\text{mod}(z)$ or $|z|$).

$$[|z|^2 = z z^*]$$

The polar angle θ is called the argument of z . It is written as $\text{arg}(z) = \theta$.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)$$

(Polar form of the complex number)

Rotation:

$$\text{Multiply by } i: i(x + iy) = -y + ix$$

$$\text{Multiply by } -i: -i(x + iy) = -y - ix$$

Therefore, a multiplication by i or $-i$ leads to a rotation of $\pi/2$ or $-\pi/2$ in the Argand Diagram.

2.5 Exponential form for complex numbers

$$z = r(\cos \theta + i \sin \theta) = r.f(\theta)$$

$$\frac{df(\theta)}{d\theta} = -\sin \theta + i \cos \theta = i(\cos \theta + i \sin \theta) = i.f(\theta)$$

$$e^{i\theta} \rightarrow \frac{de^{i\theta}}{d\theta} = ie^{i\theta}$$

$$\therefore \cos \theta + i \sin \theta = e^{i\theta}$$

Therefore:

$$\boxed{z = r(\cos \theta + i \sin \theta) = re^{i\theta}} \text{ - Euler's Formula}$$

$$z^* = re^{-i\theta}$$

$$z.z^* = re^{-i\theta} re^{i\theta} = r^2$$

2.5.1 Multiplication and Division in the Exponential Polar Form.

$$z_1 = x_1 + iy_1 = r_1 e^{i\theta}$$

$$z_2 = x_2 + iy_2 = r_2 e^{i\theta}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

2.6 Exponential form of $\sin\theta + i\cos\theta$ + De Moivre's Theorem

$$e^{i\theta} = \cos\theta + i\sin\theta \quad - 2.15$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad - 2.16$$

$$2.15 + 2.16 \rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad - 2.17$$

$$2.15 - 2.16 \rightarrow \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \quad - 2.18$$

→ These can be used to prove many trig identities.

In 2.15, let's replace θ by $n\theta$

$$e^{in\theta} = \cos n\theta + i\sin n\theta = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

$$\boxed{\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n} \quad - 2.19$$

De Moivre's Theorem

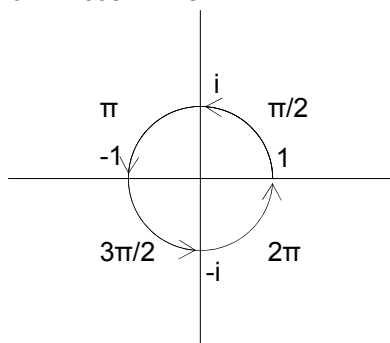
Examples:

1. What does i^i equal?

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$$

$$\left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2} = 4.81$$

2. $e^{i\pi} = \cos\pi + i\sin\pi = -1$



$$e^{i3\pi/2} = -i$$

3. $e^{i2\pi} = 1 = e^{i2n\pi} \rightarrow n = \text{integer}$.

2.6.1 Application of De Moivre's Theorem

Example 2.4

Express $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= (\cos\theta + i\sin\theta)^2 (\cos\theta + i\sin\theta)$$

$$= (\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)(\cos\theta + i\sin\theta)$$

$$= \cos^3\theta - \sin^2\theta\cos\theta + 2i\sin\theta\cos^2\theta + i\sin\theta\cos^2\theta - i\sin^3\theta - 2\cos\theta\sin^2\theta$$

$$= (\cos^3\theta - 2\sin^2\theta\cos\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\text{Real parts: } \cos 3\theta = \cos^3 \theta - 3 \sin^2 \theta \cos \theta$$

$$\text{Imaginary parts: } \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$

How about getting powers of $\sin \theta + \cos \theta$ into $\sin n\theta + \cos n\theta$?

$$\text{Let } z = e^{i\theta}, z^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$z^{-n} = \frac{1}{z^n} = e^{-in\theta} = \cos n\theta - i \sin n\theta$$

$$z^n + \frac{1}{z^n} = (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta) = 2 \cos n\theta \quad - 2.20$$

$$\text{Similarly } z^n - \frac{1}{z^n} = 2i \sin n\theta \quad - 2.21$$

Example 2.5

Take $n=1$ and use 2.21

$$z - \frac{1}{z} = 2i \sin \theta$$

$$\rightarrow \sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$$

$$\sin^2 \theta = \frac{i}{2^3} \left(z - \frac{1}{z} \right)^3 = \frac{i}{8} \left(z^2 - z + \frac{1}{z^2} \right) \left(z - \frac{1}{z} \right)$$

$$= \frac{i}{8} \left(z^3 - 2z + \frac{1}{z} - z + \frac{2}{z} - \frac{1}{z^3} \right)$$

$$= \frac{i}{8} \left(z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \right)$$

$$= \frac{i}{8} \left(z^3 - \frac{1}{z^3} \right) - \frac{3i}{8} \left(z - \frac{1}{z} \right)$$

$$= \frac{i}{8} (2i \sin 3\theta) - \frac{3i}{8} (2i \sin \theta)$$

$$\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin^3 \theta$$

Example 2.6

n th roots of unity

i.e. what are the roots of $z^n=1$?

$$e^{i2\pi} = 1, e^{i2k\pi} = 1, k = 0, 1, 2, \dots$$

$$z^n = e^{i2k\pi}, k = 0, 1, \dots, n-1$$

Take n th root of both sides

$$(z^n)^{1/n} = z = (e^{i2k\pi})^{1/n} = e^{i2k\pi/n}$$

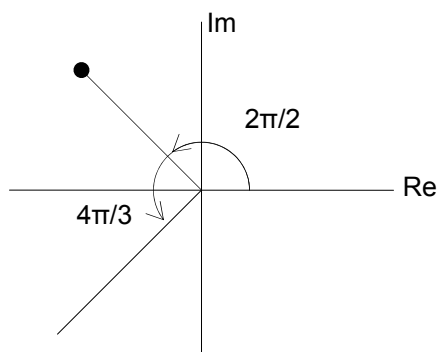
$$z = e^{i2k\pi/n}$$

$$k = 0, z = 1$$

$$k = 1, z = e^{i2\pi/n}$$

$$k = 2, z = e^{i4\pi/n}$$

$$k = n-1, z = e^{i2(n-1)\pi/n}$$



e.g. $n=3$

$z=1$

$$z = e^{2i\pi/3}$$

$$z = e^{4i\pi/3} = e^{-2i\pi/3}$$

De Moivre's Theorem

$$e^{in\theta} = \cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$$

$$e^{i2k\pi}, k = \text{integer}$$

The roots of a polynomial with real coefficients occur in conjugate pairs.

→ Remember a real number is its' own conjugate.

2.6 Hyperbolic Functions

The hyperbolic functions are the complex analogies of the trig functions.

Remember 2.17 and 2.18

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Now:-

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$x \rightarrow ix$ into $\cos \rightarrow$ gives \cosh .

$$\cos(ix) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cosh x$$

i.e.:-

$$\cos(ix) = \cosh x$$

$$i \sinh x = \sin(ix)$$

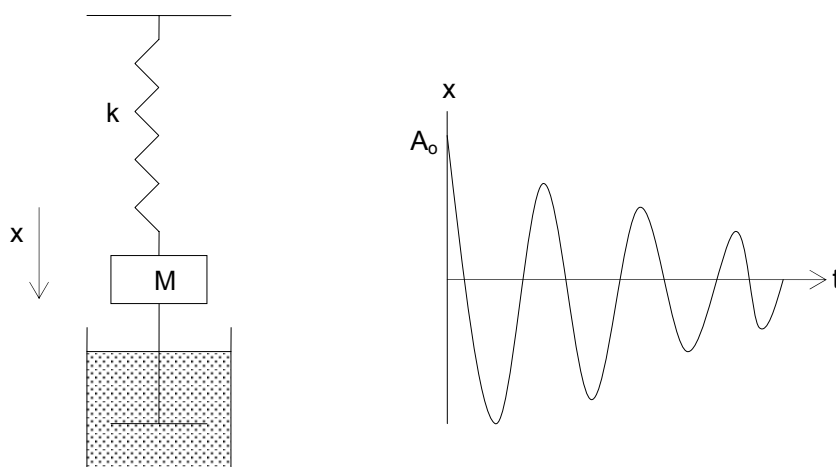
Very simply, we can show:

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

2.7 Applications of Complex Numbers in Physics

Damped Oscillator



Equation of motion:

$$F = -kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}$$

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

[With no damping $\rightarrow \frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \rightarrow$ Oscillation at frequency $\omega_o = \sqrt{\frac{k}{m}}$]

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_o^2 x = 0$$

We want to find $x(t)$

Try $\text{Re}\{z\} = x(t) = \text{Re}\{A_o e^{i\omega t}\}$

$$\frac{dz}{dt} = A_o i \omega e^{i\omega t} = A_o i \omega z$$

$$\frac{d^2z}{dt^2} = A_o \omega^2 e^{i\omega t} = -A_o \omega^2 z$$

Substitute into equation of motion (for z)

$$-\omega^2 z + i \frac{b}{m} \omega z + \omega_o^2 z = 0$$

$$-\omega^2 + i \frac{b}{m} \omega + \omega_o^2 = 0$$

i.e. quadratic in ω .

$$\omega = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

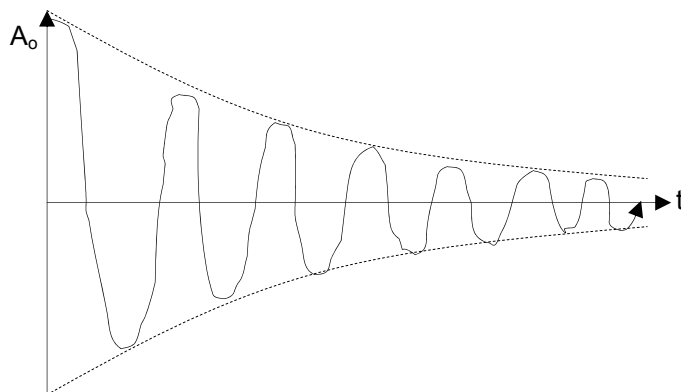
$$\omega = \frac{ib}{2m} \pm \omega'$$

Where $\omega' = \omega_o \sqrt{1 - \left(\frac{b}{2m\omega_o}\right)^2}$

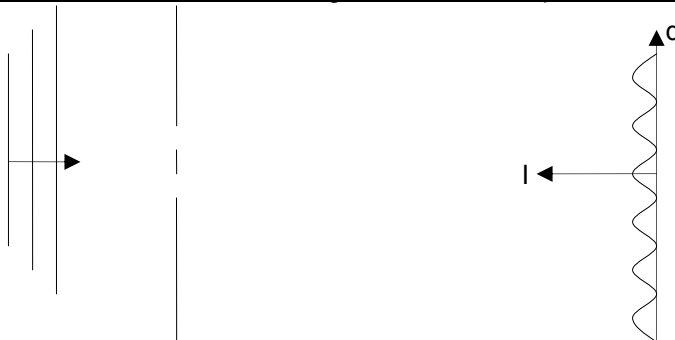
Substitute into $z = A_o e^{i\omega t}$

$$z = A_0 e^{i\left(\frac{ib}{2m} \pm \omega'\right)t} = A_0 e^{\frac{-b}{2m}t} e^{\pm i\omega't}$$

$$x(t) = \text{Re}\{z\} = A_0 e^{\left(\frac{-b}{2m}\right)t} \cos \omega' t$$



Interference of two electromagnetic waves of equal intensity



$$E_1 = A_0 \cos \omega t$$

$$E_2 = A_0 \cos(\omega t + \delta)$$

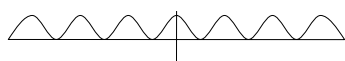
δ accounts for the path distance between two waves

$$E_1 + E_2 = A_0 (\cos \omega t + \cos(\omega t + \delta)) = 2A_0 \cos\left(\omega t + \frac{\delta}{2}\right) \cos \frac{\delta}{2}$$

