Now looking at stellar motion, rather than gas motion, in spiral galaxies.

For interstellar gas, hydrodynamic equations are closed with a gas equation of state $p = p(\rho)$, often the adiabatic form.

For the stellar fluid, use dispersion relation from f, Φ. In particular, need to know the background f_0. Use the Schwarzschild distribution function.

If disc is 'cold' (i.e. it has no random motions, just perfect circular orbits), then the result from the momentum transfer equation is analogous to that from the gas:

$$\bar{v}_{Ra} = \left(\frac{m\Omega - \omega}{\Delta} \right) k \Phi_a F$$

where $\bar{v}_{Ra}$ is the mean radial velocity, m is the multiplicity, Ω is the rotation pattern speed, $\Delta = \kappa^2 - (m\Omega - \omega)^2$. F is 1 here. This is valid if the size of the epicycle $<< \lambda_{spiral}$, the wavelength associated with the spiral motion $= \frac{2\pi}{k}$.

If the cold disc approximation is not valid, then stars passing through (R, ϕ) come from a range of equilibrium radii (R_g), and the response to the spiral wave is reduced. The value of the "fiddle factor" F, known as the reduction factor, is much less than 1.

The continuity equation gives

$$(m\Omega - \omega)\Sigma_a + k\Sigma_0 \bar{v}_{Ra} = 0$$

which relates the perturbation Σ_a to Σ_0, the background, and $\bar{v}_{Ra}$, the mean radial velocity. This is just like the gas case. Can use this equation to eliminate $\bar{v}_{Ra}$.

The Schwarzschild function f_0 comes from the solution of the collisionless Boltzmann equation with epicycles, and is

$$f_0(R, v_R, v_\phi) = \frac{\Sigma(R)}{2\pi\sigma_R\sigma_\phi} \exp\left\{ -\frac{v_R^2}{2\sigma_R^2(R)} - \frac{\left[v_\phi - v_c(R)\right]^2}{2\sigma_\phi^2(R)} \right\}$$

where the first part is the density over the product of the two velocity distributions. In the exponential, there is a radial velocity term, and an angular velocity term (where $v_c(R)$ is the circular velocity). Note that $\sigma_R << v_\phi$, and $\sigma_\phi << v_\phi$ is usually true for real disc galaxies, i.e. the random motion is much smaller than the ordered motion.

From this,

$$F\left(\underbrace{\frac{\omega - m\Omega}{\kappa}}_{S}, \underbrace{\frac{k^2\sigma_R^2}{\kappa^2}}_{\chi} \right) = F(S, \chi) = \frac{2}{\chi}\left(1 - S^2\right) e^{-\chi} \sum_{n=1}^{\infty} \frac{In\{\chi\}}{1 - \frac{S^2}{n^2}}$$

where the last part, $In\{\chi\}$, are modified Bessel functions which grow more like exponentials, rather than the usual Bessel functions which oscillate.

Another way of writing this is

$$F(S,\chi)=\frac{1-S^2}{\sin(\pi S)}\int_0^{\pi} e^{-\chi(1+\cos\tau)}\sin(S\tau)\sin\tau d\tau$$

where τ is just a dummy variable. Note that $F(S,0)=1$, i.e. when $\chi=0$.

Poisson equation (using razor-thin disc approximation) → $\Phi_a=-\frac{2\pi G\Sigma_a}{|k|}$.

Combining all of these, we get the dispersion relation:

$$(m\Omega-\omega)^2=\kappa^2-2\pi G\Sigma_0|k|F\left(\frac{\omega-m\Omega}{\kappa},\frac{k^2\sigma_R^{\ 2}}{\kappa^2}\right).$$

This is the test for stability in stellar disc.
Assumption: tightly wound pattern, $\left|\frac{kR}{m}\right|>>1$. This is usually marginally valid for real galaxies – can get nice spiral perturbations from theory, but need to look at N-body or hydro models to extend to looser patterns.

NB: Velocity distribution:

$$\sigma_v=\left[\frac{\sum_{n=1}^{N}(v_i-\bar{v})^2}{(N-1)}\right]^{1/2}$$

For exam, need to know the proofs of the first two of Newton's theorems.

Exam past paper, June 2005:
2b:

$$\langle\Theta\rangle=\Theta_c\left(1-\frac{2x}{Ry}\right)$$

Part of the equation given:

$$\frac{\rho}{R}\left[\Theta_c^{\ 2}-\langle\Theta\rangle^2\right]$$

so evaluate the square of the first equation, and put it in.

$$\langle\Theta\rangle^2=\Theta_c^{\ 2}\left(1-\frac{4x}{R_g}+\frac{4x^2}{R_g^{\ 2}}\right)\approx\Theta_c^{\ 2}-\frac{4x\Theta_c^{\ 2}}{R_g}$$

The last part is ignored because the R_g makes it small.
Substituting this in gives:

$$\frac{\rho}{R}\left[\Theta_c^{\ 2}-\Theta_c^{\ 2}+\frac{4x\Theta_c^{\ 2}}{R_g}\right]$$

The whole equation then looks like:

$$\frac{\partial}{\partial R}\left(\rho\langle\Pi\rangle^2\right)+\frac{\rho}{R}\langle\Pi\rangle^2+\frac{4x\Theta_c^{\ 2}}{R_g}\frac{\rho}{R}=0$$

Divide this by the second term to get

$$\frac{R}{\rho\langle\Pi\rangle^2}\frac{\partial}{\partial R}\left(\rho\langle\Pi\rangle^2\right)+1+\frac{4x\Theta_c^{\ 2}}{R_g\langle\Pi\rangle^2}=0$$

as required.

3d:

From part c:

$$\theta=\frac{n\pi+\omega t+\Psi\ln\left(\frac{R}{R_0}\right)}{m}$$

We want to get:

$$\Delta R=R\left(e^{\frac{4\pi}{\Psi}}-1\right)$$

Using values given in the question,

$$\theta=\frac{\omega t+\Phi\ln\left(\frac{R}{R_0}\right)}{2}$$

Take R_0 to be the distance between the centre and the first crossing of the spiral arm, and ΔR to be the distance between the parts of the spiral arms.

Take time $t=0$.

$$\theta=\frac{1}{2}\Psi\ln\left(\frac{R}{R_0}\right)$$

$\theta=2\pi$ for the second crossing. $R=R_0+\Delta R$.

$$4\pi=\Psi\ln\left(\frac{R_0+\Delta R}{R_0}\right)$$

$$e^{\frac{4\pi}{\Psi}}=1+\frac{\Delta R_0}{R_0}$$

Rearrange, and it is as required.