($\cos\left(\omega t + \frac{\delta}{2}\right)$ is ignored)

$$\text{Intensity} = |E_1 + E_2|^2 = 4A_0^2 \cos^2 \frac{\delta}{2}$$

$$4A_0^2$$



Let $E_1 = \text{Re}\{z_1\} = \text{Re}\{A_0 e^{i\omega t}\}$

$E_2 = \text{Re}\{z_2\} = \text{Re}\{A_0 e^{i[\omega t + \delta]}\}$

$$\begin{aligned}
 z_1 + z_2 &= A_o (e^{i\omega t} + e^{i(\omega t + \delta)}) \\
 I &= |z_1 + z_2|^2 = (z_1 + z_2)(z_1 + z_2)^* \\
 &= A_o^2 (e^{i\omega t} + e^{i(\omega t + \delta)})(e^{-i\omega t} + e^{-i(\omega t + \delta)}) \\
 &= A_o^2 (1 + 1 + e^{i\delta} + e^{-i\delta}) \\
 &\left[\frac{1}{2}(e^{i\delta} + e^{-i\delta}) = \cos \delta \right] \\
 &1 + \cos \delta = 2 \cos^2(\delta / 2) \\
 &= A_o^2 (2 + 2 \cos \delta) \\
 &= 2A_o^2 (1 + \cos \delta) \\
 I &= 4A_o^2 \cos^2(\delta / 2)
 \end{aligned}$$

Complex Induction in Electronic Circuits

Define:

z_r (impedance) = R Resistor

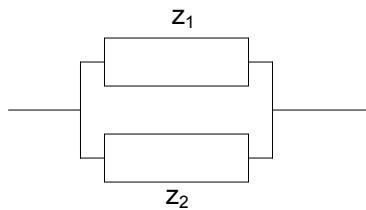
$z_c = \frac{1}{i\omega C}$ Capacitance

$z_L = i\omega L$ Coil

Impedence can be combined like resistors.

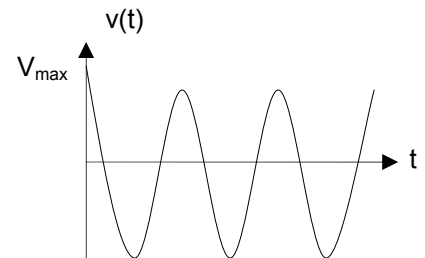
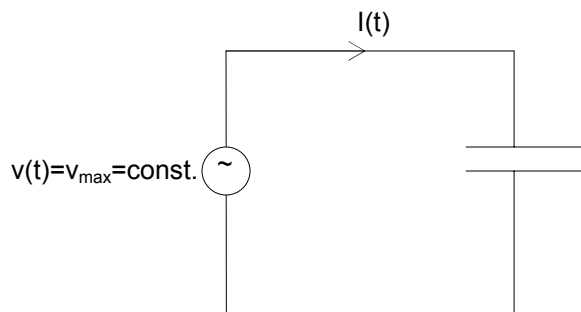


$$z_T = z_1 + z_2$$



$$\frac{1}{z_T} = \frac{1}{z_1} + \frac{1}{z_2}$$

$$\rightarrow z_T = \frac{z_1 z_2}{z_1 + z_2}$$

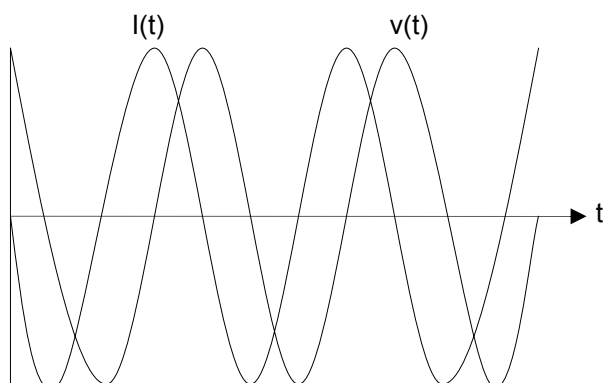


$$I = \frac{V}{R} = \text{Re} \left\{ \frac{v(t)}{z_c} \right\}$$

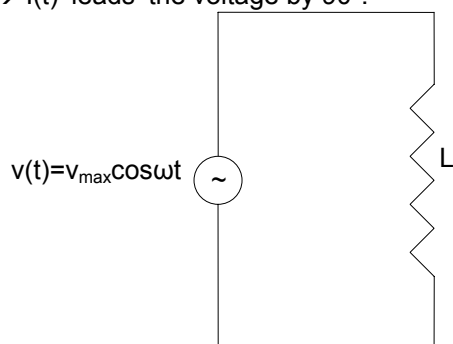
$$\text{Let } v(t) = \text{Re} \left\{ v_{\max} e^{i\omega t} \right\}$$

$$\therefore I(t) = \text{Re} \left\{ \frac{v_{\max} e^{i\omega t}}{1/i\omega C} \right\} = \text{Re} \{ i\omega C v_{\max} e^{i\omega t} \}$$

$$I(t) = \omega v_{\max} C \cdot \text{Re} \{ i \cos \omega t + \sin \omega t \} = -\omega C v_{\max} \sin \omega t$$



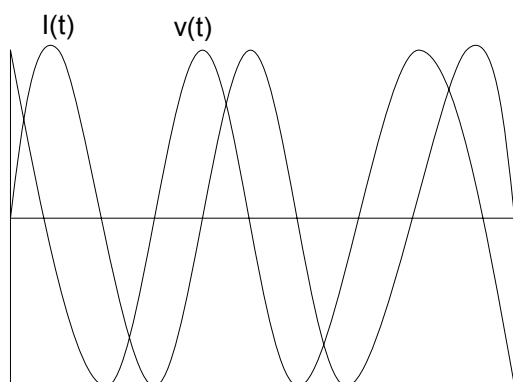
→ $I(t)$ 'leads' the voltage by 90° .



$$I(t) = \text{re} \left\{ \frac{v(t)}{z_L} \right\}$$

$$= \text{Re} \left\{ \frac{v_{\max} e^{i\omega t}}{i\omega L} \right\}$$

$$= \frac{v_{\max}}{\omega L} \sin \omega t$$

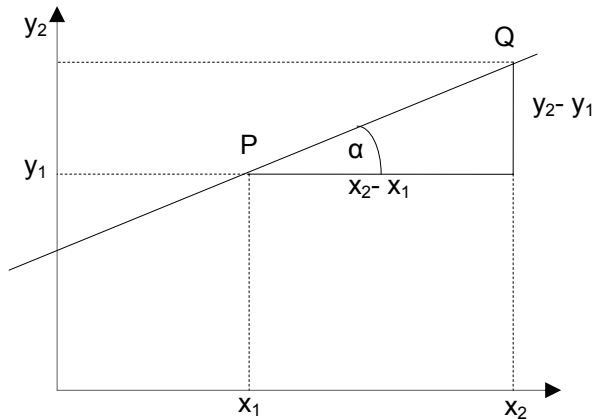


$I(t)$ 'lags' behind $v(t)$ by 90° .

3. Differential Calculus

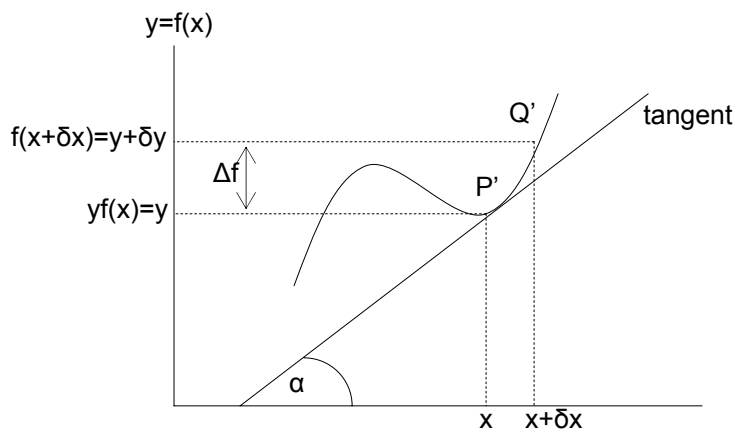
3.1 The Differential

A differential describes the rate of change of a function.



$$\tan \alpha = \frac{y_2 - y_1}{x_2 - x_1} = (\text{const.})$$

$\tan \alpha$ is the gradient



If we take P'+Q' lose together then

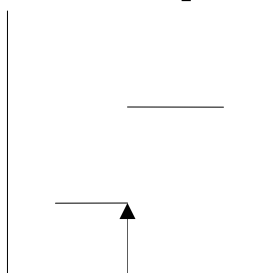
$$\tan \alpha \approx \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{\Delta f}{\Delta x}$$

As Δx gets smaller & smaller the approximation gets better. (Makes no sense when $\Delta x=0 \rightarrow 0/0$)

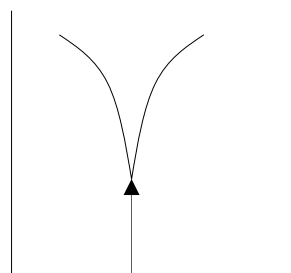
Define the first derivative of a function as:-

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \frac{d}{dx} f(x) \equiv \frac{df}{dx}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{f'(x + \Delta x) - f(x)}{\Delta x} \right] - 3.1$$



No derivative



No 1st derivative

Near to point P', Δf is given by $\Delta f \approx \frac{df(x)}{dx} \Delta x$ - 3.2

The second derivative is the gradient of the gradient.
(i.e. replace $f(x)$ by $f'(x)$ in 3.1)

$$\therefore f''(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{f'(x + \delta x) - f'(x)}{\Delta x} \right] - 3.4$$

$$\equiv \frac{df'(x)}{dx} \equiv \frac{d^2f}{dx^2} - 3.5$$

[Note $\frac{d^2f}{dx^2} \neq \left(\frac{df}{dx}\right)^2$]

$f(x)$	$f'(x)$
x^n	ax^{n-1}
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sinh ax$	$a \cosh ax$
$\cosh ax$	$a \sinh ax$

3.2 Differentiation of products – “product” rule

e.g. $f(x) = x^5 \cos 3x$

Let $u(x) = x^5$, $v(x) = \cos 3x$

$\therefore f(x) = u(x)v(x)$

From first principles:

$$f(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

Take top line

$$\begin{aligned} f(x + \Delta x) - f(x) &= u(x + \Delta x)v(x + \Delta x) - u(x)v(x) \\ &= u(x + \Delta x)v(x + \Delta x) - u(x)v(x) + u(x + \Delta x)v(x) - v(x)u(x + \Delta x) \\ &= u(x + \Delta x)[v(x + \Delta x) - v(x)] + [u(x + \Delta x) - v(x)]v(x) \end{aligned}$$

So now:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{u(x + \Delta x)[v(x + \Delta x) - v(x)] + [u(x + \Delta x) - v(x)]v(x)}{\Delta x} \right]$$

$$\boxed{f'(x) = u(x)v'(x) + u'(x)v(x)} - 3.6$$

Extending to three functions:

$$G(x) = u(x)v(x)w(x)$$

$$\boxed{G'(x) = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)} - 3.7$$

3.3 Function of a function – ‘Chain’ rule

e.g. $f(x) = \sqrt{3x^2 + 1} = \sqrt{u(x)}$

Here $u(x) = 3x^2 + 1$, the square root operates on $v(x)$

Since Δf , Δx , Δu are small

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u} \frac{\Delta u}{\Delta x} - 3.8$$

When Δf , Δx and $\Delta u \rightarrow 0$

$$\frac{df}{dx} = \frac{df}{du} \frac{du}{dx} - 3.9$$

3.4 Implicit Differentiation

So far only differentiated functions of the form $y=f(x)$

$f(x,y)=\text{constant}$

Differentiating this is called implicit differentiation.

→ Differentiate both sides and use the chain rule

- Collect terms to find $\frac{dy}{dx}$

e.g. equation of a circle $f(x,y)=C$

$$x^2 + y^2 = 4$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx} 4$$

$$2x + \frac{dy^2}{dx} = 0$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$$

Check using explicit function $y = \sqrt{4-x^2} = (4-x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

3.5 Differentiating in terms of a parameter

If $x = x(t)$ and $y = y(t)$

$$\text{then } \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} - 3.10$$

3.6 Logarithmic Differentiation

If $y = a^x$ - take logs of both sides then differentiate.

$$y = a^x$$

$$\ln y = x \ln a$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} x \ln a$$

$$\frac{1}{y} \frac{d}{dx} = \ln a$$

$$\frac{dy}{dx} = y \ln a = a^x \ln a$$

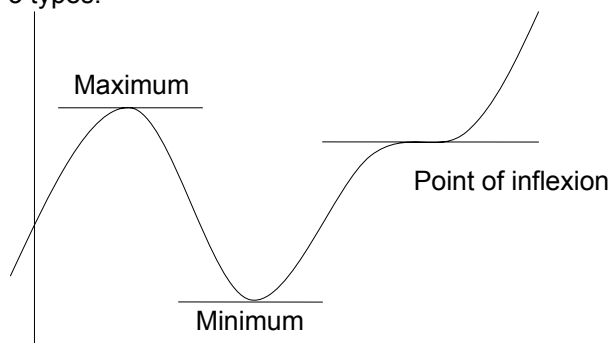
3.7 Inverse function

$$\text{Uses: } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

3.8 Maxima, Minima of points of inflexion

When $\frac{dy}{dx} = 0$ we have a stationary point.

3 types:



For a maximum, $\frac{d^2y}{dx^2} < 0$

For a minimum $\frac{d^2y}{dx^2} > 0$

For a point of inflexion $\frac{d^2y}{dx^2} = 0$ AND $\frac{d^2y}{dx^2}$ must change sign either side.

3.7 Estimating small changes

Remember $f'(x) = \frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$

For small steps $\frac{\Delta y}{\Delta x} \approx f'(x)$ {We normally work with $\frac{\delta y}{\delta x} \approx f'(x)$

$$\delta y = f'(x)\delta x$$

Example 3.10

Estimate the increase in the surface area of a circle if its' radius increases from 10cm to 11cm.

Surface area = $\pi r^2 = A = A(r)$

$$\frac{dA}{dr} = A'(r) = 2\pi r$$

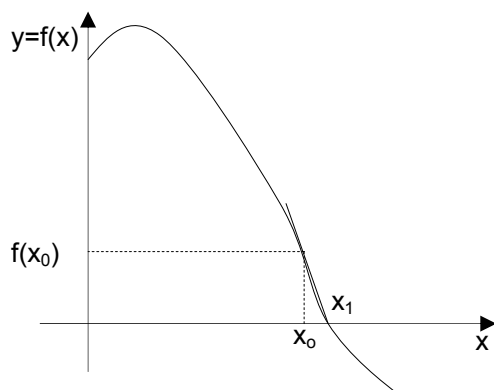
$$\Rightarrow \delta A \approx 2\pi r \delta r = 2\pi \times 0.1 \times 0.01 = 0.0063m^2$$

$$\text{Check: } \Delta A = \pi(0.11)^2 - \pi(0.1)^2 = 0.0066m^2$$

Error is less than 5%.

3.8 The Newton-Raphson method for numerically solving equations

Some equations have no analytical solution, i.e. we can't easily find the roots – in this case we need to guess the solution and improve on this guess until we get to the required accuracy.



- Make an initial guess, say x_0
- Find the gradient at $x_0 \rightarrow f'(x_0)$
- Find where $f'(x_0)$ cuts the x-axis – this is x_1 .
- Repeat until accuracy as required.

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Example: Find a root of $e^{-x} - x = 0$

If $x=0$, $f(x)=1$

If $x=1$, $f(x)=-0.63$

Guess $x_0=0.5$

$$f'(x) = -e^{-x} - 1$$

$$x_1 = 0.5 - \frac{e^{-0.5} - 0.5}{-e^{-0.5} - 1} = 0.5663$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5671$$

$$f(x_2) = 0.0000678$$

4. Series & Expansions

Notation:

Terms $u_1 + u_2 + u_3 + \dots + u_n = S_n = \text{Sum}$. Another expression is $S_n = \sum_{n=1}^N u_n$.

[Be careful: the counting index is often i, j, k, \dots]
The series can be finite or infinite.

4.1 Arithmetic Series

The difference between each term is constant.

$$S_n = a + (a + d) + (a + 2d) + \dots + (a + (N - 1)d) = \sum_{n=1}^{N-1} (a + nd)$$

$$\sum_{n=1}^{N-1} n = \frac{1}{2} N(N - 1)$$

Write down $S_n + S_n$ with the second one backwards.

$$2S_n = a + (a + d) + (a + 2d) + \dots + (a + (N - 1)d) + [a + (N - 1)d] + [a + (N - 2)d] + \dots + a \\ = [a + a + (n - 1)d] + [a + a + (n - 1)d] + \dots + [a + a + (n - 1)d]$$

N terms all the same.

$$2S_n = N[a + a + (n - 1)d]$$

$$S_n = \frac{N}{2} [\text{first.term} + \text{last.term}]$$

First term = a

Last term = $a + (N - 1)d$

Or use:

$$\sum_{n=0}^{N-1} n = \frac{1}{2} N(N - 1)$$

$$\therefore \sum_{n=0}^{N-1} a + nd = \sum_{n=0}^{N-1} a + d \sum_{n=0}^{N-1} n = Na + d \frac{1}{2} N(N - 1) \\ = \frac{N}{2} [a + a + d(N - 1)]$$

Example: sum all even integers between 0 and 500 (inclusive)

$$\sum_{n=0}^{250} 2n = \frac{1}{2} 251(0 + 500) = 62,750$$

(251 terms starting at 0)

or:

$$\sum_{n=0}^{249} = \frac{1}{2} 250(2 + 298 + 2) = 62,750$$

(250 terms starting at 2)

NB: Infinite arithmetic series always increase or decrease indefinitely \rightarrow they diverge.

4.2 Geometric series

This is a series where each term differs from its' predecessor by a common factor.

$$\sum_{n=0}^{N-1} ax^n \text{ Runs from } ax^0 \rightarrow ax^{N-1} \text{ (N terms)}$$

The sum of a geometric series is

$$\sum_{n=0}^{N-1} ax^n = a \left(\frac{1 - x^N}{1 - x} \right) - 4.3$$

Proof: Write down s_n then subtract xs_n from it.

$$S_n = 1 + x + x^2 + \dots + x^{n-1}$$

$$xS_n = x + x^2 + x^3 + x^4 + \dots + x^n$$

$$S_n - xS_n = 1 - x^n$$

$$(1-x)S_n = 1 - x^n$$

$$S_n = \left(\frac{1-x^n}{1-x} \right)$$

Infinite geometric series either diverges or converge.

$$\sum_{n=0}^{\infty} ax^n = \lim_{N \rightarrow \infty} \left(a \left(\frac{1-x^{N+1}}{1-x} \right) \right) = \left(\frac{a}{1-x} \right) \lim_{N \rightarrow \infty} (1-x^{N+1})$$

If $|x| > 1$ the series does not converge, $x > 1$ – diverge. $x < -1$ – oscillates.

$$\text{If } |x| < 1 \text{ then } \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

Example 4.2

$$\text{a) } \sum_{n=0}^6 (0.2)^n \quad \text{b) } \sum_{n=0}^5 e^n$$

a) has 7 terms – $N=7$, $x=0.2$.

$$\sum_{n=0}^6 (0.2)^n = \frac{1-0.2^7}{1-0.2} = 1.248$$

b) $N=6$, $x=3$

$$\sum_{n=0}^5 2^n = \frac{1-3^6}{1-3} = 364$$

4.3 The Binomial Expansion

Used for function of the form $(a+b)^n$.

If n is a positive integer, then:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \dots + b^n \quad - 4.5$$

There are $n+1$ terms and its' symmetrical about a and b .

Special case when $a=1 \rightarrow (1+x)^n$. ($b=x$)

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots + x^n \quad - 4.6$$

$$= \sum_{i=0}^n \frac{n!}{i!(n-i)!} \text{ Coefficient}$$

If $x \ll 1$, we can terminate the series when we have reached the required accuracy.

$(1.01)^8 \equiv (1+0.01)^8$ - may only need a few terms to get to the required accuracy.

See examples sheets & workshops.

4.4 Taylor (And Maclaurin) expansions

– Used to get a series expansion for a function.

– Very useful for approximating the function.

– Function must be continuous, single-valued, and have derivatives up to $f^{(n-1)}(x)$.

$$\left[f'(x), f''(x), \dots, f^{(n)}(x) \right]$$

Consider a polynomial $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

If we let $x=0$

$$P(0) = a_0$$

$$P'(0) = a_1$$

$$P''(0) = 2! a_2$$

$$P^{(n-1)}(0) = (n-1)! a_{n-1}$$

$$P^n(0) = n! a_n$$

We now want to approximate $f(x)$ by $P(x)$ [near to $x=0$]

$$\therefore f'(0) = P'(0)$$

$$f''(0) = P''(0)$$

The coefficients of $P(x)$ are:-

$$a_0 = P(0) = f(0)$$

$$a_1 = \frac{P'(0)}{1!} = \frac{f'(0)}{1!}$$

$$a_2 = \frac{P''(0)}{2!} = \frac{f''(0)}{2!}$$

⋮

$$a_n = \frac{1}{n!} P^{(n)}(0) = \frac{1}{n!} f^{(n)}(0)$$

So, for x sufficiently close to 0, $P(x) \sim f(x)$ and

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n f^{(n)}(0)}{n!} \quad - 4.7$$

This is the Taylor expansion of $f(x)$ about $x=0$ – Maclaurin expansions / series (special case of the Taylor expansion around $x=0$)

If we let the series become infinite, we might expect the approximation to become equality.

→ this may not be the case.

- There will be a range of validity (a range of x) for which the Taylor series converges to $f(x)$

Common Taylor expansions (about $x=0$)

i. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$ [Valid $-x < x < 1$]

ii. $e^x = + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ [Valid for all x]

$$\left[x = 1, e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right]$$

iii. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ [Valid for all x in radians]

iv. $\cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ [Valid for all x in radians]

v. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ [Valid for $-1 \leq x \leq 1$]

vi. $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \dots$ [Valid for $-1 \leq x \leq 1$, any n]

These are very useful for approximating functions.

e.g. small angle approximation for $\sin x = x$, when $x \ll 1$

Can be used to determine the values of functions of seemingly indeterminate places.

e.g. $\frac{\sin x}{x}$ What is this when $x=0$?

$$\frac{\sin 0}{0} = \frac{0}{0}!!!$$

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots$$

When $x=0$

$$\frac{\sin x}{x} = 1 \text{ (This is the same as L'Hopital's rule we used in the workshop.)}$$

4.5 Taylor series about points other than $x=0$

$x \rightarrow x-c$ You could do this substitution.

Example 4.5

In general, the Taylor series about a point $x=c$ is

$$f(x) = f(c) + \frac{1}{1!} f'(c)(x-c) + \frac{1}{2!} f''(c)(x-c)^2 + \dots \quad - 4.8$$

This is the Taylor series. Remember the Maclaurin series

$$f(x) = f(0) + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0)x^2 + \dots \text{ about } x=0.$$

5. Integral Calculus

5.1 Antidifferentiation

Consider:

i) $\frac{d}{dx} \sin x = f(x)$ - what is $f(x)$?

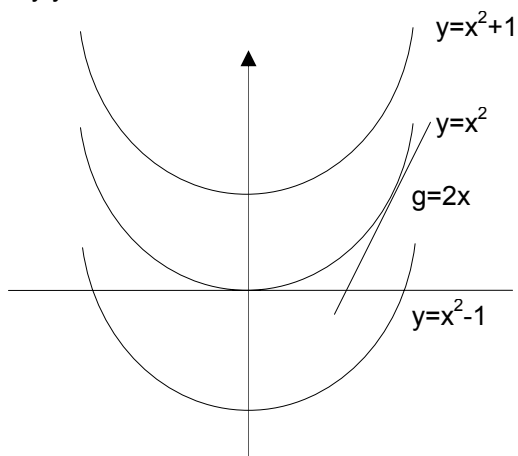
ii) $\frac{d}{dx} F(x) = \cos x$ - what is $F(x)$?

i) We know that $f(x)=\cos x$, therefore we might assume that $F(x)=\sin x$ which is true!
Therefore, $\sin x$ is an antiderivative of $\cos x$.
 $\sin x+1$ is also an antiderivative.

What are the functions, $f(x)$ whose gradient is always = $\cos x$?

Take another example: find $y(x)$ if $\frac{dy}{dx} = 2x$

Try $y=x^2$

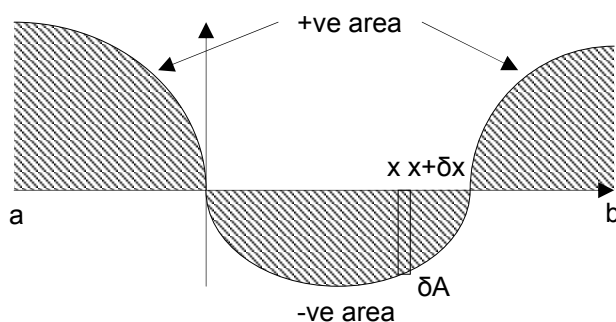


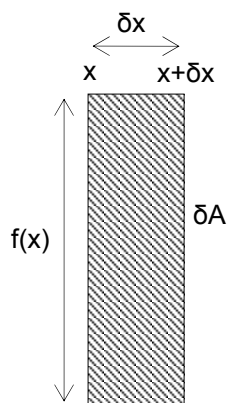
In general $y=x^2+c$ – there are an infinite number of antiderivatives.

Formally $F(x)$ is an antiderivative of $f(x)$ if $\frac{d}{dx} F(x) = f(x)$ - 5.1

All the antiderivatives are given by $F(x)+c$
Antiderivative \equiv indefinite integral

5.2 Signed Area





We want to work out the signed area contained under $f(x)$ between a and b . The small area δA is very nearly a rectangle.

$$\delta A \approx f(x)\delta x$$

$$\frac{\delta A}{\delta x} = f(x)$$

$$\text{Let } \delta x \rightarrow 0 \text{ and } \frac{\delta A}{\delta x} = f(x)$$

Rewrite $\frac{d}{dx}A(x) = f(x)$ Compare with 5.1

Therefore $A(x)$ is the antiderivative of $f(x)$

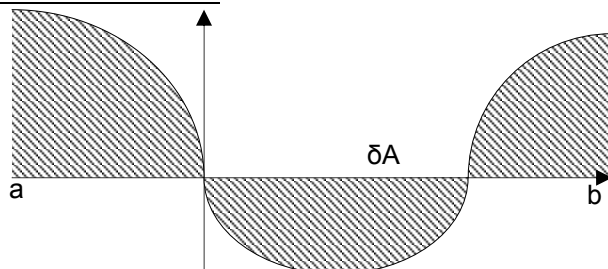
$$\therefore A(x) = f(x) + k$$

When $x=a$, $A(a)=0$, $\therefore k = -F(a)$

From a to b the signed area is $A = A(b) = F(b) - F(a)$

$$\text{Signed area} = F(b) - F(a) = [F(x)]_a^b - 5.2$$

Alternative method:



Divide the integral between a and b into N equal steps so that $\delta x = \frac{b-a}{N}$

$$\text{Therefore total area} = \sum_{x=a}^{x=b} \delta A \approx \sum_{x=a}^{x=b} f(x)\delta x$$

As $\delta x \rightarrow 0$, the approximation becomes equality.

$$A = \lim_{\delta x \rightarrow 0} \sum_{z=a}^{z=b} f(x)\delta x - 5.3$$

$$\text{The brief notation is } \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)dx - 5.4$$

Equate 5/4 and 5.2

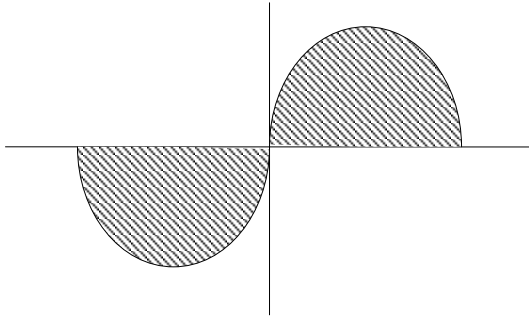
$$\int_a^b f(x)dx = [F(x)]_a^b - 5.5$$

This is called the definite integral.

$\int f(x)dx$ is the indefinite integral.

5.3 Symmetric Integrals

Take an odd function, where $f(-t) = -f(t)$ [e.g. t , t^3 , t^5 , $\sin at$, $\tan at$, ...]
e.g. $\sin t$



If we integrate about equal distances either side of the line of symmetry, the signal area = 0.

$$\rightarrow \int_{-c}^c f(t) dt = 0 \quad - 5.6$$

(for an odd function)

Similarly for an even function $f(-t) = f(t)$

$$\int_{-c}^c f(t) dt = 2 \int_0^c f(t) dt \quad - 5.7$$

(for an even function)

5.4 Definite integrals with variable limits

E.g. $I(x) = \int_c^x f(t) dt$, c is a constant.

As always $I(x) = [F(t)]_c^x = F(x) - F(c)$

[Where $F(t)$ is an antiderivative (indefinite integral) of $f(t)$]

$$\frac{dI(x)}{dx} = \frac{d}{dx} \int_c^x f(t) dt = \frac{dF(x)}{dx} = f(x)$$

or

$$\frac{d}{dx} \int_c^x f(t) dt = f(x) \quad - 5.8$$

We can also show $\frac{d}{dx} \int_x^c f(t) dt = -f(x)$ - 5.9

The general case:-

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v(x)) \frac{dv(x)}{dx} - (u(x)) \frac{du(x)}{dx} \quad - 5.10$$

5.5 Standard Integrals

$f(x)$	$\int f(x)dx$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
$\sin ax$	$-\frac{1}{a} \cos ax + c$
$\cos ax$	$\frac{1}{a} \sin ax + c$
e^{ax}	$\frac{e^{ax}}{a} + c$
$\frac{a}{x}$	$a \ln x + c$
$\sinh ax$	$-\frac{1}{a} \cosh ax + c$
$\cosh ax$	$\frac{1}{a} \sinh ax + c$

5.6 Useful Hints

5.6.1 Integrand = exact derivative

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c - 5.11$$

$$\int f'(x)f(x)dx = \frac{1}{2}(f(x))^2 + c - 5.12$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c - 5.13$$

$$\frac{d}{dx} \left(\frac{1}{2}(f(x))^2 + c \right) = \frac{2}{2} f(x)f'(x) = f(x)f'(x) \text{ as required.}$$

5.6.2 Integration using Partial Fractions

Example 5.1**Example 5.2**

$$\text{NB: } \int \frac{dx}{x^2 - a^2} = \int \frac{dx}{(x+a)(x-a)} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$$

General rules for partial fractions:

If we have a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and the degree of $p(x) <$ degree of $Q(x)$, then $Q(x)$ can be split into partial fractions as follows:

a) Each term of the form $(ax+b)$ gives a partial fraction

$$\frac{k}{(ax+b)}$$

b) Each repeated term $(cx+d)^n$ gives partial fractions

$$\frac{L_1}{(cx+d)}, \frac{L_2}{(cx+d)^2}, \frac{L_3}{(cx+d)^3}, \dots, \frac{L_n}{(cx+d)^n}$$

c) Each irreducible quadratic $px^2 + qx + r$ gives a term

$$\frac{Mx+N}{(px^2+qx+r)}$$

Where k, L_i, M and N are constants.

(See Jordan and Smith, examples 1.12→1.15)

NB: If $px^2 + qx + r$ has no real factors we complete the square and substitute – see next section.

5.6.3 Integration by Substitution

Steps to follow:- (Assuming $\int f(x)dx$)

- Examine $f(x)$ for suitable substitution
- Use $u=g(x)$ or $x=h(u)$ where g and h 'make up' $f(x)$
- Differentiate $x=h(u)$ to get $dx=h'(u)du$ or $u=g(x)$ to get $du=g'(x)dx$
- Perform the substitution (making sure the only variable is u)
- Do the integral

5.6.3.1 Letting $x=h(u)$

Example 5.3

$f(x)$	$\int f(x)dx$	Substitution
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) + c$	$x = a \sin u$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$	$x = a \tan u$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\operatorname{arcsinh} h\left(\frac{x}{a}\right) + c$	$x = a \sinh u$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arccosh} h\left(\frac{x}{a}\right) + c$	$x = a \cosh u$

5.6.3.2 Completing the Square

Example 5.4

5.6.3.3 Integrals of the form $\int F(g(x))g'(x)dx$

If we let $u=g(x)$, $du = g'(x)dx$

$$\int F(g(x))g'(x)dx = \int F(u)$$

Jordan and Smith only use two examples of this form:-

$$\int f(ax + b)dx$$

$$\int f(ax^2 + b)xdx$$

Example 5.5

5.6.3.4 Integrals of the form $\int \sin^n ax \cos^m ax dx$

Where n and m are odd (sometimes even) \pm integers.

Rules

If $n+m$ is odd, choose either:-

$$\int \sin^n ax \cos^m ax dx = \int \sin^{n-1} ax \cos^m ax \sin ax dx \text{ and use } u = \cos ax \text{ (if } n \text{ is odd)}$$

or:

$$\int \sin^n ax \cos^m ax dx = \int \sin^n ax \cos^{m-1} ax \cos ax dx \text{ and use } u = \sin ax \text{ (if } m \text{ is odd)}$$

The aim is to get $m-1$ or $n-1$ even.

If $n+m$ is even use either method.

Example 5.6

Example 5.7

5.6.4 Integration by Parts

Remember the product rule in differentiation.

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}$$

$$[(uv)' = uv' + u'v]$$

Integrate both sides

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx + c$$

Rearranging:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx + c \quad 5.14$$

Example 5.8

Aside:

How do we integrate $\int \ln x dx$?

$$\int \ln x dx = \int 1x \ln x dx$$

$$u = \ln x, \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 1, v = x$$

Example 5.9

Example 5.10

5.6.5 Reduction formulae

Show by example.

Example 5.11

Specific example:

$$\int_0^{\frac{\pi}{2}} \sin^m x dx = \frac{m-1}{m} \int_0^{\frac{\pi}{2}} \sin^{m-2} x dx - 5.15$$

If $m=6$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{6-1}{6} \int_0^{\frac{\pi}{2}} \sin^4 x dx = \left(\frac{6-1}{6}\right) \left(\frac{4-1}{4}\right) \left(\frac{2-1}{2}\right) \int_0^{\frac{\pi}{2}} \sin^0 x dx = \frac{5}{6} \frac{3}{4} \frac{1}{2} \int_0^{\frac{\pi}{2}} dx = \frac{5}{6} \frac{3}{4} \frac{1}{2} \frac{\pi}{2}$$

In a similar way we can show:

$$\int_0^{\frac{\pi}{2}} \cos^m x dx = \frac{m-1}{m} \int_0^{\frac{\pi}{2}} \cos^{m-2} x dx - 5.16$$

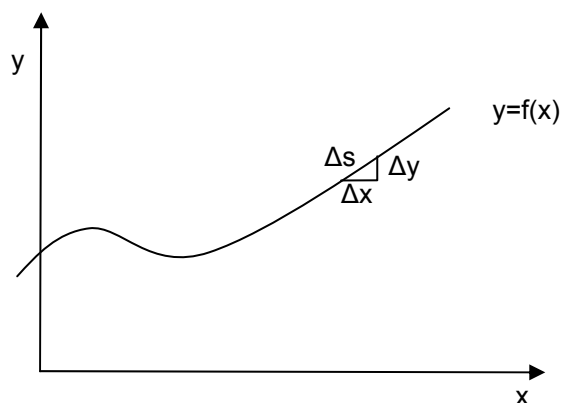
Another example:

$$\int_0^{\infty} e^{-t} t^N dt = N \int_0^{\infty} e^{-t} t^{N-1} dt = N(N-1) \int_0^{\infty} e^{-t} t^{N-2} dt$$

$$\int_0^{\infty} e^{-t} t^0 dt = \left[-e^{-t}\right]_0^{\infty} = 1$$

$$\int_0^{\infty} e^{-t} t^N dt = N(N-1)(N-2)\dots = N! \quad 5.17$$

5.7 Simple Line Integration



We call the distance along the line Δs .

When Δx , Δy , Δs are small

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Divide through by Δx

$$\frac{\Delta s}{\Delta x} \approx \frac{1}{\Delta x} \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$

$$\therefore \lim_{\Delta x \rightarrow 0} \frac{\Delta s}{\Delta x} = \lim_{\Delta x \rightarrow 0} \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}} = \sqrt{1 + \frac{ds^2}{dx^2}} = \frac{ds}{dx}$$

$$\boxed{\frac{ds}{dx} = \sqrt{1 + \frac{ds^2}{dx^2}}} \quad 5.18$$

To find the length of the curve between $x=a$ and $x=b$, integrate both sides of 5.18

$$\int_b^a \frac{ds}{dx} dx = s = \int_b^a \sqrt{1 + \frac{ds^2}{dx^2}} dx \quad 5.19$$

Example 5.12

Find the length of the curve $y = x^{3/2}$ between $x=0$ and $x=3$.

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$s = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$s = \int_0^3 \sqrt{1 + \frac{9}{4}x} dx = \int_0^3 \left(1 + \frac{9}{4}x\right)^{1/2} dx$$

$$s = \left[\left(1 + \frac{9}{4}x\right)^{1/2} \frac{2 \cdot 4}{3 \cdot 9} \right]_0^3 = 5.4$$

(Wrong in worked example)

In general for a line integral we integrate along a (defined) curve in space and the integrand is a function defined at all points.

This is denoted by:

$$\boxed{\int_C f(x, y, z) ds} \quad 5.20$$

where C is the path of integration, and ds is the parametric variable.

Example 5.13

In the x,y plane a function varies as $f(x,y)=xy^3$. Find the line integral along the path $y=2x$ between points $A=(-1,-2)$ and $B=(1,2)$.

We want to evaluate $\int_C xy^3 ds$.

We now need ds in terms of x or y (and hence dx or dy)

Choose x , we know that $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ from 5.18.

$$ds = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$

$$y = 2x, \frac{dy}{dx} = 2$$

$$\therefore ds = (1 + 4)^{\frac{1}{2}} dx = \sqrt{5} dx$$

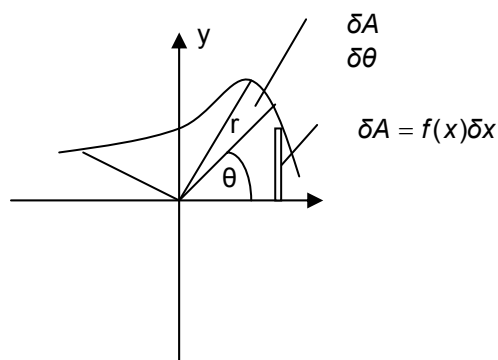
$$\therefore \int_C xy^3 ds = \sqrt{5} \int_{-1}^1 8x^4 dx$$

(Substituting $y=2x$)

$$= 8\sqrt{5} \left[\frac{x^5}{5} \right]_{-1}^1 = \frac{8\sqrt{5}}{5} [1 + 1] = \frac{16}{\sqrt{5}}$$

5.8 Integrating in polar coordinates

In plane polars we describe the function, $r = f(\theta)$



The incremental area, δA , is a narrow circular sector of radius r and angle $\delta\theta$. $\delta\theta$ is a fraction of the full circle.

$$\therefore \frac{\delta\theta}{2\pi} \pi r^2 = \delta A = \frac{1}{2} r^2 \delta\theta$$

$$A = \lim_{\delta\theta \rightarrow 0} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 \delta\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

Example 5.14

Find the area of a circle of radius 2.

(We know $A = \pi r^2 = 4\pi$)

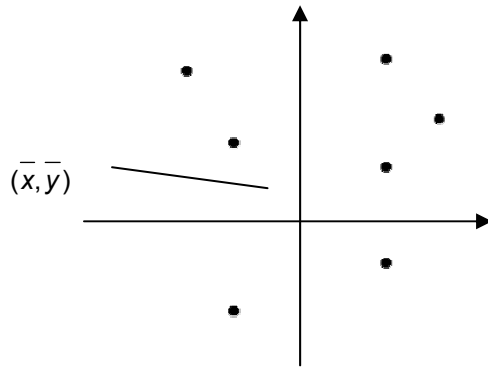
$$r = f(\theta) = 2$$

$$\therefore A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} d\theta = 2[\theta]_0^{2\pi} = 4\pi$$

5.9 Selected Applications of Integration

5.9.1 Centre of Mass

Suppose we have N particles of mass m_n with positions (x_n, y_n) , $n=1,2,\dots,N$



The balancing point is the point such that the total momentum about that point is zero. Take x and y directions separately.

$$\sum_{n=1}^n m_n (x_n - \bar{x}) = 0 \quad 5.22$$

$$\sum_{n=1}^n m_n (y_n - \bar{y}) = 0 \quad 5.23$$

$$\sum_{n=1}^n m_n x_n - \bar{x} \sum_{n=1}^n m_n = 0 \quad 5.24$$

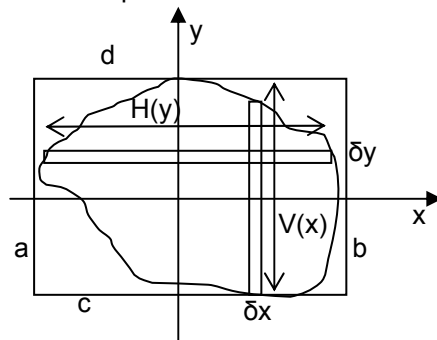
$$\sum_{n=1}^n m_n y_n - \bar{y} \sum_{n=1}^n m_n = 0 \quad 5.25$$

Total mass $M = \sum_{n=1}^N m_n$

Take 5.24 $\bar{x} = \frac{1}{M} \sum_{n=1}^N x_n m_n \quad 5.26$

$\bar{y} = \frac{1}{M} \sum_{n=1}^N y_n m_n \quad 5.27$

Assume a plate of uniform thickness and hence uniform mass/area, μ (kg.m⁻²)



The length of the strips are $V(x)$ and $H(y)$. Therefore, the mass of a strip = $\mu V(x)\delta x$ and $\mu H(y)\delta y$

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} (x - \bar{x}) \mu V(x) \delta x = 0$$

From 2.22 $\therefore \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} x \mu V(x) dx = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} x \mu V(x) dx = \mu \bar{x} A$

\therefore
where $A = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} V(x) dx$

$$\bar{x} = \frac{1}{A} \int_a^b xV(x)dx - 5.28$$

$$\bar{y} = \frac{1}{A} \int_c^d yH(y)dy - 5.29$$

5.9.2 Moment of Inertia

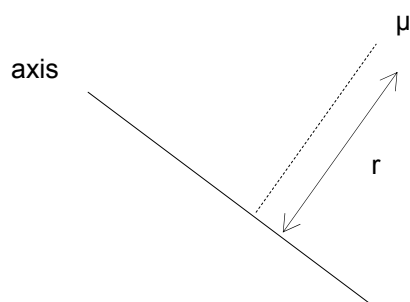
Moment of inertia is important for circular motion. It is similar to mass for linear motion. Normally given the symbol I .

Energy in linear system = $\frac{1}{2}mv^2$

Energy in a circular frame = $\frac{1}{2}I\omega^2$

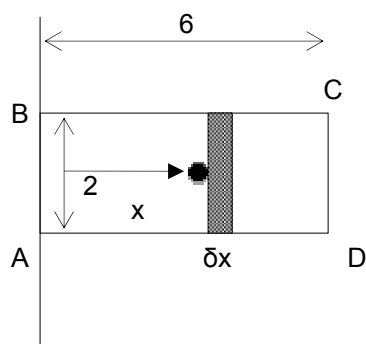
Momentum: mv , $I\omega$.

The moment of inertia of a mass m about an axis is mr^2 where r is the perpendicular distance between the axis and the mass.



Example

Find the moment of inertia of a uniform rectangular plate ABCD about the edge AB, where $AB=2$, $BC=6$, mass/area=2



Area of the shaded rectangle = $2\delta x$.

Mass of shaded rectangle = $4\delta x$

$$\delta m = 4\delta x$$

$$\delta I = 4x^2\delta x$$

$$I = \lim_{\delta x \rightarrow 0} \sum_{x=0}^{x=6} \delta I = \lim_{\delta x \rightarrow 0} \sum_{x=0}^{x=6} 4x^2\delta x = 4 \int_0^6 x^2 dx = 4 \left[\frac{x^3}{3} \right]_0^6 = 288$$

6. Differential Equations

6.1 Introduction

If $y = f(x)$, then any equation having some or all of the terms $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$ is

called an ordinary differential equation (ODE)

The order of an ODE is the highest derivative involved.

Eg:

$$\frac{dy}{dx} = y^2 \text{ First order}$$

$$\frac{d^4y}{dx^4} + \sin x \frac{d^2y}{dx^2} + 2y = 0 \text{ Fourth order}$$

$$\left(\frac{dy}{dx}\right)^3 = x^2 + 1 \text{ First order}$$

An ODE of order n has n linearly dependant solutions which form a basis of solutions.

Example 6.1

$$y \frac{dy}{dx} = -x \text{ (First-order)}$$

To solve, integrate directly.

$$\int y \frac{dy}{dx} dx = -\int x dx$$

$$\int y dy = -\int x dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + A$$

This is called the general solution.

Therefore, the solution is a set of circles of radius \sqrt{A} .

If we know some initial conditions or boundary conditions, we can pin down a specific solution. This is called an initial value problem.

E.g. if $y=4$ when $x=0$, $A=8$.

Therefore specific solution is $\frac{y^2}{2} + \frac{x^2}{2} = 8$.

6.2 1st order ODE's

i.e. of the form $\frac{dy}{dx} = f(x, y)$ – 6.1

6.2.1 1st order separable ODE's

In this case $\frac{dy}{dx} = g(x)h(y)$ – 6.2

To solve we separate the function $f(x, y)$ into $g(x)$ and $h(y)$.

We can now integrate directly.

$$\int \frac{dy}{h(y)} = \int g(x) dx \quad \text{--- 6.3}$$

Example 6.2 (a, b & c)

a) $\frac{dy}{dx} = -y^2$ $y=1$ when $x=0$

In this case $h(y) = -y^2, g(x) = 1$

$$\therefore \int \frac{dy}{y^2} = \int dy$$

$$\frac{1}{y} = x + c$$

$$y = \frac{1}{x + c}$$

$$x = 0, y = 1 \rightarrow c = 1$$

$$\therefore y = \frac{1}{x + 1}$$

$$\text{b) } \frac{dy}{dx} = x + xy = x(1 + y)$$

$$g(x) = x, h(y) = 1 + y$$

$$\int \frac{dy}{1 + y} = \int x dx$$

$$\ln|1 + y| = \frac{x^2}{2} + c$$

$$1 + y = e^{\frac{x^2}{2} + c} = e^{\frac{x^2}{2}} e^c = A e^{\frac{x^2}{2}}$$

$$y = A e^{\frac{x^2}{2}} - 1$$

$$\text{c) } x^2(1 + y) \frac{dy}{dx} = y^2(x - 1)$$

$$\frac{dy}{dx} = \frac{y^2}{(1 + y)} \frac{(x - 1)}{x^2}$$

$$\int \frac{(1 + y)}{y^2} dy = \int \frac{(x - 1)}{x^2} dx$$

$$\int \left(\frac{1}{y^2} + \frac{1}{y} \right) dy = \int \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$$

$$-\frac{1}{y} + \ln y = \ln x + \frac{1}{x} + c$$

Summary of method:

- Factorise the equation to get $g(x)h(y)$
- Separate to get $g(x)$ and $h(y)$ on different sides.
- Integrate directly to get general solution (with constant)
- If initial value problem, evaluate the constant to get the specific solution.

Example 6.3

Newton cooling

An object at temperature T in ambient air at temperature T_0 , satisfies the equation

$$\frac{dT}{dt} = -k(T - T_0) \quad (h(T) = (T - T_0), g(t) = -k)$$

$$\int \frac{dT}{T - T_0} = -k \int dt$$

$$\ln|T - T_0| = -kt + c$$

$$T - T_0 = e^{-kt + c} = A e^{-kt}$$

$$T = A e^{-kt} + T_0$$

If air temperature = 40°C, object is at 100°C when t=0

$$100 = A + 40 \rightarrow A = 60$$

$$T = 60e^{-kt} + 40 \text{ Specific solution.}$$

If the temp is 85°C after 3 minutes

$$85 = 60e^{-3k} + 40$$

$$\frac{45}{60} = e^{-3k}$$

$$\ln 0.75 = -3k$$

$$k = 0.096$$

(In minutes)

6.2.2 1st Order Linear ODEs

A linear ODE is one with no squares (etc) or products involving y and its derivatives.

They have the form:-

$$\frac{dy}{dx} + P(x)y = Q(x) - 6.4$$

E.g.:-

$$\frac{dy}{dx} + y \sin^2 x = \frac{1}{1+x} - \text{linear.}$$

$$\left(\frac{dy}{dx}\right)^2 + y = 0 - \text{non-linear}$$

$$y \frac{dy}{dx} + x = 1 - \text{Non-linear}$$

These linear equations can be solved using an integrating factor, I(x)

$$\text{Take } \frac{dy}{dx} + P(x)y = Q(x)$$

Multiply through by I(x)

$$I \frac{dy}{dx} + IPy = QI - 6.5$$

If we choose I(x) so that

$$\frac{d(Iy)}{dx} = I \frac{dy}{dx} + IPy - 6.6$$

$$\therefore \frac{d(Iy)}{dx} = QI - 6.7$$

Integrate both sides.

$$\int d(Iy) = \int QI dx$$

$$Iy = \int QI dx$$

Or

$$y = \frac{1}{I(x)} \int Q(x)I(x) dx - 6.8$$

All that remains is to find I(x).

Expand left hand side of 6.6

$$\frac{d(Iy)}{dx} = I \frac{dy}{dx} + y \frac{dI}{dx} = I \frac{dy}{dx} + IPy$$

$$IP = \frac{dI}{dx}$$

This is a separable ODE.

$$\int \frac{dI}{I} = \int P dx$$

$$\ln I = \int P dx$$

$$I = \exp\left(\int P(x)dx\right) - 6.9$$

Summary:

- Find the integrating factor $I = \exp\left(\int P(x)dx\right)$
- Rewrite as $\frac{d(Iy)}{dx} = IQ$
- Integrate to get $y = \frac{1}{I(x)} \int I(x)Q(x)dx$
[Remember the constant of integration]
- Use any initial conditions to find specific solution.

Example 6.4

Example 6.5

6.2.3 Bernoulli's Equations

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, n \neq 0,1 - 6.10$$

Rearrange:-

$$\frac{1}{y^n} \frac{dy}{dx} + \frac{1}{y^{n-1}} P(x) = Q(x)$$

$$\text{And now substitute } u = y^{1-n} \rightarrow \frac{du}{dx} = (1-n)y^{-n} \frac{dy}{dx} \rightarrow \frac{1}{y^n} \frac{dy}{dx} = \frac{1}{(1-n)} \frac{du}{dx}$$

Therefore Bernoulli's equation becomes:-

$$\frac{1}{(1-n)} \frac{du}{dx} + P(x)u = Q(x)$$

or:

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x) - 6.11$$

Therefore the integrating factor is $I(x) = \exp\left(\int (1-n)P(x)dx\right)$

Example 6.6

$$\text{Solve } \frac{dy}{dx} + \frac{y}{x} = 2x^3y^4$$

$$P(x) = \frac{1}{x}, Q(x) = 2x^3, y^n = y^4 \rightarrow n = 4$$

$$\text{Substitute } u = y^{1-n} = y^{-3}, \frac{du}{dx} = -3y^{-4} \frac{dy}{dx}$$

$$\therefore \frac{1}{y^4} \frac{dy}{dx} + \frac{1}{y^3x} = 2x^3$$

From 6.11 (or by rearranging)

$$\frac{du}{dx} - \frac{3u}{x} = -6x^3$$

Integrating factor

$$I(x) = \exp\left(\int P(x)dx\right) = \exp\left(\int \frac{-3}{x} dx\right) = \exp(-3 \ln x) = \exp(\ln x^{-3}) = x^{-3}$$

$$\frac{d(Iu)}{dx} = IQ = I(-6x^3)$$

$$x^{-3}u = \int e^{-3}(-6x^3)dx = -6 \int dx$$

$$x^{-3}u = -6x + c$$

$$u = -6x^4 + cx^3 = y^{-3}$$

$$\underline{y = (-6x^4 + cx^3)^{-1/3}}$$

6.2.4 Clever substitutions

Example 6.7

Solve $\sec^2 y \frac{dy}{dx} + x \tan y = Q(x)$

Substitute $w = \tan y$, $\frac{dw}{dx} = \sec^2 y \frac{dy}{dx}$

Therefore the ODE becomes $\frac{dw}{dx} + xw = Q(x)$, which is linear in w .

Another useful trick:-

$$\frac{dy}{dx} = f(ax + by + c) \quad - 6.12$$

If we substitute $u = ax + by + c$ then $\frac{du}{dx} = a + b \frac{dy}{dx}$. Therefore the ODE becomes:-

$$\frac{du}{dx} = a + b \frac{dy}{dx} = a + bf(u)$$

$$\frac{du}{dx} = a + bf(u) \quad - 6.13$$

which is separable.

Example 6.8

Solve $\frac{dy}{dx} = (x + y + 1)^2$

Therefore $a = b = c = 1$

$$u = x + y + 1 \rightarrow \frac{dy}{dx} = 1 + \frac{du}{dx}$$

$$\therefore \frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + u^2$$

$$\int \frac{du}{(1+u^2)} = \int dx$$

$$\tan^{-1} u = x + c$$

$$\underline{\tan^{-1}(x + y + 1) = x + c}$$

Homogenous ODE is of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad - 6.14$$

Solve by substitution $v = \frac{y}{x}$, $y = xv(x)$

so that $\frac{dy}{dx} = v(x) + x \frac{dv}{dx}$

Therefore 6.14 becomes

$$v + x \frac{dv}{dx} = f(v)$$

$$x \frac{dv}{dx} = f(v) - v$$

which is separable.

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

If $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ and $f(x,y)$ and $g(x,y)$ homogenous and of the same degree the ODE

is still homogenous – use the same substitution.

A function is said to be homogenous of degree m if $f(kx, ky) = k^m f(x, y)$ - 6.15

Example 6.9

Summary of solving 1st order ODE's by substitution

1. Check to see if it is linear in a function y – solve as linear ODE
2. $\frac{dy}{dx} = (ax + by + c)$ → substitute $v = ax + by + c$ → the ODE becomes separable.
3. If the ODE is homogenous, i.e. $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ substitute $y = xv(x)$ → separable
4. Equations of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ homogenous if f and g are of the same degree
→ substitute $y = xv(x)$

6.3 Second Order ODE

$$\frac{d^2 z}{dt^2} = mg \text{ If we integrate twice } \frac{dz}{dt} = mgt + c_1, z = \frac{mgt^2}{2} + c_1 t + c_2$$

There are two constants requiring 2 initial conditions.

6.3.1 Special Cases

The general form of a 2nd order ODE is:-

$$\frac{d^2 y}{dt^2} = f\left(y, t, \frac{dy}{dt}\right) \text{ - 6.16}$$

If y is absent

$$\frac{d^2 y}{dt^2} = f\left(t, \frac{dy}{dt}\right)$$

This is now a 1st order ODE in $\frac{dy}{dt}$

Substitute $v = \frac{dy}{dt}$. Our ODE becomes $\frac{dv}{dt} = f(t, v)$

Therefore use the techniques of our last section.

If t is absent

$$\frac{d^2 y}{dt^2} = f\left(y, \frac{dy}{dt}\right)$$

Again substitute $v = \frac{dy}{dt}$ $\frac{d^2 y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = \frac{dv}{dy} v$

Therefore the ODE becomes:-

$$v \frac{dv}{dy} = f(y, v)$$

This is now a 1st order of v in terms of y .

If both t and dy/dt are absent:

$$\frac{d^2y}{dt^2} = f(y)$$

Use $\frac{dy}{dt}$ as an integrating factor.

Multiply through by $\frac{dy}{dt}$

$$\frac{d^2y}{dt^2} \frac{dy}{dt} = f(y) \frac{dy}{dt}$$

$$\frac{d\left(\frac{1}{2}\left(\frac{dy}{dt}\right)^2\right)}{dt} = f(y) \frac{dy}{dt}$$

Integrate both sides:

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 = \int f(y)dy + \text{const.}$$

6.4 2nd Order ODEs with const. coefficients

The general form is:

$$\boxed{\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2y = f(t)} \quad - 6.17$$

If we set $f(t) = 0$ so that $\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2y = 0$ and find $y_c(t)$ a solution.

If we substitute this into the ODE we get 0

Then find a solution for the full ODE $y_p(t)$

$$y(t) = y_c(t) + y_p(t)$$

$y_c(t)$ is the complimentary function (CF)

$y_p(t)$ is the particular integral (PI)

The complete solution is the sum of the CF and the PI.

6.4.1 Finding the CF

The CG is the solution of the auxiliary equation (or homogenous equation)

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2y = 0 \quad - 6.18$$

Try a solution $y = Ae^{\lambda x}$

$$\frac{dy}{dx} = \lambda Ae^{\lambda x}, \quad \frac{d^2y}{dx^2} = \lambda^2 Ae^{\lambda x}$$

$$\lambda^2 Ae^{\lambda x} + a_1 \lambda Ae^{\lambda x} + a_2 Ae^{\lambda x} = 0$$

$$\lambda^2 + a_1 \lambda + a_2 = 0 \quad - 6.19$$

Therefore we can find λ by solving the quadratic equation.

There are three possibilities:

1. The roots are real and independent λ_1 and λ_2
2. The roots are real and identical, λ_1
3. The roots are complex $\lambda = \alpha \pm i\beta$

1. Roots are real and distinct

$$y_1 = A_1 e^{\lambda_1 x}$$

$$y_2 = A_2 e^{\lambda_2 x}$$

$$\text{Therefore } y_x(x) = A_1 e^{\lambda_1 x} + y_2 = A_2 e^{\lambda_2 x} \quad 6.21$$

2. Roots are real and identical

$$y_c(x) = (A_1 x + A_2) e^{\lambda x} \quad 6.21$$

(Left as exercise to prove by substitution)

3. Roots are complex, $\lambda = \alpha \pm i\beta$

$$y_c(x) = A_1 e^{(\alpha+i\beta)x} + A_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x} (A_1 e^{i\beta x} + A_2 e^{-i\beta x})$$

$$= e^{\alpha x} (B_1 \cos \beta x + B_2 \sin \beta x)$$

$$= e^{\alpha x} C \cos(\beta x + \varphi)$$

where φ is a constant.

(Remember 1.4, harmonic function

$$a \cos y + b \sin u = c \cos(u + \varphi), c = \sqrt{a^2 + b^2}, \tan \varphi = \left(\frac{-b}{a} \right)$$

Example 6.11

6.4.2 Finding the Particular Integral (PI)

The form of the PI depends on the form of $f(x)$

$$\left[\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x) \right]$$

Use the following rules:

1. If $f(x) = ae^{\alpha x}$, try $y_p(x) = be^{\alpha x}$

2. If $f(x) = a_1 \cos \alpha x + a_2 \sin \alpha x$ (a_1 or a_2 could be zero), try

$$y_p(x) = b_1 \cos \alpha x + b_2 \sin \alpha x$$

3. If $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ try $y_p(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$

4. If $f(x)$ is a sum or product of any of the above, try $y_p(x)$ as the most general sum or product of the individual trial solutions.

5. If the trial solution contains any terms already in the CF, multiply the trial PI by x and try again.

We then substitute the PI into the full ODE to evaluate the constants.

The complete solution of the full ODE is the sum of the CF and PI.

Summary:

- Write down the auxiliary equation
- Find the CF containing 2 arbitrary constants
- Choose a trial form of the PI depending on $f(x)$
- Check to see no terms are common in PI and CF
- If terms are common multiply the PI by x
- Substitute the PI into the full ODE to evaluate the coefficients.
- Write down the full solution = CF + PI.
- Use any initial conditions to evaluate the constants.

Example 6.12

Next example is where the PI has a term in the CF.

Example 6.13

7. The Harmonic Oscillator

7.1 Unforced, Undamped Oscillator

In Simple Harmonic Motion (SHM), the restoring force is proportional to the displacement.

i.e. $F = -kx$ or:

$$M \frac{d^2x}{dt^2} = -kx \quad \text{- 7.1}$$

$$\rightarrow M \frac{d^2x}{dt^2} + kx = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Common to write $\frac{k}{m} = \omega_o^2$

$$\text{Therefore} \quad \frac{d^2x}{dt^2} + \omega_o^2 x = 0 \quad \text{- 7.2}$$

$$\text{Or} \quad \ddot{x} + \omega_o^2 x = 0$$

Therefore the auxiliary equation is:-

$$\lambda^2 + \omega_o^2 = 0$$

$$\lambda = \pm i\omega_o$$

$$x_c(t) = e^0 (A_1 e^{i\omega_o t} + A_2 e^{-i\omega_o t})$$

$$\text{or} \quad \underline{x_c(t) = B_1 \cos \omega_o t + B_2 \sin \omega_o t}.$$

7.2 Now let's add a forcing term – the forced oscillator

$$\ddot{x} + \omega_o^2 x = F(t) \quad \text{- 7.3}$$

$F(t)$ is the forcing or driving term, and is time-dependent.

Let's use $F(t) = F_o \cos \omega t$.

$$\text{Therefore} \quad \ddot{x} + \omega_o^2 x = F_o \cos \omega t$$

$$\text{The CF is} \quad x_c(t) = A \cos \omega_o t + B \sin \omega_o t$$

$$\text{Let's use} \quad x_p(t) = a \cos \omega t + b \sin \omega t$$

This is fine so long as $\omega \neq \omega_o$ otherwise we must multiply the PI by t .

Case 1, $\omega \neq \omega_o$

$$x_p(t) = a \cos \omega t + b \sin \omega t$$

$$\dot{x}_p(t) = -a\omega \sin \omega t + b\omega \cos \omega t$$

$$\ddot{x}_p(t) = -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t$$

Substitute into the ODE

$$-a\omega^2 \cos \omega t - b\omega^2 \sin \omega t + \omega_o^2 (a \cos \omega t + b \sin \omega t) = F_o \cos \omega t$$

$$a(\omega_o^2 - \omega^2) \cos \omega t + b(\omega_o^2 - \omega^2) \sin \omega t = F_o \cos \omega t$$

$$\therefore b = 0, a = \frac{F_o}{(\omega_o^2 - \omega^2)}$$

The complete general solution is:-

$$x(t) = A \cos \omega_o t + B \sin \omega_o t + \frac{F_o}{(\omega_o^2 - \omega^2)} \cos \omega t \quad \text{- 7.4}$$

$$A \cos \omega_o t + B \sin \omega_o t \rightarrow \text{Free response}$$

$$\frac{F_o}{(\omega_o^2 - \omega^2)} \cos \omega t \rightarrow \text{Forced response}$$

Case 2, $\omega = \omega_0$

Now we need to multiply the PI by t .

$$x_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t)$$

$$\dot{x}_p(t) = a \cos \omega_0 t + b \sin \omega_0 t + t(-a \omega_0 \sin \omega_0 t + b \omega_0 \cos \omega_0 t)$$

$$\begin{aligned} \ddot{x}_p(t) &= -a \omega_0 \sin \omega_0 t + b \omega_0 \cos \omega_0 t - a \omega_0 \sin \omega_0 t + b \omega_0 \cos \omega_0 t + t(-a \omega_0^2 \cos \omega_0 t - b \omega_0^2 \sin \omega_0 t) \\ &= 2(-a \omega_0 \sin \omega_0 t + b \omega_0 \cos \omega_0 t) + t(-a \omega_0^2 \cos \omega_0 t - b \omega_0^2 \sin \omega_0 t) \end{aligned}$$

Substituting into the ODE gives:-

$$a = 0, b = \frac{F_0}{2\omega_0}$$

$$x_p(t) = \frac{F_0}{2\omega_0} t \sin \omega_0 t$$

Therefore complete solution is:-

$$x(t) = a \cos \omega_0 t + b \sin \omega_0 t + \frac{F_0}{2\omega_0} t \sin \omega_0 t \quad \text{- 7.5}$$

$a \cos \omega_0 t + b \sin \omega_0 t \rightarrow$ Free response

$\frac{F_0}{2\omega_0} t \sin \omega_0 t \rightarrow$ Driven response

We therefore see that the second term increases with time – theoretically to infinity. In reality there will be some damping.

7.3 Damped oscillator – No forcing term

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x = 0$$

$$\text{Damping} \propto \frac{dx}{dt}$$

Auxiliary equation is:

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

$$\text{Roots } \lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Three cases:-

1. $\gamma^2 > \omega_0^2$ - Roots are real and negative
2. $\gamma^2 = \omega_0^2$ - Roots are real, coincidental and equal to $-\gamma$
3. $\gamma^2 < \omega_0^2$ - Roots are complex, $\lambda = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$

Case 1, $\gamma^2 > \omega_0^2$ - Heavy Damping

$$x_c(t) = Ae^{-|\lambda_1|t} + Be^{-|\lambda_2|t} \quad \text{- 7.7}$$

i.e. there is no oscillation, the amplitude simply decays exponentially.

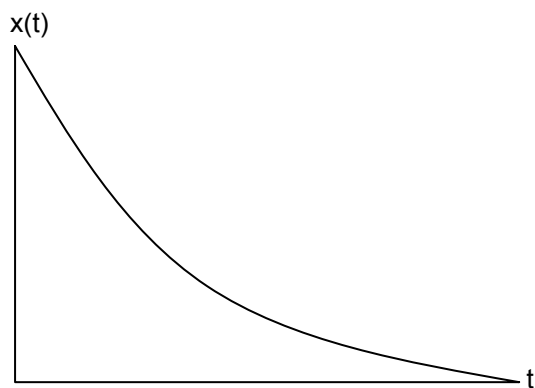
Case 2, $\gamma^2 = \omega_0^2$ - Critical Damping

The roots are coincident, real and equal to $-\gamma$

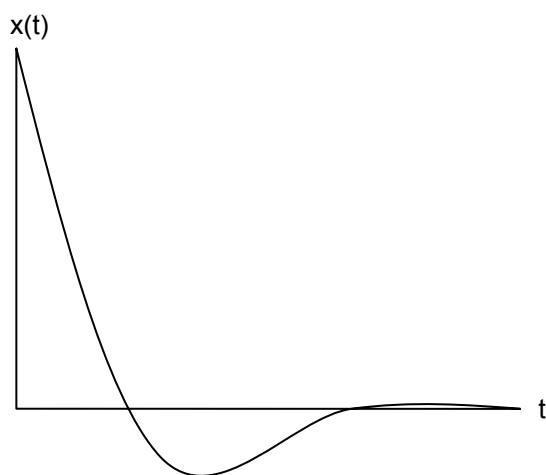
$$\text{Therefore } x_c(t) = (At + B)e^{-\gamma t} \quad \text{- 7.8}$$

Again no oscillation, but depends on A and B.

$$\text{e.g. } x_c(t) = (2t + 1)e^{-3t}$$



$$\text{or } x_c(t) = (-5t + 1)e^{-3t}$$



Case 3: $\gamma^2 < \omega_o^2$

$$\begin{aligned} \lambda_{1,2} &= -\gamma \pm \sqrt{\gamma^2 - \omega_o^2} \\ &= -\gamma \pm i\sqrt{\omega_o^2 - \gamma^2} \end{aligned}$$

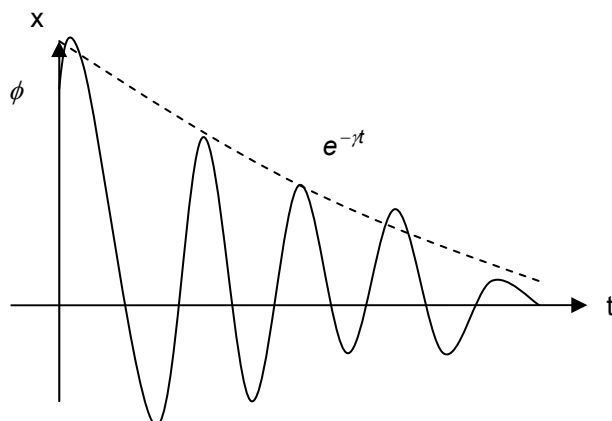
Remember if $\lambda = \alpha \pm i\beta$

$$x_c(t) = e^{\alpha t} (A \cos \beta t + B \sin \beta t)$$

$$\therefore x_c(t) = e^{-\gamma t} \left(a \cos\left(\sqrt{\omega_o^2 - \gamma^2} t\right) + B \sin\left(\sqrt{\omega_o^2 - \gamma^2} t\right) \right)$$

$$= e^{-\gamma t} C \cos\left(\sqrt{\omega_o^2 - \gamma^2} t + \phi\right)$$

Therefore the solution oscillates at frequency $\sqrt{\omega_o^2 - \gamma^2}$ and not ω_o .
Also the amplitude decays exponentially.



7.4 Damped, Forced Oscillator

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_o^2 x(t) = F(t) = F_o \cos \omega t = \text{Re}\{F_o e^{i\omega t}\} \quad -7.10$$

This time we are going to find the particular integral using complex exponentials. As always, we can use complex variables in place of our real ones, as long as we remember to take the real part at the end.

Our ODE becomes:

$$\ddot{X}(t) + 2\gamma\dot{X}(t) + \omega_o X(t) = F_o e^{i\omega t}$$

Where $X(t)$ is a complex variable and $x(t) = \text{Re}\{X(t)\}$

Now our trial solution is $X_p(t) = Ce^{i\omega t}$, where C is a complex coefficient.

$$\dot{X}_p(t) = i\omega Ce^{i\omega t}$$

$$\ddot{X}_p(t) = -\omega^2 Ce^{i\omega t}$$

Substitute into the ODE:

$$-\omega^2 Ce^{i\omega t} + 2\gamma i\omega Ce^{i\omega t} + \omega_o^2 Ce^{i\omega t} = F_o e^{i\omega t}$$

$$C(-\omega^2 + 2\gamma\omega i + \omega_o^2) = F_o$$

$$C = \frac{F_o}{2\gamma\omega i + (\omega_o^2 - \omega^2)}$$

Get into standard form by multiplying top and bottom by the complex conjugate \rightarrow

$$C = \frac{F_o((\omega_o^2 - \omega^2) - 2\gamma\omega i)}{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2} \quad -7.12$$

Therefore our CF is

$$X_p(t) = \frac{F_o((\omega_o^2 - \omega^2) - 2\gamma\omega i)}{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2} e^{i\omega t} = \frac{F_o((\omega_o^2 - \omega^2) - 2\gamma\omega i)}{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2} (\cos \omega t + i \sin \omega t)$$

We want $x_p(t) = \text{Re}\{X_p(t)\}$

$$X_p(t) = \frac{F_o}{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2} [(\omega_o^2 - \omega^2)\cos \omega t + 2\gamma\omega \sin \omega t] \quad -7.13$$

Our complete solution is the sum of the CF and the PI.

Note all the CFs decay away with time – these are known as transients.

In the long term only the forced solution exists.

Back to $X_p(t)$:

This is a harmonic function of the form:-

$$a \cos u + b \sin u = c \cos(u + \phi)$$

$$\text{where } c = \sqrt{a^2 + b^2}, \phi = \arctan\left(-\frac{b}{a}\right)$$

$$a = (\omega_o^2 - \omega^2), b = 2\gamma\omega$$

$$\therefore c = \sqrt{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2}$$

$$\therefore x_p(t) = \frac{F_o}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t + \phi) \quad - 7.14$$

$$\text{where } \phi = \arctan\left(\frac{-2\gamma\omega}{\omega_o^2 - \omega^2}\right) \quad - 7.15$$

When is the amplitude a maximum?

Max when $(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2$ is a minimum.

Let $g(\omega) = (\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2 = \omega_o^4 - 2\omega_o^2\omega^2 - \omega^4 + 4\gamma^2\omega^2$

Differentiate and set to zero

$$\frac{dg(\omega)}{d\omega} = -4\omega_o^2\omega + 4\omega^3 + 8\gamma^2\omega = 0$$

$$\Rightarrow \omega^2 = \omega_o^2 - 2\gamma^2 \quad - 7.16$$

$$\text{or } \omega = \sqrt{\omega_o^2 - 2\gamma^2}$$

This is known as the resonance condition.

Remember γ is the damping.

Note that without damping, resonance is when $\omega = \omega_o$.

Substitute this resonance condition into 7.14 and the amplitude will be:-

7.14 is $x_p(t) = A \cos(\omega t + \phi)$

$$x_p(t) = \frac{F_o}{2\gamma(\omega_o^2 - \gamma^2)^{1/2}} \cos(\omega t + \phi) \quad - 7.17$$

